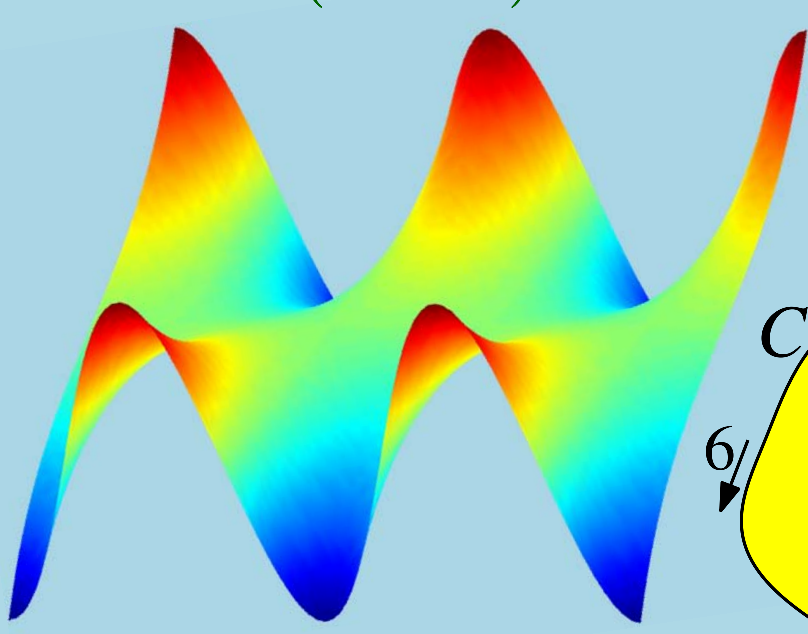
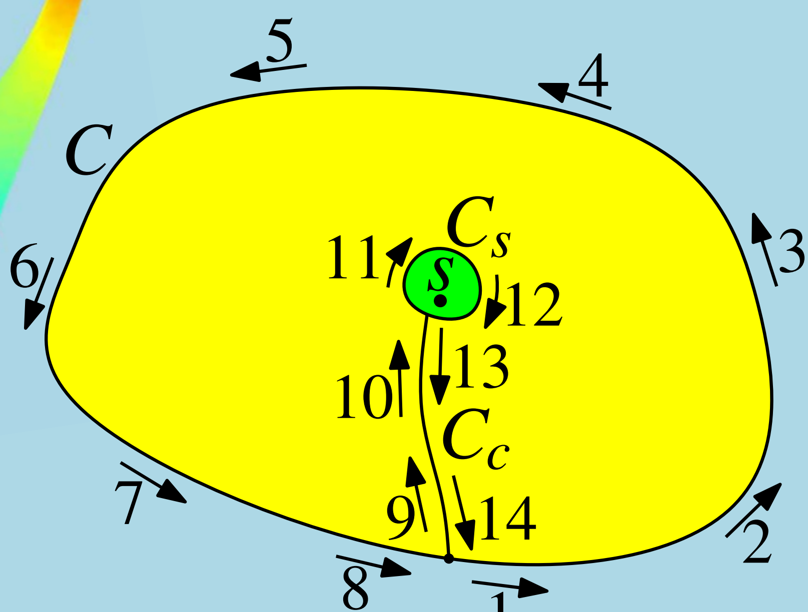
Elementary Complex Analysis

$\mathrm{Re}(\sinh z)$



$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f$$



$$f(z_0) = \frac{1}{i2\pi} \oint_C \frac{f(z)}{z - z_0} dz$$

$$f^{(n)}(z_0) = \frac{n!}{i2\pi} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

$$\oint_C w \, dz = \oint_{C_s} w \, dz$$

Taha Sochi

Preface

This book is about complex analysis which is a vital and fascinating branch of mathematics that has many applications in pure and applied mathematics as well as science and engineering. The book is similar in style, structure and objectives to my previous book "Introduction to the Mathematics of Variation". So, the book is basically a collection of solved problems with a rather modest theoretical background presented in the main text and hence it is largely based on the method of "learning by example and practice". However, solved problems are also used occasionally to address important theoretical aspects and fill potential gaps in the required theoretical background.

The motivating vision and objective in the writing of this book are primarily to provide a clear, interdependent and well structured exposition of this subject that benefits those who are at the level of undergraduate in general mathematics (especially real analysis) and have interest in extending their knowledge to complex analysis. Accordingly, simple language and method of presentation supported by a collection of clear and well defined symbols and terminology are used throughout the book to provide maximum transparency and accessibility even though some of these means may compromise the technicality of the book to a certain degree. Furthermore, many examples and cross references are provided within the main text and solved problems to enable the reader to understand the presented materials and appreciate their significance and relations to each other with minimum effort. We also structured and ordered the book within chapters, sections, subsections and problems in such a way that, in our view, provides maximum clarity and optimal graduality (like a gently rising stairway rather than a steep ladder or cliff).

The book covers the basics of the mathematics of complex numbers and variables (such as the meaning and basic operations of complex numbers) as well as advanced topics (like complex contour integration and Cauchy's theorem) and their applications. As hinted earlier, the structure of the book is that each topic investigated within a section or a subsection is introduced by a rather compact (but mostly sufficient) theoretical background. The theory is then followed by a number of solved problems in which the topic is further investigated, elaborated, discussed and applied. Cross referencing is used extensively to avoid repetition as well as to provide essential clarifications about relations and links (as indicated above).[1]

The materials in this book require a decent background in general mathematics (mostly in real analysis or calculus) at the level of the first or second year of a university program in mathematics or science. Knowledge and familiarity with the arithmetic and algebra of complex numbers and variables (as well as mathematical skills in dealing with these topics) will be a big advantage for the reader of this book although no such background is assumed in the writing and preparation of the book. As indicated above, the materials in this book have mainly practical objectives, and hence the intended audience of the book are essentially readers and students of science, engineering and applied mathematics. The book can be used as a text or as a reference for an introductory course on this subject as part of an undergraduate curriculum in mathematics or physics or engineering. The book can also be used as a source of supplementary pedagogical materials used in tutorial sessions associated with such a course.

Taha Sochi
London, May 2021

[1] The cross references in the electronic versions of the book are hyperlinked.

Contents

Nomenclature

In the following list, we define the common symbols, notations and abbreviations which are used in the book as a quick reference for the reader. The list may exclude what is used locally and casually. Also, we did not include the symbols and notations of standard functions (like e^z and $\cos y$) due to their common use and familiarity, but we included those which have distinct meaning or conventional use (e.g. $\log_e x$ and $\operatorname{Ln} z$) to avoid confusion.

!	factorial
* (asterisk)	complex conjugate
′ (prime)	total derivative (usually with respect to z)
$\cup$, $\cap$	union, intersection
$\forall$	for all
1D, 2D, 3D	one-dimensional, two-dimensional, three-dimensional
$\lvert a\rvert$, $\lvert z\rvert$	absolute value of $a \in \mathbb{R}$, modulus (or magnitude) of $z \in \mathbb{C}$
a_{-1}, ${}_k a_{-1}$	residue of a complex function (in its Laurent series expansion around a point)
A_{-1}, ${}_n A_{-1}$	residue of a product of complex functions
arg, Arg	argument, principal argument
C	curve (usually in the complex plane)
C	constant
$\mathbb{C}$	the set of complex numbers
D	domain in the complex plane
D_f	domain of function f
f, F	function
$f_1, f_2, \cdots, f_n$	functions
$f^{(n)}$	n^{th} derivative of f (i.e. $d^n f/dz^n$)
i	imaginary unit
iff	if and only if
$\operatorname{Im}(z)$	imaginary part of z
$\ln(z)$	natural logarithm of z
$\operatorname{Ln}(z)$	principal value or branch of $\ln(z)$
$\log_e x$	real natural logarithm of x (as defined in calculus for real positive x)
mod	modulus (or modulo) in modular arithmetic
n	integer number (possibly restricted, e.g. positive or non-negative)
N	neighborhood in the complex plane
p, p_1, p_2	powers (or exponents or indices)
P, P_n	polynomial, polynomial of order n
r, θ	modulus and argument of a complex number in polar form
R	region in the complex plane
R	radius
R_f	range of function f
$\mathbb{R}$	the set of real numbers
$\operatorname{Re}(z)$	real part of z

S	set of complex numbers (or points in the complex plane)
S, S_1, S_2	series
u, v	real and imaginary parts of a complex number in Cartesian form (i.e. $w = u + iv$)
w	complex number with Cartesian form $u + iv$
x, y	real and imaginary parts of a complex number in Cartesian form (i.e. $z = x + iy$)
z	complex number with Cartesian form $x + iy$ and polar form $re^{i\theta}$
θ_p	the principal value of the argument of a complex number in polar form
ρ, ϕ	polar coordinates of plane

Chapter 1
Preliminaries

In this chapter we introduce the topic of complex analysis and its subject (which is complex numbers, variables and functions especially analytic functions) and outline its characteristic features and mathematical foundations. We also provide some general remarks and outline basic conventions, notations, terminology and concepts that are needed in the presentation of this subject in this book. A list of essential and commonly used identities (with their proofs) is also appended at the end of this chapter to facilitate and substantiate the upcoming investigations and discussions.

1.1 Introductory Remarks, Conventions and Notations

We present in the following bullet points a number of general remarks related to conventions, notations and commonly occurring issues in this book.

• As indicated in the Preface, this book has mainly practical objectives and hence we generally disregard and avoid issues of highly theoretical nature which are usually found in textbooks and articles on pure mathematics.

• We may use "number" to mean "quantity" (i.e. a specific number or a general variable). Hence, "complex number" for example could mean $1 + i$ (which is a specific number) as well as $z = x + iy$ (which is a general variable). The ultimate meaning should be determined by the context which is generally clear and transparent.

• Based on the previous point, we use "complex number" in many contexts to mean complex number or/and variable.

• All the angles (representing for instance rotations and arguments of complex numbers) in this book are measured in radians (not in degrees). In this regard, we follow the common convention about the significance of the sign of angles, i.e. negative means clockwise and positive means anticlockwise. Also, "rotation" without identifying its center generally means "around the origin".

• Terms like "complex functions" generally mean "complex-valued functions of complex variables". Similarly, terms like "real functions" mean "real-valued functions of real variables".

• The terms related to 2D shapes (such as "circle", "square" and "rhombus") may be used conveniently to refer to surfaces (and hence they include their interior regions) or to refer to curves (and hence they represent their perimeters only). For instance, the "area of a circle" is an attribute of the circle as a surface, while the "curvature of a circle" is an attribute of the circle as a curve.[2]

• The arguments of some functions (like the natural logarithm ln) may be parenthesized or not according to convenience and clarity. Hence, we may write $\ln(z)$ or $\ln z$ if there is no potential confusion. We may also use different types of brackets (or containers) like round or square to provide maximum clarity in the specific context (with consideration of practical and aesthetic issues as well). In brief, the use (or non-use) of brackets and selecting their shape and size is generally determined by clarity and convenience rather than rules and conventions.

• In this book we deal with single complex variable problems, i.e. problems in which we have one independent complex variable (which is usually labeled z) and one dependent complex variable (which is usually labeled w). Those who are interested in multi complex variable problems should look elsewhere.

• We distinguish between imaginary numbers as the numbers of the form iy (with y being real) and complex numbers as the numbers of the form $x + iy$ (with x and y being real). In other words, imaginary numbers are complex numbers whose real part is zero. Accordingly, we have real numbers (like 1 and $-\sqrt{5}$), imaginary numbers (like $i\pi$ and $-i2$) and complex numbers (like $-2 + i5$ and $\sqrt{2} - i\sqrt{3}$). However,

[2] It should be noted that for circle we usually use "disk" to include the interior region.

in the extended sense of "complex numbers", real and imaginary numbers are no more than special cases of complex numbers although in certain contexts (and for specific purposes) they may be treated as disjoint sets (and hence "complex numbers" means those complex numbers that are not real or imaginary).
• We generally use "imaginary part" of a complex number to mean the part that is multiplied by i and hence it opposes the "real part" which is the part that is not multiplied by i (noting that both theses parts are real). For instance, if $z = x + iy$ then the real part is x and the imaginary part is y. We may also use "imaginary component" of a complex number to mean the part that contains i and hence it opposes the "real component" which is the part that does not contain i (noting that the real component is real while the imaginary component is imaginary). For instance, if $z = x + iy$ then the real component is x while the imaginary component is iy.
• Based on the two previous points, the distinction between "imaginary number", "imaginary part" and "imaginary component" should be made very clear to avoid confusion and misunderstanding.
• For the sake of clarity and convenience, the term "scalar" (rather than "real") may be used in some contexts to counter the term "complex" noting the similarity of complex numbers with vectors (and hence the similarity of real numbers with scalars). We note that scalar in this sense can include both the real and imaginary parts (noting that both of these are real).
• "Origin-centered" means centered on the origin of coordinates and "origin-punctured" means excluding the origin. Also, "unit circle" and "unit disk" mean circle and disk of radius 1.
• The definitions and identifications of the terminology and concepts of complex analysis (see for example § 1.5) are not agreed upon in general and hence the literature of complex analysis, like any other discipline, is not entirely consistent in this regard. This also applies to the mathematical conventions adopted by the authors in this field. The reader should therefore be aware of the specific choices and conventions of each particular author. In this book, we tried to pick the best and most common of those choices and conventions although this objective may not be reached (as we hoped) occasionally.
• In many situations certain conditions and requirements (such as continuity or differentiability to a certain order or being open or bounded) are necessitated by the contexts and circumstances. In such situations it should be understood that these conditions and requirements are satisfied even if they are not imposed explicitly. For example, when we discuss theorems and formulations that require second order differentiability then the functions involved in these theorems and formulations should have second order derivatives with no need for explicit declaration. This is to reduce avoidable congestion in the text and eliminate unwanted distraction.
• As we will see, complex numbers are commonly represented by the points of a coordinated plane (which is the complex plane that is usually coordinated by Cartesian or polar systems). Hence, "complex numbers" and "points of complex plane" may be used interchangeably. For instance, we may say: a given complex number is on a circle or inside it or on a parabolic curve although what is actually on or inside these curves are the points representing that number.
• We commonly use expressions like the positive or negative (or non-positive) real or imaginary axis meaning the positive or negative (or non-positive) parts of these axes.
• Since the scope of this book is complex analysis, we take the results and facts of real analysis (as well as any needed results and facts from other disciplines and branches) for granted although we may try to prove or verify some for specific and limited purposes.
• Some mathematical symbols and operations may have more than one meaning, and hence we generally rely on the context (as well as the vigilance of the reader) for identifying the intended meaning (although we may notify the reader in certain perplexing circumstances). For example, the double-bar symbol in $|\alpha|$ means modulus when α is a complex number (e.g. $|z|$) but it means absolute value when α is a real number and hence there should be no confusion about the intended meaning.[3] In fact, in this example we could use two symbols but we avoided this for simplicity and aesthetic considerations (as well as the obvious closeness and strong similarity between the two meanings).
• The domain of validity and applicability of mathematical formulations (such as functions, equations and systems of equations) is assumed to be the largest possible (within the given contexts and circumstances)

[3] Taking the modulus of a complex number and taking the absolute value of a real number may be seen as identical operations, but in our view they should not.

if no explicit definition of the domain is given.
• "Equation" and "equality" refer to mathematical relations symbolized with "$=$". "Equation" may also be used to refer to equivalence or identity relations (symbolized with "$\equiv$"). "Inequality" refers to mathematical relations symbolized with "$>, <, \geq, \leq$". "Semi-inequality" may be used to refer to mathematical relations symbolized with "$\geq, \leq$". "Non-equality" refers to mathematical relations symbolized with "$\neq$".
• The quadrants of rectangular Cartesian systems (coordinating complex planes) are identified according to their order ($1^{st}, 2^{nd}, 3^{rd}, 4^{th}$) as $++, -+, --, +-$.
• We use $\log_e x$ for the natural logarithm function of $x \in \mathbb{R}$ as defined in calculus (i.e. it is specifically *real* function of *real* variables greater than zero) while we use $\ln z$ for the natural logarithm function of $z \in \mathbb{C}$ (i.e. it is generally *complex* function of *complex* variables). More details about the distinction between $\log_e$ and ln will be given in the appropriate positions in the book (see for example § 2.2).
• The convention in this book is that the range of the real "function" $\arctan(y/x)$ (where x and y are real) is the interval $-\pi < \theta \leq \pi$ where the intervals:

$$-\pi < \theta < -\frac{\pi}{2} \qquad -\frac{\pi}{2} < \theta < 0 \qquad 0 < \theta < \frac{\pi}{2} \qquad \frac{\pi}{2} < \theta < \pi \tag{1}$$

correspond respectively to the $3^{rd}, 4^{th}, 1^{st}, 2^{nd}$ quadrants (i.e. $\frac{-y}{-x}, \frac{-y}{+x}, \frac{+y}{+x}, \frac{+y}{-x}$). The reason for this convention is to have a unique and non-ambiguous determination of the *principal argument* (i.e. $\operatorname{Arg} z \equiv \theta_p$) of complex numbers in polar form (in accord with our convention about the range of θ_p). Accordingly, we have:

$$\operatorname{Arg} z \equiv \theta_p = \arctan\left(\frac{y}{x}\right) \qquad (z \in \mathbb{C} \text{ and } x, y \in \mathbb{R} \text{ and } -\pi < \theta_p \leq \pi) \tag{2}$$

where $z = x + iy$. In fact, there are many issues (some of which are problematic) about this, but these issues will be investigated and clarified gradually as we progress in our investigation of complex analysis (see for example § 1.8.2).
• For the sake of diversity, we deliberately use different approaches and methods of presentation and discussion although we try to keep the style similar in general. This also applies to solving the mathematical problems where we deliberately use different methods and approaches when we have the freedom to do so. The purpose of this is to diversify the contents and provide the reader with more thorough techniques for tackling the problems of complex analysis. In fact, in some cases we solved some problems more than once using different methods of solution to achieve this purpose.

1.2 General Background about Complex Analysis

The subject of complex analysis is a vital mathematical branch that has many applications in pure and applied mathematics, science and engineering. In some sense, it is an extension and expansion to the real (or ordinary) analysis as represented by ordinary calculus and hence it heavily relies on the differential and integral calculus of real variables. Complex analysis has several advantages over real analysis. For example, 2D problems (in mathematics, science and engineering) can be modeled, formulated and solved more compactly and easily by the symbolism, formalism and techniques of complex analysis where the real and imaginary parts represent the two dimensions in one go. It is also a powerful tool for solving difficult real definite integrals and hence it is a vital aid and supplement to real analysis. In fact, it can be used to provide solutions to some real definite integrals that cannot be solved (analytically or at all) by the methods and techniques of real analysis (see for example § 5.4 and 7.3).

However, complex analysis (like anything else in this world) has also limitations. For example, it usually requires more complex mathematical machinery and techniques (as well as notations and symbolism) and could lead to rather confusing formulations and results (in comparison to the more intuitive real analysis) although this extra cost is compensated by the considerable ease and handiness that it affords.[4] Moreover, it is inherently restricted to 2D (and in fact to *flat* 2D) representations and correlations although certain extensions and generalizations can be made. So in brief, although complex analysis is an exceptionally powerful and valuable addition and extension to real analysis (and to mathematics in general) it is not the

[4] In fact, complex analysis can be the only way for tackling and solving some problems as indicated above.

ultimate solution and hence other mathematical methods and disciplines are still needed to tackle certain mathematical problems in analysis that cannot be tackled by complex analysis.

The subject matter of complex analysis is the investigation of the mathematical rules and applications of the set of complex numbers (as well as variables belonging to this set and the functions that correlate these variables) which are distinguished by the form $x+iy$ where x and y are real numbers/variables and i is the imaginary unit defined as the (positive) square root of -1, i.e. $i=\sqrt{-1}$. The fundamental principles of complex analysis are generally similar to the fundamental principles of real analysis and, in fact, they are based on them with certain modifications and extensions dictated by the rather strange (or unique) behavior of i which is meaningless in real analysis and hence it is not found there. Accordingly, we can say that complex analysis is the mathematical subject that investigates the rules and behavior of the set of real numbers $\mathbb{R}$ plus the number i (including any algebraic combination of $\mathbb{R}$ and i such as $3+i7$ where $+$ has its usual meaning and $i7$ means the product $i \times 7$). In other words, complex analysis investigates the rules and behavior of the numbers $\mathbb{C}$ which are made of the union $\mathbb{R} \cup i$ and their arithmetic and algebraic combinations.[5]

So, from this perspective complex analysis is just an extension of real analysis, and this should ease the task of developing and applying complex analysis because this task can be achieved by taking the principles and rules of real analysis to their logical consequences and implications by the addition of i with its characteristic behavior. Thus, if we are smart enough and we know real analysis very well (as well as knowing the properties and the rules of manipulation of i) then we can, in principle, derive and develop the entire complex analysis from real analysis by just following the logic. However, there are many tricks and obscure consequences and implications that cannot be easily captured and obtained by simple logic, and this, in fact, is the main task in the investigation of complex analysis where certain rules, methods, techniques and theorems are developed and obtained as a result of applying intricate mathematical tricks and techniques associated with sophisticated logic and reasoning.

Problems

1. Make a simple argument to support the statement that if a *general* complex formulation (such as equation or function) is to be correct then it should resemble its real counterpart (assuming the existence of this counterpart).

   Answer:[6] As we indicated earlier and will investigate later, real numbers (and variables) are a subset of complex numbers (and variables) and hence if a general complex formulation is to be correct then it should be valid on its entire domain including the real numbers and this should produce the real version of the complex formulation. Yes, if a complex formulation is not general (i.e. it excludes real numbers because it applies, for instance, on strictly complex numbers) then we may not have a real counterpart (let alone having similar real and complex formulations).

   Note: we can call this statement "the correspondence principle of complex analysis" (which resembles the correspondence principle of quantum physics). In fact, this statement (or principle) should provide a basis for an initial and quick test for any general complex formulation that we obtain from our mathematical reasoning and manipulation, i.e. if the formulation failed to produce its real counterpart when the complex variable is replaced by a real variable then it should be rejected.

1.3 Complex Numbers and Variables

"Complex number" may be defined generically as an ordered pair of real numbers where the first member of this pair (say x) is called the real part while the second member (say y) is called the imaginary part

[5] We note that for the purpose of introducing the subject of complex numbers and variables at this early stage in the book, the above description and characterization of complex analysis is focused on the domain of complex analysis. To be more technical, thorough and representative of the real essence of complex analysis, we may say: the subject matter of complex analysis is the investigation of complex functions of complex variables and their properties and behavior as well as the direct and indirect consequences of these properties and behavior (noting that this investigation requires extensive background investigation and knowledge about complex numbers and variables and their properties and behavior).

[6] This answer is pedagogical but not rigorous (and some terminology may not be very clear at this stage). However, more clarifications about this issue (and related issues) will come (see for instance § 1.11).

and hence the complex number can be expressed as (x, y). The imaginary part is usually distinguished by being a multiplicand of the imaginary unit i (with i being the square root of -1, i.e. $i = \sqrt{-1}$) and hence the pair can be represented as a sum of the real part plus the imaginary part times i (or a sum of the real component and the imaginary component; see § 1.1). So, if z is a complex number with real part x and imaginary part y then it is commonly represented as $z = x + iy$.[7] It is common to use $\text{Re}(z)$ to refer to the real part of z and $\text{Im}(z)$ to refer to its imaginary part. So, if $z = 3 + i6$ then $\text{Re}(z) = 3$ and $\text{Im}(z) = 6$ (not $i6$). It should be emphasized that the imaginary part is also real (as indicated) since it is the multiplicand of i (without i). However, we use "imaginary component" to refer to the piece of the complex number that contains i, as explained in § 1.1. We may also use terms like "imaginary numbers" or "pure imaginary numbers" to refer to numbers that have only imaginary component (like $i2$). So, the distinction between the terms "imaginary part", "imaginary component" and "imaginary number" should be clear.

Complex numbers have two main common representations or forms: Cartesian representation and polar representation. The Cartesian representation is based on splitting the number to real and imaginary parts (corresponding to the x and y coordinates of the familiar orthonormal Cartesian coordinate system) while the polar representation is based on splitting the number to modulus and argument[8] (corresponding to the ρ and ϕ coordinates of the familiar polar coordinate system). So, if z is a complex number then its Cartesian representation takes the form $z = x + iy$ (where x is the real part, y is the imaginary part and i is the imaginary unit) while its polar representation takes the form $z = re^{i\theta}$ (where $r \equiv |z|$ is the modulus, θ is the argument and i is the imaginary unit with e representing the exponential function). It should be noted that these two representations may also be given in the form of ordered pairs, i.e. $z = (x, y)$ in the Cartesian representation and $z = (r, \theta)$ in the polar representation. However, in this book we generally avoid using this form of representation except in exceptional circumstances.

Based on the above symbolic representations, a complex number (in its Cartesian form) can be represented graphically as a position vector in a 2D plane coordinated by an orthonormal Cartesian system where the real part of the number is represented by the x coordinate (or component) of the vector while the imaginary part is represented by the y coordinate (or component). Similarly, a complex number (in its polar form) can be represented graphically as a position vector in a 2D plane coordinated by a polar system where the modulus of the number is represented by the ρ coordinate of the vector while the argument is represented by the ϕ coordinate. Complex numbers are commonly symbolized with z and hence the plane on which complex numbers are represented graphically in Cartesian or polar forms is commonly known as the z plane.[9]

As we will see, any complex number z has a conjugate which is commonly symbolized with z^* (i.e. asterisk or star symbol).[10] Graphically, the conjugate of a complex number represents the mirror reflection of that number (as a position vector or as a point) in the x axis (according to the Cartesian representation) and hence it is obtained by reversing the sign of its imaginary part, i.e. if $z = x + iy$ then $z^* = x - iy$. Similarly, the conjugate graphically represents (according to the polar representation) the number itself (i.e. in magnitude) but rotated in opposite sense and hence the conjugate is obtained by reversing the sign of the argument of the number, i.e. if $z = re^{i\theta}$ then $z^* = re^{-i\theta}$.

Problems

1. List the common forms of expression of a complex number z (based on the Cartesian and polar forms of representation).
   Answer: We may list the following forms:
   - Cartesian form $z = x + iy$ (where x is the real part of z and y is its imaginary part).
   - Cartesian pair form $z = (x, y)$.

[7] In fact, i may precede or succeed y. In this book, we use the former for clarity despite the disadvantage of separating the minus sign when the imaginary part is negative.
[8] Or magnitude and phase angle.
[9] The z plane may also be called the complex plane. However, as we will see later (refer for example to § 1.5 and § 1.11) we normally have more than one complex plane (usually the z plane and the w plane) and hence the "complex plane" is more general than the "z plane" as it includes planes other than the z plane.
[10] It is also commonly symbolized with $\bar{z}$ (i.e. bar) but in this book we use z^* only.

• Polar exponential form $z = re^{i\theta}$ (where r is the modulus and θ is the argument).
• Polar pair form $z = (r, \theta)$.
• Polar trigonometric form $z = r(\cos\theta + i\sin\theta)$.

Note 1: in engineering j is commonly used to represent the imaginary unit i and hence we have $z = x + jy$, $z = re^{j\theta}$ and $z = r(\cos\theta + j\sin\theta)$.

Note 2: θ in $z = re^{i\theta}$ and $z = r(\cos\theta + i\sin\theta)$ should be in radians (if they are to be treated and manipulated freely by standard mathematical operations).[11]

Note 3: what we call "Cartesian form" may be called in the literature "rectangular form". What we call "Cartesian pair form" may be called in the literature "Cartesian form" or "Cartesian coordinates form" or "component form" or "component notation". What we call "polar exponential form" may be called in the literature "exponential form". What we call "polar pair form" may be called in the literature "phasor form" or "angle form" (noting that this form may also be given as $r\angle\theta$ or other similar notations). What we call "polar trigonometric form" may be called in the literature "polar form".

2. Complex number is intrinsically a 2D object. Explain how this 2D characteristic is expressed and represented symbolically and graphically.
   Answer: This is achieved by splitting the number to real and imaginary parts (according to the Cartesian representation) and by splitting the number to modulus and argument (according to the polar representation).
3. Outline the relation between the complex number in its polar representation and the employed coordinate system.
   Answer: The modulus r and the argument θ of a complex number in its polar representation correspond respectively to the ρ and ϕ coordinates in the polar system (with some distinctions with regard to the range of θ and ϕ that will become clear later on).
4. Express the imaginary unit i as a complex number (in various forms).
   Answer: In Cartesian (sum) form $i = 0 + i$. In Cartesian pair form $i = (0, 1)$. In polar exponential form $i = e^{i\pi/2}$. In polar pair form $i = (1, \pi/2)$. In polar trigonometric form $i = \cos(\pi/2) + i\sin(\pi/2)$.
5. How to represent real numbers (like -6 and π) and pure imaginary numbers (like $-i3$ and $i9$) as complex numbers in Cartesian pair form?
   Answer: Real numbers (which may be symbolized by x) have zero imaginary part and hence they are represented in Cartesian pair form as $(x, 0)$. Imaginary numbers (which may be symbolized by iy) have zero real part and hence they are represented in Cartesian pair form as $(0, y)$.
6. What is common between real numbers and pure imaginary numbers?
   Answer: It is the number zero because zero (as a complex number) is represented in Cartesian pair form as $(0, 0)$ and hence it fits the characteristic of real numbers and the characteristic of imaginary numbers (as given in Problem 5) at the same time. In other words, it can be seen as a real number and as a pure imaginary number (as well as a complex number) and hence it is unique in this regard. This originates from the fact that the number zero is on the real axis (which represents the real numbers) and on the imaginary axis (which represents the imaginary numbers) as well as on the complex plane (which represents the complex numbers) and hence it is common to all (since the origin of coordinates which represents the number zero is common to all).
7. A given point in the Cartesian plane is represented uniquely by a single complex number in Cartesian form but it is represented non-uniquely by infinitely-many complex numbers in polar form. Discuss this issue.
   Answer: This can be easily explained by the relationship between the Cartesian coordinate system and the polar coordinate system in conjunction with elementary trigonometric facts. In brief, if a given

[11] This is based on the fact that these functions (i.e. exponential, cosine and sine) are generally subject to standard analytical operations (such as differentiation, integration, power series expansion, taking inverse, etc.) and hence they should be based on naturally (rather than conventionally) measured variables for these operations to apply correctly and in their standard forms (noting that these standard analytical operations are derived and formulated using natural variables, i.e. radian in this case). As indicated earlier, in this book we use radians exclusively for measuring angles and rotations and quantifying arguments in all forms of polar representation (as well as any other purpose).

point in the Cartesian plane is represented by a complex number z where $z = x + iy$ in Cartesian form and $z = r\cos\theta + ir\sin\theta$ in polar trigonometric form (see Problem 1) then on comparing these two forms we get:

$$x = r\cos\theta = r\cos(\theta + 2n\pi) \qquad \text{and} \qquad y = r\sin\theta = r\sin(\theta + 2n\pi)$$

where the addition of $2n\pi$ (with n being an integer) is justified by the periodicity of the cosine and sine functions. As we see, adding $2n\pi$ affects the polar form (since $\theta \neq \theta + 2n\pi$ in general) but not the Cartesian form. Hence, by adding $2n\pi$ we get infinitely-many complex numbers in polar form representing the same point in the Cartesian plane without affecting the Cartesian form of that point. In other words, the point is represented by a single complex number in Cartesian form but by infinitely-many complex numbers in polar form.
Note: based on the above investigation, the argument θ in the polar form of a complex number z representing a given point in the Cartesian plane is commonly given as:

$$\arg(z) \equiv \theta = \theta_p + 2n\pi \equiv \operatorname{Arg}(z) + 2n\pi$$

where $\theta_p \equiv \operatorname{Arg}(z)$ is known as the principal argument which is commonly taken to be in the range $-\pi < \theta_p \leq \pi$ (or $0 \leq \theta_p < 2\pi$ according to another convention which we do not use in this book).

8. Define the principal argument formally in all parts of the z plane.
Answer: We may define it as follows (taking $-\pi < \operatorname{Arg} z \leq \pi$ and $-\frac{\pi}{2} < \operatorname{Arctan}\left(\frac{y}{x}\right) < \frac{\pi}{2}$ and considering the sign of y/x as combined):

$$\begin{aligned}
&\operatorname{Arg} z = \operatorname{Arctan}\left(\frac{y}{x}\right) && \text{(in the first and fourth quadrants)}\\
&\operatorname{Arg} z = \operatorname{Arctan}\left(\frac{y}{x}\right) + \pi && \text{(in the second quadrant)}\\
&\operatorname{Arg} z = \operatorname{Arctan}\left(\frac{y}{x}\right) - \pi && \text{(in the third quadrant)}\\
&\operatorname{Arg} z = \operatorname{Arctan}\left(\frac{y}{x}\right) = 0 && \text{(on the positive real axis)}\\
&\operatorname{Arg} z = \pi && \text{(on the negative real axis)}\\
&\operatorname{Arg} z = \frac{\pi}{2} && \text{(on the positive imaginary axis)}\\
&\operatorname{Arg} z = -\frac{\pi}{2} && \text{(on the negative imaginary axis)}\\
&\operatorname{Arg} z \;\text{ is not defined} && \text{(on the origin)}
\end{aligned}$$

1.4 Mathematical Foundations of Complex Analysis

In this section we summarize the main general mathematical foundations required to establish the mathematics of complex numbers (in addition to the basic definitions and conventions of this mathematics such as the definition of i and the rules of its manipulation).[12] In brief, we use the following three real power series as our starting point:[13]

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{3}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{4}$$

[12] In fact, the mathematical foundations of complex analysis should include the subject of limits (which is investigated in § 1.9). However, we deferred the investigation of limits for structural reasons.

[13] These series expansions are established in the textbooks of the calculus of real variables (as represented here by x). In fact, these series expansions can be easily obtained from the standard form of the Maclaurin series.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{5}$$

Now, if we accept that the series of e^x applies to iy (i.e. by replacing x with iy where y is real) and the rule of addition of exponents applies to complex numbers as to real numbers, then on using the Cauchy product of two power series and the binomial theorem we get:

$$\begin{aligned} e^z &\equiv e^{x+iy} = e^x e^{iy} = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \times \left(\sum_{n=0}^{\infty} \frac{(iy)^n}{n!}\right) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{x^m}{m!} \frac{(iy)^{n-m}}{(n-m)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} x^m (iy)^{n-m} = \sum_{n=0}^{\infty} \frac{(x+iy)^n}{n!} \end{aligned}$$

that is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \tag{6}$$

which is the same as the series of e^x (as given by Eq. 3) with z replacing x. Now, from Eq. 6 with $z = iy$ (or more directly from our aforementioned acceptance that the series of e^x applies to iy) we get:

$$\begin{aligned} e^{iy} &= \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots \\ &= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} + \cdots = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \\ &= \cos y + i \sin y \end{aligned} \tag{7}$$

where we used Eqs. 4 and 5 (with y replacing x).

Now, if we extend the definitions of the trigonometric cosine and sine functions to include those with complex arguments (so we can make sense of $\cos z$ and $\sin z$ and their alike) then we should link them to the above definition and formulation of the complex exponential function, and hence we extend the series expansions of Eqs. 4 and 5 to include complex arguments.[14] In fact, all we need to do is to see if the proposed extension is consistent with the above definition and formulation. Now, if in Eq. 7 we replace y (which is real) by z (which is complex) then we should have $e^{iz} = \cos z + i \sin z$ (where $\cos z$ and $\sin z$ are the proposed extended complex functions). So, all we need to do is to check if this will produce a (complex) series expansion that is consistent with the (real) series expansions of Eqs. 3-5. Now, on using the power series of Eq. 6 with iz replacing z, we get:

$$\begin{aligned} e^{iz} &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \cdots \\ &= 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} + \cdots = \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) \end{aligned}$$

that is

$$e^{iz} = \cos z + i \sin z \tag{8}$$

where in the last step we used the power series expansion of cosine and sine functions in the forms given by Eqs. 4 and 5 but with z (which is complex) replacing x (which is real).[15] This means that the proposed

[14] "Arguments" here and in similar contexts means arguments of functions (i.e. z in $\cos z$ and $\sin z$) and not the arguments of complex numbers (in the context of representing a complex number by its modulus and argument as explained in § 1.3). In fact, the readers should be aware of this multiple use of "argument" throughout the book to avoid misunderstanding.

[15] In fact, we also used the assumption that the rules of algebraic addition apply to infinite series as to finite sums. We should also note the triviality of the above check which is done for the purpose of demonstration and consolidation.

extension will lead to $e^{iz} = \cos z + i \sin z$ (which is consistent with Eq. 7 which we derived earlier) provided that the series of e^z, $\cos z$ and $\sin z$ are given by the forms of Eqs. 3-5. So, we simply have total consistency because all our definitions, conventions, assumptions, equations and series expansions are consistent with each other (as well as being consistent with our heritage from calculus and real analysis).

To sum up, starting from the real series expansions of Eqs. 3-5 (supported by a few intuitive and acceptable assumptions and definitions as well as the definition of i and the rules of its manipulation) we extended the real series expansions of Eqs. 3-5 to their complex equivalents. In other words, we can now use the series expansions of the exponential, cosine and sine functions as given by Eqs. 3-5 but with the complex variable z replacing the real variable x. We can also use the newly manufactured relation between the exponential, cosine and sine functions which is a very powerful tool in complex analysis. This achievement, which may seem limited in its theoretical and practical significance and value, is very important and has far reaching consequences. In fact, it will open a wide gate for importing a large part of the mathematics of real analysis and facilitate the task of developing the mathematics of complex analysis because, as we will see, large parts of complex analysis are based on these functions and their series and hence the correspondence between the real and complex forms of these functions and series will naturally extend the mathematics of real analysis to its complex counterpart. In other words, complex analysis will in essence be an extension and generalization of real analysis (as explained earlier in § 1.2) and hence we can import loads of rules, formulae, methods, techniques, etc. from real analysis and convert or adapt them for the use in complex analysis with minimum effort.

Problems

1. Verify the following identities (where z, z_1, z_2 are complex and x, y are real):

   (**a**) $e^z = e^x (\cos y + i \sin y)$. (**b**) $\cos(-z) = \cos z$. (**c**) $\sin(-z) = -\sin z$.

   (**d**) $e^{-iz} = \cos z - i \sin z$. (**e**) $e^{z_1} e^{z_2} = e^{z_1 + z_2}$. (**f**) $e^{z_1}/e^{z_2} = e^{z_1 - z_2}$.

   (**g**) $\cos z = \frac{e^{iz}+e^{-iz}}{2}$. (**h**) $\sin z = \frac{e^{iz}-e^{-iz}}{i2}$.

   Answer: We should note first that in Problems like this what is basically required (in part at least) is to "verify" rather than "prove" because the given relations are either based on assumptions that we accepted earlier or can be obtained as instances of the above-explained extension of the mathematics of real analysis to its complex counterpart. So, what we actually do is to verify the consistency of our definitions, formulations, rules, relations, results, implications, conclusions, etc.

   (**a**) From the definition of z (i.e. $z = x + iy$) and the rules of exponents as well as Eq. 7 we obtain:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y) \tag{9}$$

   (**b**) From the series expansion of $\cos z$ (with $-z$ replacing z) we have:

$$\cos(-z) = \sum_{n=0}^{\infty} \frac{(-1)^n (-z)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n} z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \cos z$$

   (**c**) From the series expansion of $\sin z$ (with $-z$ replacing z) we have:

$$\sin(-z) = \sum_{n=0}^{\infty} \frac{(-1)^n (-z)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1} z^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = -\sin z$$

   (**d**) This is an instance of Eq. 8, that is:

$$e^{-iz} = e^{i(-z)} = \cos(-z) + i \sin(-z) = \cos z - i \sin z \tag{10}$$

   where we use $\cos(-z) = \cos z$ and $\sin(-z) = -\sin z$ which are verified in parts (b) and (c).

   (**e**) We use Eq. 9 (as well as known rules of real exponential functions and familiar real trigonometric identities some of which are verified earlier), that is:[16]

$$e^{z_1} e^{z_2} = e^{(x_1+iy_1)} \times e^{(x_2+iy_2)} = e^{x_1} (\cos y_1 + i \sin y_1) \times e^{x_2} (\cos y_2 + i \sin y_2)$$

[16] In fact, this is just a confirmation of the consistency of our aforementioned assumption that the sum rule of exponents applies to complex variables as to real variables (and hence the verification is a trivial demonstration).

$$\begin{aligned} &= e^{x_1+x_2}\left[(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i\,(\cos y_1 \sin y_2 + \sin y_1 \cos y_2) \right] \\ &= e^{x_1+x_2}\left[\cos(y_1+y_2) + i \sin(y_1+y_2) \right] = e^{(x_1+x_2)+i(y_1+y_2)} \\ &= e^{(x_1+iy_1)+(x_2+iy_2)} = e^{z_1+z_2} \end{aligned}$$

(**f**) From part (a) and known facts we have:

$$\begin{aligned} \frac{1}{e^{z_2}} &= \frac{1}{e^{x_2+iy_2}} = \frac{1}{e^{x_2}(\cos y_2 + i \sin y_2)} = \frac{e^{-x_2}(\cos y_2 - i \sin y_2)}{(\cos y_2 + i \sin y_2)(\cos y_2 - i \sin y_2)} \\ &= \frac{e^{-x_2}(\cos y_2 - i \sin y_2)}{\cos^2 y_2 + \sin^2 y_2} = e^{-x_2}(\cos[-y_2] + i \sin[-y_2]) = e^{-x_2-iy_2} = e^{-z_2} \end{aligned}$$

Hence, from the result of part (e) we get:

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1} \times \frac{1}{e^{z_2}} = e^{z_1} e^{-z_2} = e^{z_1+(-z_2)} = e^{z_1-z_2}$$

(**g**) We use Eqs. 8 and 10, that is:

$$\frac{e^{iz}+e^{-iz}}{2} = \frac{(\cos z + i \sin z) + (\cos z - i \sin z)}{2} = \frac{2\cos z}{2} = \cos z \tag{11}$$

(**h**) We use Eqs. 8 and 10, that is:

$$\frac{e^{iz}-e^{-iz}}{i2} = \frac{(\cos z + i \sin z) - (\cos z - i \sin z)}{i2} = \frac{i2 \sin z}{i2} = \sin z \tag{12}$$

1.5 General Terminology and Concepts of Complex Analysis

As indicated earlier, complex numbers are commonly notated with z ($= x + iy$ with x being the real part and y being the imaginary part) and represented by points of a plane (where the horizontal dimension of the plane represents the real part while the vertical dimension represents the imaginary part). Accordingly, this plane is called the **complex plane** or the **z plane**.[17] The (finite) complex plane plus infinity (which is usually represented by a single point) is called the **extended complex plane**. A set S representing points z in the complex plane that satisfies the inequality $|z - z_0| < \rho$ (with z_0 being a given point in the plane and ρ being a positive real number) is called a **neighborhood** of z_0 (or more informatively "a ρ-neighborhood of z_0"). This set obviously represents the interior of a circular disk (of radius ρ) centered on z_0 and hence it may also be called **open disk**.[18] A neighborhood of a given point z_0 excluding z_0 itself is called **deleted neighborhood of z_0** (which may also be called **deleted disk** or **punctured disk**). This neighborhood is obviously represented by the relation $0 < |z - z_0| < \rho$. A **boundary point** of a set S in the complex plane is a point whose every neighborhood contains at least one point inside S and at least one point outside S. The **boundary** of a set S in the complex plane is the set of all the boundary points of S. A point z_0 in the complex plane is described as an **interior point** of a given set S (made of points in the complex plane) if there is a neighborhood of z_0 that lies entirely inside S. If all the points of a given set S in the complex plane are interior points then S is an **open set**. A point z_0 in the complex plane is described as an **isolated point** (with respect to a given property) if it is the only point in its neighborhood that have this property (i.e. it is not part of a curve or a 2D patch that have this property).[19]

[17] As indicated earlier, "complex plane" is more general than "z plane".

[18] In fact, the definition of neighborhood (as given above and as it is generally stated in the literature) is restricted to the case where ρ is constant and it identifies the boundary of the neighborhood in all directions. Otherwise, the neighborhood (in a more general sense) is *contained* in this disk (noting that this more general sense should not be excluded by a convention).

[19] For instance, an isolated singularity (of a given function) is the only singular point in its neighborhood (for that function), i.e. the function is analytic in some punctured disk around the singularity. Similarly, an isolated zero (of a given function) is the only point in its neighborhood at which that function vanishes, i.e. the function is non-zero in some punctured disk around the zero.

A curve in the complex plane is commonly called **contour**.[20] A t-parameterized continuous curve (or contour) represented as $z(t)$ is **closed** if $z(t_1) = z(t_2)$ (where $t_1 \leq t \leq t_2$ with t_1 and t_2 being the parameters of its end points) and it is **open** otherwise.[21] A closed contour can be tracked clockwise or anticlockwise.[22] The common convention is that the positive sense of tracking is anticlockwise while the negative sense of tracking is clockwise. Accordingly, a closed curve is described as **positively oriented** if its sense of tracking is anticlockwise and **negatively oriented** if its sense of tracking is clockwise. A curve is described as **simple** if it does not cross or touch itself.[23] A **polygonal path** (or line or curve) is an unbroken (or continuous) curve made (possibly) of segments of straight lines.[24]

A set S in the complex plane[25] is described as **connected** if every pair of points in S can be connected by a polygonal path that lies entirely in S. A **domain** is an open connected (2D) set in the complex plane. A domain in the complex plane with some or none or all of its boundary may be called a **region**. A region that includes its boundary is a **closed region** while a region that excludes its boundary is an **open region**.[26] A region that lies inside a disk of finite radius is described as **bounded region**; otherwise it is **unbounded**. A region in the complex plane is **simply-connected** if every closed curve in the region can shrink continuously (i.e. without leaving the region) to a point in the region. Accordingly, a region with a hole[27] in it is not simply-connected because a closed curve surrounding the hole cannot shrink to a point in the region continuously since it must leave the region at the hole if it should shrink to a point in the region (or alternatively it shrinks to a point in the hole which is outside the region). A region in the complex plane is described as **multiply-connected** if it can be divided into a finite set of simply-connected regions (although it as a whole is not simply-connected).

As in real analysis where we have real functions (or functional relationships) that correlate real dependent variables to real independent variables such as $y \equiv f(x) = x + 1$ $(x, y \in \mathbb{R})$, in complex analysis we have **complex functions** that correlate complex dependent variables to complex independent variables such as $w \equiv f(z) = z + 1$ $(z, w \in \mathbb{C})$. Also, as in real analysis where real functions are seen as mappings (or transformations) from the x real line onto the y real line, in complex analysis complex functions are seen as mappings from the ***z* complex plane** onto the ***w* complex plane**. Accordingly, complex function can be defined generically by the following relation:

$$w(z) = f(z) = u(x, y) + iv(x, y) \tag{13}$$

where w $(= u + iv)$ and z $(= x + iy)$ are complex variables while u, v, x, y are real variables (with each of u and v being a function of x and y in general as indicated in the above equation) and i is the imaginary unit.

A complex function $f(z)$ that is defined in a neighborhood of a given point z_0 (except possibly at z_0)

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[20] In fact, "contour" may be used by some for a more specific type of curve (e.g. closed). So, the convention of each author should be followed.

[21] In this statement we are assuming the curve to be "finite", i.e. it does not go to infinity. We also assume t to be real continuous variable that takes all the values of the interval $t_1 \leq t \leq t_2$.

[22] It may be sensible to identify the sense (or direction) of tracking of some open curves (which have obvious rotational curvature or they are part of or similar to known closed curves like circles and squares) by "clockwise" and "anticlockwise" although the sense (or direction) of tracking is usually identified by identifying the start and end points of tracking.

[23] We note that the curves in this book are generally simple unless stated otherwise.

[24] In fact, being straight should not be necessary (although it is labeled "polygonal").

[25] We should note that a phrase like "in the complex plane" here and in similar contexts is not a restriction on the definition and its validity and applicability but because in complex analysis (which is the subject of interest to us in this book) we are mainly interested in objects and sets that belong to the complex plane. Otherwise, many of the definitions and identifications in this section (and even elsewhere) are broad and they apply to surfaces in general (whether plane or not and whether in the complex plane or not) and some even extend beyond surfaces.

[26] We may also define "partially closed region" and "partially open region" accordingly. We should also note that in most cases the sensibility of the given statements requires the "region" to be open (where we usually rely on the context and this understanding for this extra "open" condition). In fact, "domain" should be the right word to replace "region" in most cases but "region" is preferred for its generality and clarity (noting that there are several conventions and opinions about the meaning of "domain" in complex analysis as well as multiple usages in general).

[27] "Hole" here (and in similar contexts) is general and hence it includes, for instance, a single point and a curve.

in the z plane is said to have a **limit L at z_0**, symbolized as

$$\lim_{z \to z_0} f(z) = L \tag{14}$$

if for each real number $\varepsilon > 0$ there is a real number $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.[28] A complex function $f(z)$ is described as **continuous at a point z_0** in the z plane if[29]

$$\lim_{z \to z_0} f(z) = f(z_0) \tag{15}$$

A function is described as **continuous over a region** (or a curve) if it is continuous at the points of that region (or curve). A function is described as **continuous on a boundary point** z_0 of a region R if the limit of Eq. 15 does exist for $z \in R$. A function is described as **continuous** if it is continuous at all points in its domain of definition.[30] A complex function $f(z)$ that is defined in a neighborhood of a given point z_0 is said to be **differentiable at z_0** and it has a **derivative f' at z_0** given by the limit

$$f'(z_0) \equiv \left.\frac{df}{dz}\right|_{z=z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{16}$$

if this limit does exist.[31] A function is described as **differentiable over a region** (or a curve) if it is differentiable at the points of that region (or curve). A function is described as **differentiable** if it is differentiable over its entire domain. A complex function $f(z)$ is described as **analytic at a point z_0** in the complex plane if $f(z)$ is continuously differentiable at z_0 and at every point in some neighborhood of z_0.[32] A function is described as **analytic over a region** (or a curve) if it is analytic at the points of that region (or curve). A function is described as **analytic** if it is analytic over its entire domain.[33] A complex function that is analytic at every point in the (finite) z plane is described as an **entire function** (or **integral function**). An analytic function is described as **analytically continued** if its domain of definition is extended beyond its original extent by being identical in value to another function (which is defined on a larger domain) on a connected set (in their common domain) and this process is called **analytic continuation**. A function is described as **bounded** (over a region in the z plane or over the entire z plane) if its magnitude is finite at all points (of the region or the z plane); otherwise it is **unbounded**.[34] A function is described as **singular at a point z_0** in the complex plane if it does not behave well at z_0 in one way or another such as by being unbounded at z_0 (e.g. by having non-zero

[28] In some of these statements we use "if" although we can use "*iff*" because we are in the process of making definitions and identifications rather than making mathematical statements and theorems.

[29] In simple terms, "continuous" means "having no sudden jump".

[30] Statements like this may be extended to the entire complex plane (when the domain extends this far). However, we should note that expressions like "entire complex plane" may need to be restricted (in certain contexts and circumstances) to the "finite complex plane". More discussion about these issues will be given later (see for example Problem 3).

[31] As we will see later (refer to § 1.9), the uniqueness of the value of the limit and its independence of the direction of approach to z_0 is a necessary condition for its existence.

[32] We should note that "differentiable at z_0 and at every point in some neighborhood of z_0" may contain some sort of repetition which we can justify by our desire to be very clear and transparent. We should also note that another condition is usually imposed in the definition of *analytic function* that is being single-valued (and hence only single-valued functions and single-valued branches of multi-valued functions should be considered) noting that the condition of being single-valued is more appropriate to belong to "*function*" than to "*analytic*" and hence this condition should not be specific to analytic functions. Also, it should be obvious that for a function to be analytic at a point or on a region it should beforehand be well defined there (and this should be understood wherever this term occurs with no necessity for stating this condition explicitly).

[33] We should note that being analytic may also be defined by having a power series representation (which is obviously related to differentiability). In fact, a function is analytic at a point *iff* it can be represented by a (Taylor) power series at that point (i.e. converging in a neighborhood of that point). These issues will be investigated further later on (see for instance § 5.1).

[34] "Bounded" and "unbounded" may also be used to characterize sets (regardless of representing functions or not) and regions (as explained earlier). So, the reader should be aware of this distinction to avoid confusion and misunderstanding. We should also note that a complex function (or number or variable) may be described as finite *iff* both its real and imaginary parts are finite. In fact, being finite (in this context) and being bounded are equivalent although the use of one term or the other may be more appropriate according to the context and situation.

numerator and zero denominator) or by being undefined at z_0 (e.g. by having zero numerator and zero denominator which is meaningless or indeterminate) or by ceasing to be differentiable or analytic at z_0.[35] Accordingly, a point in the complex plane at which a function $f(z)$ becomes singular (e.g. by having zero denominator) is described as **singular point** or **singularity** of f. A point z_0 is described as a **zero of a function** $f(z)$ if f vanishes at z_0, i.e. $f(z_0) = 0$. An analytic function $f(z)$ has a **zero of order n** at z_0 ($n > 0$) if $f^{(n)}(z_0) \neq 0$ but

$$f(z_0) = f^{(1)}(z_0) = \cdots = f^{(n-1)}(z_0) = 0$$

where the parenthesized index (in the superscript) stands for derivative (with respect to z) of that order. If none of the derivatives of f vanish at z_0 then z_0 is a zero of order 1 and is commonly called **simple zero**. A singular point z_0 of an analytic function is a **pole of order n** if the reciprocal of the function has a zero (or rather isolated zero) of order n at z_0.[36]

A **branch** of a complex multi-valued function f (e.g. the logarithm function $\ln z$ which is investigated in § 2.2; also see Problem 11 of 1.8.7 for another example) is a single-valued function F that is similar in form to f (e.g. the principal logarithm function $\mathrm{Ln}\, z$ which is investigated in § 2.2) and it is continuous on its domain.[37] A **branch cut** of a multi-valued function f is a set of points (usually a line) in the complex plane that must be removed from the domain of f to create a branch of f (also see Problem 20).[38] A **branch point** is a point z_0 in the complex plane at which a multi-valued function passes from one branch to another when going around an arbitrarily-small closed curve around z_0 (also see Problem 21 of the present section as well as Problem 24 of § 2.2). In fact, these concepts and terms are usually associated with functions like the square root and the complex logarithm which will be investigated later (see for example § 1.8.11, § 1.11 and § 2.2) and hence more clarifications about these issues are to be expected in the future.

Problems

1. Classify the following sets as bounded/connected/open or otherwise:
   (**a**) The strip $-2 \leq y \leq 8$.
   (**b**) The interior of a disk with center $(-3, 0)$ and radius 5.
   (**c**) The set represented by $|x| > 6$.
   (**d**) The set represented (in polar form) by $5 < r \leq 9$.
   Answer:
   (**a**) This is unbounded, connected and closed.[39]

[35] Whether lack of analyticity (i.e. by lack of differentiability specifically in the above sense with no other reason for not behaving well) implies having singularity or not seems to be a contentious issue and hence it is a matter of opinion and convention. However, in this book we generally regard lack of analyticity (in the sense of lack of differentiability) as a cause for being singular (and this seems to be the dominant convention). In fact, some authors define "singular point" of a complex function as a point in the complex plane at which the function fails to be analytic noting that this definition is essentially practical (for the benefit of complex analysis which is mostly interested in analytic functions and analyticity) rather than being a definition for the concept of "singular point". Anyway, we chose the above definition to be more representative and general from practical and conceptual perspectives.

[36] A "pole of order n" may be defined differently (see § 3.3) although these definitions are generally equivalent.

[37] We should note that discussing the issue of "branch" and its associates is a little bit difficult and messy especially at this early stage in the book where some of the concepts and terms are not fully explained and may not be clear to the reader. However, we feel obliged to do so for the sake of completeness (as well as having to do so somewhere in the book and it seems that nowhere is fully free of problems and complications). We should also note that whether it is appropriate (or not) for a multi-valued relation to be called "function" is a matter of convention and opinion and it differs from one author to another. Anyway, for the sake of convenience it is still possible to use "function" loosely for multi-valued relations while preserving its technical sense for single-valued relations (as we generally do in this book). Accordingly, a multi-valued "function" is a union of single-valued functions (where the quotation marks indicate its loose or non-technical usage).

[38] It should be noted that we may have more than one branch cut (and even infinitely-many branch cuts); see Problem 25 of § 2.2. However, it is important to be aware that in the literature the plurality of "branch cuts" (and similar plurality expressions like "each branch cut" or "any branch cut") may mean two different things, i.e. once it means a single branch cut represented by a single curve but it is attributed to different branches and hence it is plural from this perspective, and once it means multiple branch cuts represented by multiple curves and hence it is plural from this perspective (as in Problem 25 of § 2.2 which we indicated above).

[39] Being closed (i.e. totally) may be debated.

(**b**) This is bounded, connected and open.
(**c**) The relation $|x| > 6$ is equivalent to the union $(x < -6) \cup (x > 6)$, i.e. the strip (or half-plane) to the left of the vertical line $x = -6$ and the strip (or half-plane) to the right of the vertical line $x = 6$. Therefore, this set is unbounded, unconnected and open.
(**d**) This set is the origin-centered ring (or annulus) with interior radius 5 and exterior radius 9 where its interior boundary (i.e. the circle $r = 5$) is excluded and its exterior boundary (i.e. the circle $r = 9$) is included. It is bounded, connected (but not simply-connected) and neither open nor closed (because it is open regarding its interior boundary and closed regarding its exterior boundary and hence we may describe it as partially open or/and partially closed).

2. Give some examples of continuous functions.
   Answer: The polynomial, exponential, and trigonometric and hyperbolic cosine and sine functions are continuous over the entire complex plane.[40] The reciprocal (or inverse) function (i.e. $1/z$) is continuous over the entire complex plane except at $z = 0$. The sums, differences, products and compositions of continuous functions are also continuous (with some conditions).
3. Give some examples of entire functions.
   Answer: Common examples are polynomial, exponential, and trigonometric and hyperbolic cosine and sine functions. Their sums, differences, products and compositions (as well as their derivatives and integrals) should also be included (with some conditions).[41]
   Note 1: it should be obvious that by definition "entire" is an attribute of complex functions[42] and hence when we talk about entire functions (of certain types like those given above) we should mean complex functions of complex variables. For example, the exponential functions that are entire should be restricted to the complex exponentials of complex variables (i.e. excluding for instance the exponentials that are defined on the *entire* real or imaginary axis exclusively) because if an exponential has the form e^x or the form e^{iy} (with x and y being real) then it will not be analytic (let alone be entire).[43] This should similarly apply to "analytic". So, when we talk about the analyticity or entirety of exponential functions we mean the complex exponentials of complex variables, i.e. of the form e^z where $z = x + iy$ (with x and y being real). This should apply to all types of entire functions (as indicated above) as well as analytic functions (i.e. in the sense of complex analysis). For more details, see Problem 4 of § 3.1. Also see § 4.5.
   Note 2: most of the above entire functions (and their alike) blow up at infinity (and hence they are singular there) and this may cast a shadow on their entirety. However, this is not the case because infinity should be excluded (when talking abut entirety and its alike) since it is not an ordinary region or point in the z plane. So, in the definition of analyticity and entirety infinity must be excluded (unless it is specifically required in the particular instance or context) although in other contexts and circumstances it may be or must be included (also see Problem § 2 of § 4.5). Accordingly, we may define entire function as "a complex function that is analytic at every point in the *finite* z plane" because otherwise we will not have entire functions (other than constant functions according to Liouville's theorem which will be investigated in § 4.5).
4. Referring to the above definitions (as given in the text), it seems that identifying the attributes of complex functions (like continuity, analyticity and entirety) is very difficult job. Comment on this.
   Answer: Yes, this is the case if we have to rely only on these definitions. However, these definitions are needed only to identify the attributes of some basic forms of functions (e.g. exponentials

[40] In Problem 7 of § 1.9 it is shown that analytic function is continuous and in Problem 4 of § 3.1 it is shown that these functions are entire and hence this statement is established (although may not be in the best and direct way).

[41] The purpose of this Problem and its answer is to have examples of generic forms and types of entire functions without going through technical (and confusing) details. So, to be rigorous we need (as indicated above) to apply certain restrictions (some of which will be clarified in the future). We also note that the above also applies (by priority) to "analytic" (in place of "entire").

[42] In fact, even "analytic" (in the technical sense of complex analysis) is an attribute of complex functions although it may be used (rather laxly) to describe real functions (noting the difference between the analyticity in the two cases). This should partly explain the wide usage of "holomorphic" in complex analysis in place of "analytic" (see Problem 11).

[43] In fact, being defined exclusively on a subset of the complex plane (rather than the entire complex plane as required by the definition of entire) should be sufficient to exclude such a function from being entire regardless of being analytic or not.

or polynomials). Some general rules are then used to extend the rather limited results that we obtained from the definitions. For example, combinations (such as sums, differences and products) of continuous/analytic/entire functions are also continuous/analytic/entire, and hence we do not need to investigate these attributes of such combinations from first principles and by applying elementary definitions. This should reduce the task of identifying the attributes of complex functions substantially since the identification by using the aforementioned general rules is usually easy.

5. What is the relation between the set of analytic functions and the set of continuous functions?
Answer: Analyticity implies continuity but continuity does not imply analyticity. In other words, analytic functions are continuous but continuous functions are not necessarily analytic. So, the set of analytic functions is a subset of the set of continuous functions. Also, see Problem 7 of the present section as well as part (a) of Problem 7 of § 1.9 and Problem 13 of § 3.1.
6. Compare the differentiability of real and complex functions.
Answer: Real functions can be differentiable only to a certain order (see the upcoming note), while complex (analytic) functions have derivatives of all orders (as will be shown in Problem 6 of § 4.3). So, as soon as we establish the analyticity of a given complex function we can differentiate it as many times as we wish knowing that all these derivatives are analytic and hence we can enjoy their analyticity. In fact, this makes complex functions more favorable and useful than real functions and represents a big advantage for complex analysis over real analysis.
Note: in the literature of real analysis we can find many examples of real functions that are differentiable only to a certain order. For instance, if $f(x) = x\,|x| = x\sqrt{x^2}$ then we have (using the product rule of differentiation):
$$\frac{df}{dx} = \frac{d\left(x\sqrt{x^2}\right)}{dx} = \sqrt{x^2} + x\frac{x}{\sqrt{x^2}} = \sqrt{x^2} + \frac{x^2}{\sqrt{x^2}} = \sqrt{x^2} + \sqrt{x^2} = 2\sqrt{x^2} = 2\,|x|$$
and hence f is differentiable (i.e. it has first order derivative) over the entire real line including the origin. However, df/dx is not differentiable (i.e. f does not have second order derivative) at the origin because:
$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d(2\,|x|)}{dx} = 2\frac{d\,|x|}{dx} = 2\frac{d\sqrt{x^2}}{dx} = 2\frac{x}{\sqrt{x^2}} = 2\frac{x}{|x|}$$
which has a discontinuity at $x = 0$ since it is equal to -1 on the negative real line and to $+1$ on the positive real line. In fact, $x/\,|x|$ is singular at $x = 0$ and it is not defined there (since it has the indeterminate form $0/0$).
7. Give examples of complex functions f which are:
(**a**) Continuous and analytic.
(**b**) Continuous but not analytic.
(**c**) Neither continuous nor analytic.
Answer:[44]
(**a**) Polynomial, exponential and trigonometric and hyperbolic cosine and sine functions are continuous and analytic (everywhere). See Problem 4 of § 3.1 and part (a) of Problem 7 of § 1.9.
(**b**) The functions $f(z) = z^*$ and $f(z) = |z|$ are continuous but not analytic (everywhere). See Problem 13 of the present section as well as Problems 7 and 13 of § 3.1.
(**c**) The functions $f(z) = \frac{1}{z}$ and $f(z) = \frac{1}{z^2+1}$ are neither continuous nor analytic (the first at the origin and the second at $\pm i$).
8. What "continuous" means when we say: a complex function $f(z)$ is continuous at point z_0?
Answer: It means:
$$\lim_{z\to z_0} f(z) \text{ exists} \qquad \& \qquad f(z_0) \text{ exists} \qquad \& \qquad \lim_{z\to z_0} f(z) = f(z_0)$$

[44] This answer is rather terse but it is sufficient for the purpose of this Problem (at this stage in the book). More investigations about these issues will be presented later on (see for example § 3.1).

9. What "analytic" means when we say: a complex function $f(z)$ is analytic at point z_0?
Answer: It means:

$f(z)$ is defined at z_0 & $f(z_0)$ is single-valued & $f(z)$ has derivative at and around z_0

Also, see footnote [32] on page 17.

10. What "regular" function means?
Answer: "Regular" function is commonly used in the literature as synonymous to "analytic" function. However, due to uncertainty and misuse of "regular" (where it is used confusingly in different meanings) we avoid the use of this term in this book.

11. What "holomorphic" function means?
Answer: The term "holomorphic" may be used as synonymous to "analytic" (possibly in a more specific sense or context that presumes being complex). However, there seems to be multiple use of "holomorphic" and therefore we avoid using this term in this book (preferring the rather older and possibly clearer term "analytic" although it may be less specific). By the way, holomorphic functions may also be labeled or described as "regular" which we also avoid in this book (see Problem 10).
Note: in this context, it is useful to be aware of the term "meromorphic" which means a function that is analytic over a region (which could be the entire complex plane) except at some isolated singularities (or poles specifically; see § 3.3). It is noteworthy that some authors use "meromorphic" as a synonym for "holomorphic" although, strictly, it is not.

12. List some of the properties and rules of continuous complex functions.
Answer: For example:
(**a**) Sums, differences, products, compositions and moduli of continuous functions are also continuous.
(**b**) Quotient of continuous functions is also continuous where the denominator does not vanish.
(**c**) A function is continuous *iff* its real and imaginary parts are continuous (see Problem 8 of § 1.11).
(**d**) A function is continuous if it is analytic (see part a of Problem 7 of § 1.9).

13. The modulus of a continuous function is continuous. Justify this.
Answer: If $f(z) = u+iv$ (where u and v are real) is a continuous function then u and v are continuous (see point c of Problem 12) and hence the sum of their squares (i.e. $u^2+v^2 = u\times u+v\times v$) is continuous (see point a of Problem 12). Therefore, the modulus (i.e. the square root $|f| = \sqrt{u^2+v^2}$ which is a composition) should also be continuous (see point a of Problem 12).

14. Investigate the continuity of the following complex functions:
(**a**) $z^3 + \sin^2(z^2)$. (**b**) e^{3+z^2-z}. (**c**) $\tan z$. (**d**) $\frac{\cosh z}{e^z}$.
Answer:
(**a**) z^3 is a polynomial function and hence it is continuous everywhere, while $\sin^2(z^2)$ is a product of $\sin(z^2)$ by itself $\left[\text{i.e. } \sin^2(z^2) = \sin(z^2)\times\sin(z^2)\right]$ with $\sin(z^2)$ being a composition of sine and polynomial functions both of which are continuous everywhere. Hence, their sum $z^3 + \sin^2(z^2)$ is continuous everywhere.
(**b**) e^{3+z^2-z} is a composition of exponential and polynomial functions both of which are continuous everywhere and hence e^{3+z^2-z} is continuous everywhere.
(**c**) $\tan z$ is a quotient of $\sin z$ and $\cos z$ (since $\tan z = \frac{\sin z}{\cos z}$) both of which are continuous everywhere. However, because it is a quotient we should exclude the points at which its denominator $\cos z$ vanishes, i.e. $z = \left(n + \frac{1}{2}\right)\pi$ (see part a of Problem 14 of § 2.3). Hence, $\tan z$ is continuous everywhere except at these points.
(**d**) $\frac{\cosh z}{e^z}$ is a quotient of $\cosh z$ and e^z both of which are continuous everywhere. Moreover, e^z does not vanish and hence $\frac{\cosh z}{e^z}$ is continuous everywhere.[45]

15. Verify if the following functions are entire or not:
(**a**) $f(z) = \cosh z - e^z$. (**b**) $f(z) = z^4\cos z^2$. (**c**) $f(z) = 2z\sin z + \frac{3}{z}$.
Answer: The verification is essentially based on the fact that the sums, differences, products and compositions of entire functions are also entire.

[45] From the relation $e^z = e^{x+iy} = e^x e^{iy}$ we can see that e^z vanishes nowhere in the (finite) complex plane because $e^x \neq 0$ for any $x \in \mathbb{R}$ while e^{iy} is of unity modulus.

(**a**) This is a difference of hyperbolic cosine and exponential functions both of which are entire and hence f is entire.
(**b**) This is a product of a polynomial (i.e. z^4) and a composition of a cosine and a polynomial (i.e. $\cos z^2$) all of which are entire and hence f is entire.
(**c**) Although $2z \sin z$ is entire (since it is a product of $2z$ and $\sin z$ both of which are entire) $\frac{3}{z}$ is not entire (since it is singular at $z = 0$) and hence f is not entire.

16. Investigate the functions $e^{\ln z}$ and $\ln e^z$ from the perspective of entirety.
Answer: We my write $e^{\ln z} = \ln e^z = z$ and hence we may conclude that both are entire since $f(z) = z$ is entire. However, there is an important difference between $e^{\ln z}$ (whose domain excludes 0 at least) and $\ln e^z$ (whose domain includes the entire z plane) and hence it may be argued that $\ln e^z$ is entire but $e^{\ln z}$ is not because $\ln e^z$ is defined (and analytic) on the entire complex plane while $e^{\ln z}$ is not.
Note 1: the above result (if approved) should imply that although the composition of entire functions is entire the opposite may not be true, i.e. the composition of functions some of which are not entire is not necessarily not entire. So, $\ln e^z$ is entire although ln is not.
Note 2: it may be argued that the exponential and logarithm functions are inverses and hence $e^{\ln z}$ and $\ln e^z$ are just symbols for the function $f(z) = z$ rather than being functions of their own (noting that some equate $\ln e^z$ to $z + i2n\pi$ although this should not affect the issue of entirety).
Note 3: the relations $e^{\ln z} = z$ and $\ln e^z = z$ will be investigated in § 2.2. However, they should be recognized from a general background in analysis because the exponential and logarithm functions are inverses. Also, see Problem 6 of § 1.8.10.
17. What is the relation between being analytic at a point and being singular at that point?
Answer: Singularity is a cause for non-analyticity (taking notice of the footnote in the end of this answer). However, there are two possibilities with regard to the impact of non-analyticity on singularity:[46]
• Non-analyticity is a cause for singularity: in this case being non-singular means being analytic (as well as being analytic means being non-singular) since being non-analytic means being singular. So, the two (i.e. non-analyticity and singularity or analyticity and non-singularity) are equivalent although singularity (in other circumstances) may be caused primarily by other causes such as being indeterminate or unbounded.
• Non-analyticity is not a cause for singularity: in this case a function that is singular at a point cannot be analytic at that point but a function that is not singular at a point is not necessarily analytic at that point. Accordingly, singularity implies non-analyticity but non-analyticity does not necessarily imply singularity. To put it differently, all analytic points are non-singular while only some non-analytic points are singular.[47] So, we do not have analytic singular but we do have the other three combinations (i.e. analytic non-singular, non-analytic singular, and non-analytic non-singular).
Note: in this book we adopt the convention that non-analyticity (in the sense of lack of differentiability) is a (primary) cause for being singular and hence we accept the first possibility and reject the second (see footnote [35] on page 18). Anyway, in our view this is a trivial issue and it is a matter of terminology and convention with no real impact on the actual mathematics of the subject.[48]
18. What is the relation between being unbounded at a point and being singular at that point?
Answer: Being unbounded implies being singular although being singular does not necessarily imply being unbounded since singularity may be caused by a reason other than being unbounded (e.g. by being undefined or potentially by being non-analytic in the sense of being non-differentiable).
19. Manipulate Eq. 16 to get another expression for $f'(z_0)$.

[46] These two possibilities are about whether the lack of differentiability specifically (in the given technical sense) with no other reason for not behaving well is a cause for being singular or not (noting that being singular by other causes should imply being non-analytic in the sense of lack of differentiability).
[47] We use here lax expressions like "analytic points" for simplicity and clarity.
[48] We also refer the reader to footnote [199] on page 193 and Problem 6 of § 5.2 for further details about the nature of singularities and how they affect the analyticity (noting that some types of "singularity" do not actually affect the analyticity and hence the above discussions and generalizations should be understood considering this fact which possibly can have an impact on the terminology and how it is used). However, it should be noticed that this type of "singularity" (i.e. which does not harm analyticity) may not be regarded by some as a singularity (or at least as a genuine singularity).

Answer: From Eq. 16 $\big[$noting that $\Delta z = z - z_0$ and hence $f(z_0 + \Delta z) = f(z)\big]$ we have:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{17}$$

20. Provide more clarification about the removal of branch cut.
Answer: As indicated in the text, the purpose of removing the branch cut from the domain of multi-valued functions (which are made single-valued) is to achieve continuity and analyticity of these functions over their (modified) domain. Accordingly, the domain of a (single-valued) branch is smaller than the domain of the (multi-valued) original function. So in brief, in the first step of the process of making a branch we select a single value of the original (multi-valued) function over its entire domain (and exclude all other values) to achieve single-valuedness, while in the second step we remove the branch cut from the domain of the (single-valued) function to achieve continuity (and analyticity) over the (narrowed) domain. The result of these two steps is to have a (single-valued and continuous) branch.
21. Define “branch point”.
Answer: There are several (or even many) ways for defining branch point (resulting in different definitions in the literature). For example:
• It is a point z_0 in the complex plane at which a multi-valued function passes from one branch to another when going around an arbitrarily-small closed curve around z_0.
• It is a point z_0 in the complex plane at which a multi-valued function is discontinuous when going around an arbitrarily-small closed curve around z_0.
• It is a point z_0 in the complex plane at which a multi-valued function has no single-valued branch in any neighborhood of z_0.
In our view, these definitions (and their alike, or at least some of them) are neither rigorous (or even sufficiently technical) nor comprehensive and hence for the concept of “branch point” to be fully appreciated a familiarity with this concept in its different locations, contexts and instances throughout the subject of complex analysis is required. Also, see Problem 24 of § 2.2.
22. Determine the singularities of the following complex functions:
(**a**) $f(z) = \frac{1}{z^3+3z}$. (**b**) $f(z) = \csc^2 z$. (**c**) $f(z) = \frac{1}{z^2+2z+2}$.
Answer: The singularities of these three functions are where their denominators vanish (noting that $\csc^2 z = \frac{1}{\sin^2 z}$).
(**a**) The denominator is $z^3 + 3z = z(z - i\sqrt{3})(z + i\sqrt{3})$ which vanishes when $z = 0$ or $z = \pm i\sqrt{3}$. Hence, these are the singularities of f.
(**b**) We have $\csc^2 z = \frac{1}{\sin^2 z}$ where $\sin^2 z$ vanishes when $z = n\pi$ (with n being integer; see part b of Problem 14 of § 2.3). Hence, these are the singularities of f.
(**c**) The denominator is $z^2 + 2z + 2$ which vanishes when $z = -1 \pm i$. Hence, these are the singularities of f.
Note: the singularities of the functions in parts (a) and (c) are poles of order 1 (i.e. simple poles). This should be obvious from the given definition of “pole of order n” (see the text). The singularities of the function in part (b) are poles of order 2 (i.e. double poles). This will be investigated further later on (see for instance part e of Problem 7 of § 5.4). However, since $\sin z$ has simple zeros at $z = n\pi$ (according to the given definition noting that at $z = n\pi$ we have $\sin z = 0$ but $\frac{d \sin z}{dz} \neq 0$) then $\sin^2 z$ has double zeros there (which can be confirmed by noting that at $z = n\pi$ we have $\sin^2 z = \frac{d \sin^2 z}{dz} = 0$ but $\frac{d^2 \sin^2 z}{dz^2} \neq 0$) and hence $\csc^2 z$ should have double poles at $z = n\pi$.

1.6 Mathematical Representation of Sets and Shapes in the Complex Plane

The use of the terminology and notation of complex variables facilitates the mathematical representation of sets of complex numbers and geometric shapes in the complex plane. For example:
• The equation $|z - z_0| = \rho$ (where z_0 is a given complex number and ρ is a given positive real number) represents a circle in the complex plane with center z_0 and radius ρ.

• The inequality $|z - z_0| < \rho$ represents the interior of a disk in the complex plane with center z_0 and radius ρ. This set may also be called open disk or neighborhood of z_0.
• The relation $|z - z_0| \le \rho$ represents a disk in the complex plane with center z_0 and radius ρ.
• The equation $z = C + iy$ (where C is a real constant and $-\infty < y < \infty$) represents the (vertical) line $x = C$. This line can also be represented more compactly as $\operatorname{Re} z = C$.
• The equation $z = x + iC$ (where $-\infty < x < \infty$ and C is a real constant) represents the (horizontal) line $y = C$. This line can also be represented more compactly as $\operatorname{Im} z = C$.
• The intersection $(\operatorname{Im} z \ge 0) \cap (0 < \operatorname{Re} z \le 2)$ represents the semi-infinite strip bordered by the x axis, the y axis and the line $x = 2$ (with the inclusion of the bottom and right boundaries and the exclusion of the left boundary).
• The equation $z^* = z$ (or $\operatorname{Im} z = 0$) represents the real axis (i.e. the x axis).
• The equation $z^* = -z$ (or $\operatorname{Re} z = 0$) represents the imaginary axis (i.e. the y axis).
• The equation $\operatorname{Re}(z) = \operatorname{Im}(z)$ represents the line $y = x$.
• The equation $z = \pi e^{i\theta}$ $(0 \le \theta \le \pi)$ represents the upper half of the origin-centered circle of radius π.
• The equation $z = 6 - i + 2e^{i\theta} = (-\pi < \theta \le \pi)$ represents the circle with center $6 - i$ and radius 2.
• The relation $-\frac{\pi}{2} \le \arg(z) \le 0$ represents the quarter-plane between the negative imaginary axis and the positive real axis (i.e. the fourth quadrant) including its boundaries.

As we see in the above examples (and will see much more later on), the mathematical representation of sets and shapes employs both the Cartesian form of representation and symbolism (which is based on the real-imaginary split of complex numbers) and the polar form of representation and symbolism (which is based on the modulus-argument split of complex numbers). It also employs different types of mathematical relations (such as equality, inequality, intersection of sets, etc.) and notions (such as radius, conjugate, etc.). This versatility facilitates the mathematical representation of sets and shapes in the complex plane and makes it a powerful tool for clear, compact, elegant and flexible way of expression. This, in turn, facilitates the formulation and manipulation of mathematical relations.

Problems

1. Interpret the following relations (which employ the terminology and notation of complex variables):

(**a**) $|z| = 1$. (**b**) $|z - 2 + i3| < 6$. (**c**) $|z + 1 - i9| \le 2$.

(**d**) $\operatorname{Re}(z) = 4$. (**e**) $\operatorname{Im}(z) = -2$. (**f**) $\operatorname{Re}(z) \le -1$.

(**g**) $\operatorname{Im}(z) > 12$. (**h**) $8 < \operatorname{Re}(z) \le 10$. (**i**) $|z| > 2$.

(**j**) $2 \le |z| \le 3$. (**k**) $z = i2 + e^{i\theta}\ \big(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\big)$. (**l**) $\big(0 < \arg z < \frac{\pi}{2}\big) \cap \big(|z| < 3\big)$.

(**m**) $\big(|z| < 5\big) \cup \big(|z| > 11\big)$. (**n**) $z \le e^{(\log_e 9) + i\theta}\ (-\pi < \theta \le \pi)$. (**o**) $3\,|z - 2|^2 - 5\,|z + i|^2 = 0$.

Answer: These relations represent the following:
(**a**) The origin-centered unit circle.
(**b**) The interior of the disk with center $2 - i3$ and radius 6.
(**c**) The disk with center $-1 + i9$ and radius 2.
(**d**) The (vertical) line $x = 4$.
(**e**) The (horizontal) line $y = -2$.
(**f**) The half-plane to the left of the line $x = -1$ (including this line).
(**g**) The half-plane above the line $y = 12$ (excluding this line).
(**h**) The infinite vertical strip bordered by the line $x = 8$ on the left and the line $x = 10$ on the right (excluding the former and including the latter).
(**i**) The exterior of the origin-centered circle of radius 2.
(**j**) The origin-centered ring (or annulus) bordered by the circles $|z| = 2$ and $|z| = 3$ (including the two borders).
(**k**) The right half of the unit circle with center $i2$.
(**l**) The interior of the origin-centered quarter-disk in the first quadrant with radius 3.
(**m**) The z plane excluding the origin-centered ring of inner radius 5 and outer radius 11.
(**n**) The origin-centered disk of radius 9.

(**o**) The circle with center $-3 - i\frac{5}{2}$ and radius $\sqrt{75/4}$ because:

$$
\begin{aligned}
3\,|z-2|^2 - 5\,|z+i|^2 &= 0 \\
3\,|(x-2)+iy|^2 - 5\,|x+i(y+1)|^2 &= 0 \\
3\left[(x-2)^2+y^2\right] - 5\left[x^2+(y+1)^2\right] &= 0 \\
3\left[x^2-4x+4+y^2\right] - 5\left[x^2+y^2+2y+1\right] &= 0 \\
3x^2-12x+12+3y^2-5x^2-5y^2-10y-5 &= 0 \\
x^2+6x+y^2+5y-\frac{7}{2} &= 0 \\
x^2+6x+9+y^2+5y+\frac{25}{4} &= \frac{7}{2}+9+\frac{25}{4} \\
(x+3)^2+\left(y+\frac{5}{2}\right)^2 &= \frac{75}{4}
\end{aligned}
$$

2. What are the mathematical representations (in complex notation) of the following curves and regions in the complex plane:
(**a**) The circle with center $(-4, 7)$ and radius $\sqrt{\pi}$.
(**b**) The exterior region of the disk with center (α, β) and radius γ (noting that α, β, γ are real).
(**c**) The region bordered from below by the parabola $y = x^2$ (i.e. the region contained in this parabola).
(**d**) The origin-centered elliptical disk[49] with semi-axes a and b with a being along the x direction and b being along the y direction (noting that a and b are real).
(**e**) The square with vertices $2+i3$, $5+i3$, $2+i6$ and $5+i6$.
(**f**) The triangle with vertices $(0, 0)$, $(11, 0)$ and $(0, 8)$.
(**g**) The part of the z plane between the lines $x = 0$ and $y = x$ (i.e. in the first and third quadrants) including the boundaries.
Answer:
(**a**) $|z + 4 - i7| = \sqrt{\pi}$. This is obvious because a circle with center z_0 and radius ρ is given by $|z - z_0| = \rho$.
(**b**) $|z - \alpha - i\beta| > \gamma$. This is obvious because a disk with center z_0 and radius ρ is given by $|z - z_0| \le \rho$.
(**c**) $\operatorname{Im} z > (\operatorname{Re} z)^2$. This is because $y = x^2$ means in complex notation $\operatorname{Im} z = (\operatorname{Re} z)^2$ and hence the region above this curve is represented by the relation $y > x^2$, i.e. $\operatorname{Im} z > (\operatorname{Re} z)^2$.
(**d**) $\left(\frac{\operatorname{Re} z}{a}\right)^2 + \left(\frac{\operatorname{Im} z}{b}\right)^2 \le 1$. This is because this ellipse is given by $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ which in complex notation is $\left(\frac{\operatorname{Re} z}{a}\right)^2 + \left(\frac{\operatorname{Im} z}{b}\right)^2 = 1$.
(**e**) As a curve, the square is given by the following union of intersections:

$$
\Big(\{\operatorname{Re} z = 2\} \cap \{3 \le \operatorname{Im} z \le 6\}\Big) \cup \Big(\{\operatorname{Re} z = 5\} \cap \{3 \le \operatorname{Im} z \le 6\}\Big) \cup
$$
$$
\Big(\{\operatorname{Im} z = 3\} \cap \{2 \le \operatorname{Re} z \le 5\}\Big) \cup \Big(\{\operatorname{Im} z = 6\} \cap \{2 \le \operatorname{Re} z \le 5\}\Big)
$$

As a region, the square is given by the following intersection:

$$
\Big(2 \le \operatorname{Re} z \le 5\Big) \cap \Big(3 \le \operatorname{Im} z \le 6\Big)
$$

(**f**) As a curve, the triangle is given by the following union of intersections:

$$
\Big(\{\operatorname{Re} z = 0\} \cap \{0 \le \operatorname{Im} z \le 8\}\Big) \cup \Big(\{\operatorname{Im} z = 0\} \cap \{0 \le \operatorname{Re} z \le 11\}\Big) \cup
$$

[49] We mean by "elliptical disk" the ellipse and its interior region.

$$\left(\{0 \le \operatorname{Re} z \le 11\} \cap \left\{\operatorname{Im} z = 8 - \frac{8}{11}\operatorname{Re} z\right\}\right)$$

As a region, the triangle is given by the following intersection:

$$\Big(\operatorname{Re} z \ge 0\Big) \cap \Big(\operatorname{Im} z \ge 0\Big) \cap \Big(\operatorname{Im} z \le 8 - \frac{8}{11}\operatorname{Re} z\Big)$$

(**g**) $\left(-\frac{3\pi}{4} \le \arg z \le -\frac{\pi}{2}\right) \cup \left(\frac{\pi}{4} \le \arg z \le \frac{\pi}{2}\right)$. This is because this part of the z plane consists of two eighth-planes: the eighth-plane between these lines in the third quadrant which is represented by $-\frac{3\pi}{4} \le \arg z \le -\frac{\pi}{2}$, and the eighth-plane between these lines in the first quadrant which is represented by $\frac{\pi}{4} \le \arg z \le \frac{\pi}{2}$. Hence, this part of the z plane is represented by the union of these eighth-planes.

1.7 Graphic Representation of Sets and Shapes in the Complex Plane

Based on the mathematical representation of sets and shapes which was investigated in § 1.6, sets of complex numbers and geometric shapes in the complex plane can be represented graphically in a corresponding way. For example, the closed disk $|z| \le 1$ and the open half-plane $\operatorname{Re}(z) > 1$ can be represented graphically by the left and right frames of Figure 1. More examples will be given in the Problems and in the forthcoming parts of the book. Iterating what have been said in § 1.6, the graphic representation of sets and shapes employs both the Cartesian form and the polar form as well as different types of mathematical relations and geometric objects, and this versatility facilitates the graphic representation of sets and shapes in the complex plane and makes it a powerful tool for clear, impressive, elegant and flexible way of demonstration. This, in turn, helps in the development and understanding of mathematical relations and formulations.

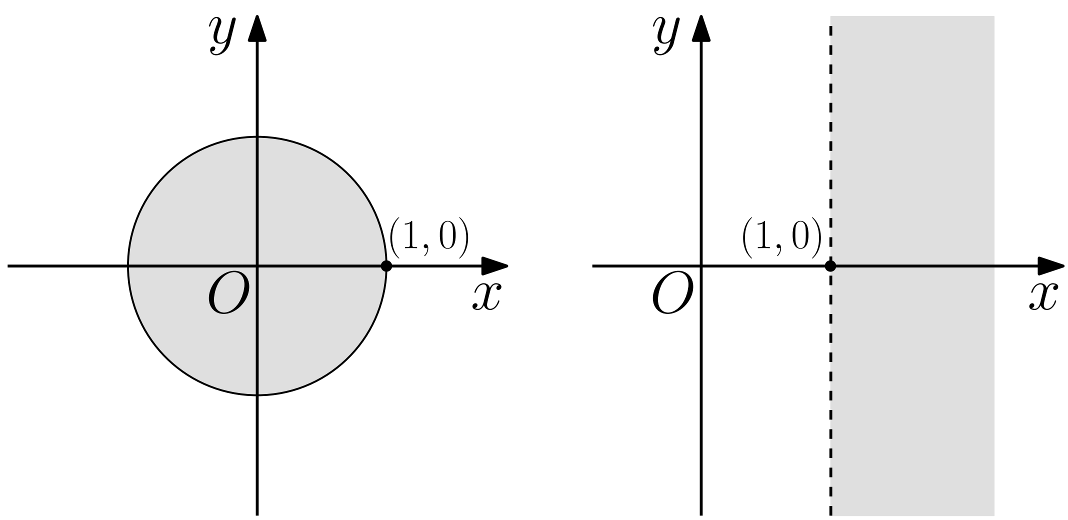

Figure 1: Graphic representation of the closed disk $|z| \le 1$ (left frame) and the open half-plane $\operatorname{Re}(z) > 1$ (right frame). The represented sets are shaded noting that solid boundary means included and dashed boundary means excluded. See § 1.7.

Problems

1. Make graphic representations of the following sets and shapes in the complex plane:

(**a**) $1 \le |z - 3 - i3| < 2$.

(**b**) $\big(\operatorname{Re} z > 1\big) \cap \big(\operatorname{Im} z \ge 3\big)$.

(**c**) $\big(0 \le \arg z \le \frac{\pi}{4}\big) \cap \big(|z| < 5\big)$.

(**d**) $\big(|z| \le |4e^{i\theta}|\big) - \big(\operatorname{Im} z > 3\big) \quad \big(0 \le \theta \le \frac{\pi}{2}\big)$.

Answer: See Figure 2.

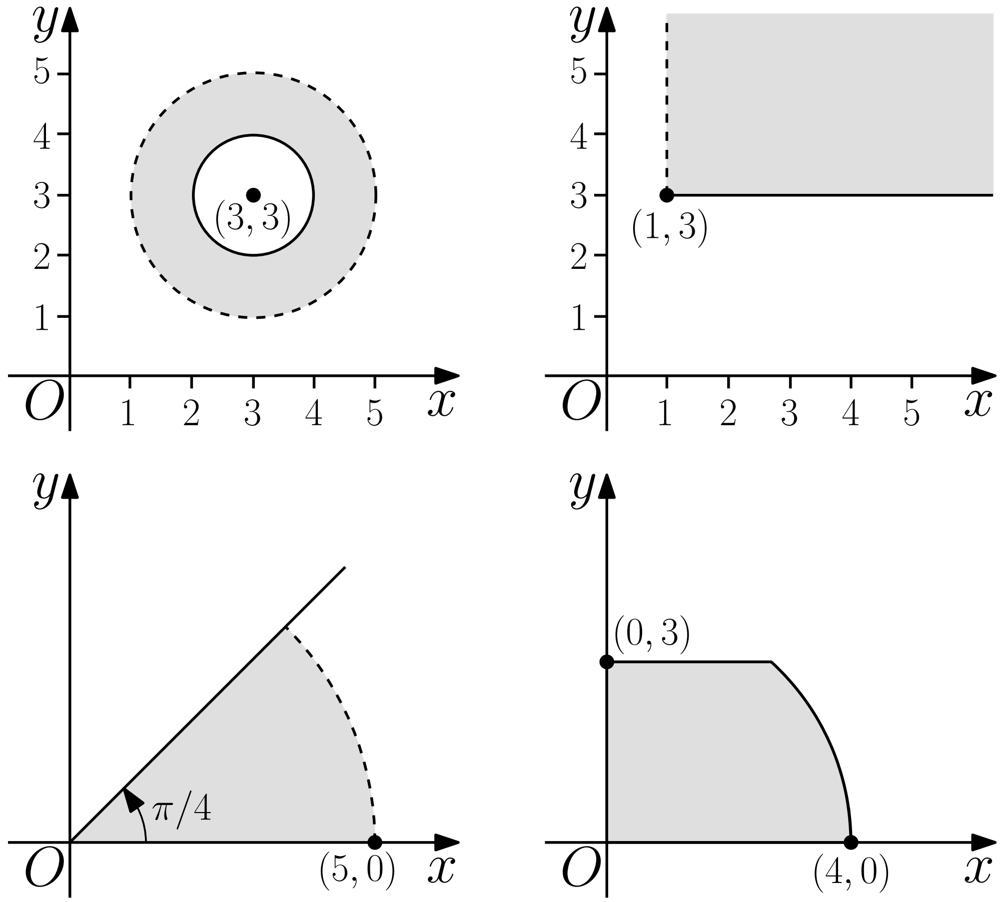

Figure 2: Graphic representations of the circular ring $1 \le |z - 3 - 3i| < 2$ (upper left frame), the quarter-plane $\big(\mathrm{Re}\, z > 1\big) \cap \big(\mathrm{Im}\, z \ge 3\big)$ (upper right frame), the interior of the eighth-disk $\big(0 \le \arg z \le \frac{\pi}{4}\big) \cap \big(|z| < 5\big)$ (lower left frame), and the truncated quarter-disk $\big(|z| \le |4e^{i\theta}|\big) - \big(\mathrm{Im}\, z > 3\big)$ $\big(0 \le \theta \le \frac{\pi}{2}\big)$ (lower right frame). The represented sets are shaded noting that solid boundary means included and dashed boundary means excluded. It is obvious that the quarter-plane extends to infinity on the right and the top. See Problem 1 of § 1.7.

2. What are the mathematical representations of the shaded regions (in the complex plane) plotted in Figure 3?
   Answer: The shaded region in the upper left frame is represented mathematically by the following difference of sets:

$$\Big[|z| \quad \le \quad \rho\Big] - \Big[\Big(\mathrm{Im}\, z \le \{\mathrm{Re}\, z + \rho\}\Big) \cap \Big(\mathrm{Im}\, z \ge \{\mathrm{Re}\, z - \rho\}\Big) \cap \Big(\mathrm{Im}\, z \le \{-\mathrm{Re}\, z + \rho\}\Big) \cap \Big(\mathrm{Im}\, z \ge \{-\mathrm{Re}\, z - \rho\}\Big)\Big]$$

   The shaded region in the upper right frame is represented mathematically by the following difference of sets:

$$\left[\left(\frac{\mathrm{Re}\, z - \alpha}{a}\right)^2 + \left(\frac{\mathrm{Im}\, z - \beta}{b}\right)^2 < 1\right] - \left[\mathrm{Im}\, z < \left\{B - \frac{B}{A}\mathrm{Re}\, z\right\}\right]$$

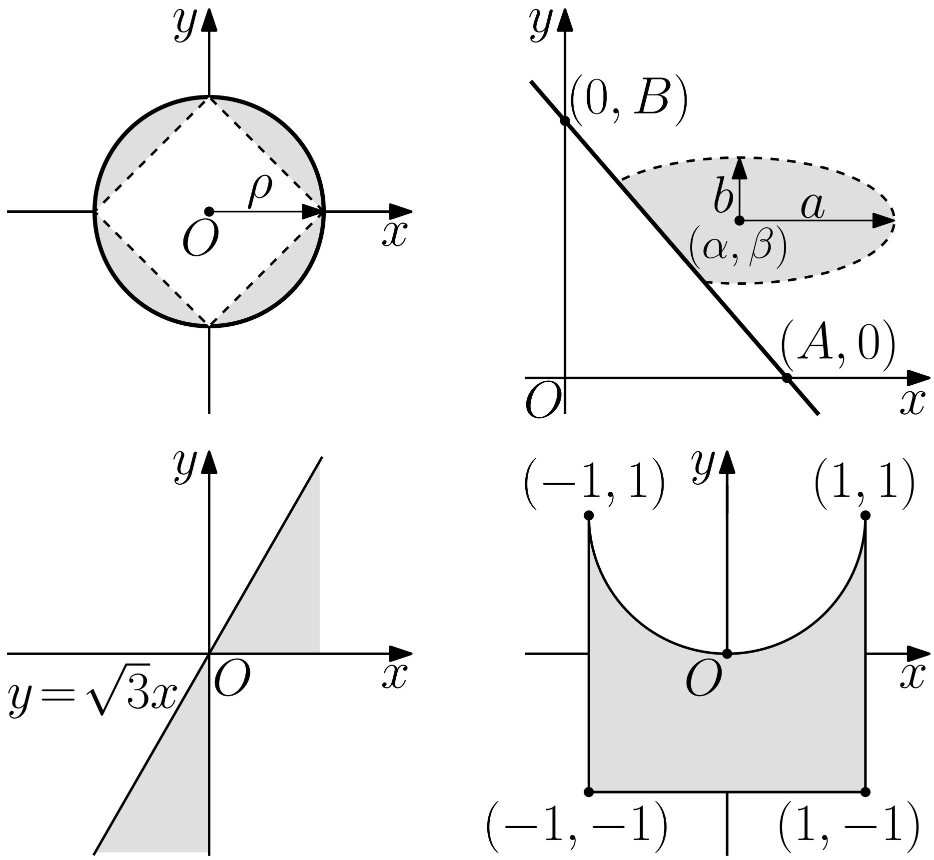

Figure 3: The shaded region in the upper left frame is an origin-centered disk (of radius ρ) excluding the inscribed square. The shaded region in the upper right frame is an elliptical disk (of semi-axes a and b and with center $\alpha + i\beta$) excluding the region to the left of (or below) the straight line with an x intercept A and a y intercept B. The shaded region in the lower left frame is the combination of the infinite sector between the line $y = \sqrt{3}x$ and the negative imaginary axis (in the third quadrant) and the infinite sector between the line $y = \sqrt{3}x$ and the positive real axis (in the first quadrant). The shaded region in the lower right frame is the square with vertices at $\pm(1 \pm i)$ excluding the overlapping part of the interior of the unit disk centered on i. Solid boundary means included and dashed boundary means excluded. See Problem 2 of § 1.7.

The shaded region in the lower left frame is represented mathematically by the following union of sets:

$$\left(-\frac{2\pi}{3} \le \arg z \le -\frac{\pi}{2}\right) \cup \left(0 \le \arg z \le \frac{\pi}{3}\right)$$

The shaded region in the lower right frame is represented mathematically by the following difference of sets:

$$\left[\left(-1 \le \operatorname{Re} z \le 1\right) \cap \left(-1 \le \operatorname{Im} z \le 1\right)\right] - \left[|z - i| < 1\right]$$

1.8 General Aspects and Rules of Complex Numbers

In the following subsections we outline some of the general aspects (such as mathematical properties) and rules of complex numbers and variables.

1.8.1 Relationship between Real, Imaginary and Complex Numbers

Real numbers are a subset of complex numbers, i.e. a real number is a complex number whose imaginary part is zero. In the z plane, real numbers are represented by the x axis (in the Cartesian representation) and by the union of the semi-lines $\theta = 0$ and $\theta = \pi$ (in the polar representation). In fact, most of the general properties of numbers are common to both real and complex numbers (see for example Problem 1). However, some properties of real numbers (like ordering) do not apply to complex numbers. For example, it is meaningful to write $2 < 3$ or $x_1 \geq x_2$ but it is meaningless to write $1 + i < 2 + i$ or $z_1 > z_2$. So, the property of ordering (apart from equality like $z_1 = z_2$ which is not really an ordering relation) is not defined on the set of complex numbers (although ordering of their moduli or arguments or real or imaginary parts are defined since these are real numbers). Similarly, some properties of complex numbers like conjugation (i.e. having conjugate) do not apply to real numbers (see Problem 2).

Like real numbers, imaginary numbers are a subset of complex numbers, i.e. an imaginary number is a complex number whose real part is zero. In the z plane, imaginary numbers are represented by the y axis (in the Cartesian representation) and by the union of the semi-lines $\theta = -\pi/2$ and $\theta = \pi/2$ (in the polar representation). Regarding the relation between real and imaginary numbers, they can be seen as disjoint sets if we exclude the number zero which is common to both (as well as to complex numbers).

Problems

1. Give examples of general properties that are common to real and complex numbers.
   Answer: For example:
   • Reflexivity, symmetry and transitivity.
   • Commutativity and associativity of addition and multiplication of numbers.
   Note: although the properties of reflexivity, symmetry and transitivity essentially belong to the equality relation, they can also characterize the numbers (since it is meaningful to say: real/complex numbers are reflexive, symmetric and transitive with regard to the equality relation). Similarly, although the properties of commutativity and associativity essentially belong to the addition and multiplication operations, they can also characterize the numbers (since it is meaningful to say: real/complex numbers are commutative and associative with regard to the addition and multiplication operations).
2. The claim in the text that real numbers have no conjugates may be challenged by the fact that real numbers are their own conjugates. Deliberate on this issue.
   Answer: In fact, conjugation applies to real numbers (since each real number is its own conjugate, noting that real numbers are a subset of complex numbers) but in a trivial way that makes this property appear nonsensical. To be more precise, real numbers as (a subset of) complex numbers have conjugates (which are their own), but real numbers as real numbers (i.e. on their own) have no conjugates. Accordingly, the number $3 + i0$ has a conjugate which is $3 - i0$ but the number 3 has no conjugate (since it has no imaginary part for the concept of conjugate to have a sensible meaning). So, as long as we are concerned with real numbers exclusively (as it is the case in real analysis), we can correctly claim (as we did in the text) that conjugation does not apply to real numbers (i.e. as real numbers) because "conjugation" requires having an imaginary part (or component) which is meaningless within the domain of real numbers and real analysis.

1.8.2 Relationship between Cartesian and Polar Representations

If z is a complex number with Cartesian representation $z = x + iy$ and polar representation $z = re^{i\theta}$ then the modulus and argument of z are given respectively by:

$$r \equiv |z| = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta \equiv \arg z = \arctan\left(\frac{y}{x}\right) \tag{18}$$

where in these relations we transform from Cartesian to polar.[50] Similarly, we have:

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta \tag{19}$$

[50] We should note that the relation $\theta \equiv \arg z = \arctan(y/x)$ is rather loose and it is waiting more investigation.

where in these relations we transform from polar to Cartesian using the identity $e^{i\theta} = \cos\theta + i\sin\theta$ (or using the well-known relationships between polar and Cartesian coordinates noting that these relationships are based on simple geometric and trigonometric relations). The relationship between Cartesian representation and polar representation is demonstrated in Figure 4.

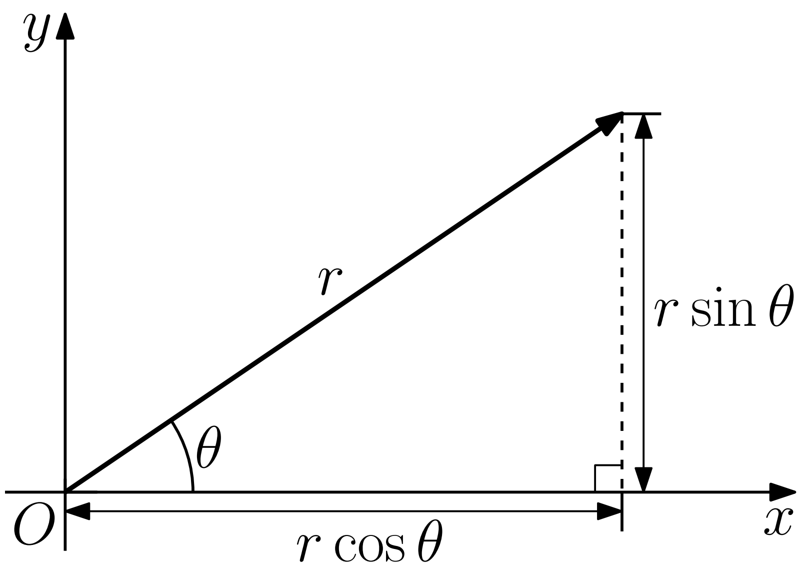

Figure 4: Graphic illustration of the relationship between the Cartesian form and the polar form of complex numbers. See § 1.8.2.

In this context, it is necessary to draw the attention to the following important points:

• For correct and unique determination of $\arg z$ of a complex number (i.e. to which quadrant it belongs) the signs of x and y should be considered separately (and not as combined in y/x). Accordingly, $\arctan(y/x)$ is in the first quadrant if $x > 0$ and $y > 0$ but it is in the third quadrant if $x < 0$ and $y < 0$ although in both cases y/x is positive. Similarly, $\arctan(y/x)$ is in the fourth quadrant if $x > 0$ and $y < 0$ but it is in the second quadrant if $x < 0$ and $y > 0$ although in both cases y/x is negative.[51]

• As noted earlier, the common convention (which we follow) is that when measuring angles, clockwise sense of rotation is regarded negative and anticlockwise sense is regarded positive.

• While the Cartesian form of a given complex number (as represented by a unique point in the complex plane) is unique, the polar form is not because we can add or subtract an integer multiple of 2π (i.e. $2n\pi$) to the argument (or phase angle) θ of a complex number without affecting its real or imaginary part $\big[$noting that $x = r\cos\theta = r\cos(\theta + 2n\pi)$ and $y = r\sin\theta = r\sin(\theta + 2n\pi)\big]$.

• Recalling the previous point, there are certain conventions about the range of the "principal value" of the argument (or phase angle) of a complex number. The most common of these conventions are $-\pi < \theta_p \le \pi$ and $0 \le \theta_p < 2\pi$ where θ_p is the principal value (or principal argument). We note that in this book we follow the convention $-\pi < \theta_p \le \pi$.

• Recalling again the previous points, some readers may note a lack of rigor in the relation $\theta \equiv \arg z = \arctan(y/x)$ regarding the range of θ and arctan as well as the definition of $\arg z$. To be more rigorous, we may write $\theta_p \equiv \operatorname{Arg} z = \arctan(y/x)$, but unfortunately even this will not solve the problem completely considering that arctan as a single-valued function does not have a sufficient range, while arctan as an inverse of tan over its entire domain should not be restricted to the principal value of θ, i.e. $\theta_p \equiv \operatorname{Arg} z$.[52]

[51] The reader should also note in this context the upcoming points which are related to this point. Also, see Problem 8 of § 1.3.

[52] In fact, if we define the range of arctan to be between $-\pi$ and π then we may have a rather rigorous relation: $\theta \equiv \arg z = \arctan(y/x) + 2n\pi$ noting that $\theta = \theta_p + 2n\pi$ and $\theta_p \equiv \operatorname{Arg} z$ and hence $\theta_p = \arctan(y/x)$ although arctan is not a (single-valued) function. We may also use the single-valued function Arctan (as we did in Problem 8 of § 1.3) but this

Anyway, these rather trivial and pointless complications and deliberations should not be considered seriously although it is useful for the reader to be aware of (especially from a pedagogical perspective). Accordingly, in the following (and indeed in the book in general) we will feel broadly relaxed about such issues as long as the ultimate meaning is obvious relying on this understanding and the awareness of the triviality of such matters.

Problems

1. Find the polar form of the following numbers (which are given in Cartesian form):

   (**a**) $z = -45$. (**b**) $z = -ie^2$. (**c**) $z = 3 - i7$. (**d**) $z = i12 - 6$. (**e**) $z = -(\sqrt{5} + i)$.

   Answer: We remark that in the following, the principal polar form (which employs the principal value of the argument) corresponds to $n = 0$ (noting that n is an integer). We also remark that in some cases we divide by zero for the sake of demonstration only (with no intention to commit this taboo which may upset some mathematicians!).

   (**a**)
$$\begin{aligned} r &\equiv |z| = \sqrt{x^2 + y^2} = \sqrt{(-45)^2 + 0^2} = 45 \\ \theta_p &\equiv \operatorname{Arg} z = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{0}{-45}\right) = \pi \end{aligned}$$

   Hence, $z = re^{i\theta} = re^{i(\theta_p + 2n\pi)} = 45e^{i(\pi+2n\pi)}$.

   (**b**)
$$\begin{aligned} r &\equiv |z| = \sqrt{x^2 + y^2} = \sqrt{0^2 + (-e^2)^2} = e^2 \\ \theta_p &\equiv \operatorname{Arg} z = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{-e^2}{0}\right) = -\frac{\pi}{2} \end{aligned}$$

   Hence, $z = re^{i\theta} = re^{i(\theta_p + 2n\pi)} = e^2 e^{i(2n\pi - \pi/2)} = e^{2+i(2n\pi-\pi/2)}$.

   (**c**)
$$\begin{aligned} r &\equiv |z| = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-7)^2} = \sqrt{58} \\ \theta_p &\equiv \operatorname{Arg} z = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{-7}{3}\right) \simeq -1.1659 \end{aligned}$$

   Hence, $z = re^{i\theta} = re^{i(\theta_p + 2n\pi)} \simeq \sqrt{58}e^{i(2n\pi - 1.1659)}$.

   (**d**)
$$\begin{aligned} r &\equiv |z| = \sqrt{x^2 + y^2} = \sqrt{(-6)^2 + 12^2} = \sqrt{180} \\ \theta_p &\equiv \operatorname{Arg} z = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{12}{-6}\right) = \arctan\left(\frac{2}{-1}\right) \simeq 2.03444 \end{aligned}$$

   Hence, $z = re^{i\theta} = re^{i(\theta_p + 2n\pi)} \simeq \sqrt{180}e^{i(2.03444 + 2n\pi)}$.

   (**e**)
$$\begin{aligned} r &\equiv |z| = \sqrt{x^2 + y^2} = \sqrt{\left(-\sqrt{5}\right)^2 + (-1)^2} = \sqrt{6} \\ \theta_p &\equiv \operatorname{Arg} z = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{-1}{-\sqrt{5}}\right) \simeq -2.7211 \end{aligned}$$

   Hence, $z = re^{i\theta} = re^{i(\theta_p + 2n\pi)} \simeq \sqrt{6}e^{i(2n\pi - 2.7211)}$.

2. Find the Cartesian form of the following numbers (which are given in polar form):

   (**a**) $z = 11e^{-i3\pi}$. (**b**) $z = \pi e^{i10\pi}$. (**c**) $z = 14e^{i3\pi/2}$. (**d**) $z = 36e^{i\pi/4}$. (**e**) $z = e^{3+i7\pi/6}$.

   Answer:

   (**a**) $z = 11\Big[\cos(-3\pi) + i\sin(-3\pi)\Big] = 11\Big[-1 + i0\Big] = -11$

leads to rather lengthy and messy relationships (and hence we avoid it).

(**b**) $z = \pi\Big[\cos(10\pi) + i\sin(10\pi)\Big] = \pi\Big[1 + i0\Big] = \pi$

(**c**) $z = 14\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right] = 14\Big[0 - i\Big] = -i14$

(**d**) $z = 36\Big[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\Big] = 36\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = 18\sqrt{2} + i18\sqrt{2}$

(**e**) $z = e^3\left[\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right] = e^3\left[-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right] \simeq -17.3946 - i10.0428$

3. The Cartesian form of a given complex number (as represented by a unique point in the complex plane) is unique, but the polar form is not unique. Discuss this issue.
Answer: The key point in this discussion is the restriction "as represented by a unique point in the complex plane" because the numbers $re^{i\theta}$ and $re^{i(\theta+2n\pi)}$ as abstract numbers are obviously two different and unique numbers and they are as much different as the numbers $re^{i\theta}$ and $re^{i(\theta+2ne)}$ for example. Yes, when we consider the representation of all complex numbers by a single (or unique) complex "Cartesian plane" then we can see that the polar form is not unique because infinitely-many numbers fall on the same point in this unique plane while the Cartesian form is unique because each point in this unique plane is represented by a unique "Cartesian number". Accordingly, if we represent all the complex numbers by infinitely-many "parallel" complex "Cartesian planes" then the corresponding points in these parallel planes will have the same Cartesian representation but different polar representations. So, we may say that the polar representation of complex numbers is richer than the Cartesian representation. Anyway, as hinted earlier (by using for instance "Cartesian plane") this issue essentially originates from the graphic representation of complex numbers by coordinated planes and the relation between the Cartesian and polar coordinates (since non-uniqueness originates from the fact that the real and imaginary parts, which are based on the Cartesian representation, are not uniquely related to the polar representation since adding or subtracting an integer multiple of 2π to the argument of a complex number does not affect its real or imaginary part). So, if we use for instance a coordinated plane in which the horizontal axis represents the "signed magnitude" while the vertical axis represents the argument then this polar-like representation of complex numbers should also be unique (although we will need to replace some of the familiar relations, formulations and representations which are based on the traditional polar and Cartesian representations to cope with this new situation).
To sum up, when we start from a Cartesian representation (where each "Cartesian number" is represented uniquely by a single point of the "Cartesian plane") then the polar representation that is based on this "primary" Cartesian representation is not unique because infinitely-many "polar numbers" correspond to a single "Cartesian number". However, if we start from a polar-like representation of complex numbers (with disregard to any other "primary" representation) then the polar-like representation of complex numbers is also unique. Accordingly, being non-unique can be seen as a consequence of the relation of the polar representation to the Cartesian representation (i.e. being a subsidiary of the Cartesian representation) rather than being an intrinsic attribute of the polar (or polar-like) representation itself and on its own.
4. As explained already, adding $2n\pi$ to the argument of a complex number does not affect the number itself (or rather does not affect its real and imaginary parts and hence its Cartesian form). What about the potential effect of this addition on the functions of this complex number (or rather variable)?
Answer: While adding $2n\pi$ to the argument of a complex number does not affect the number itself (as a geometric entity represented by a specific point in the complex "Cartesian plane"), it may affect the function of that number. For example, while $f(z) = z = x + iy$ or $f(z) = \mathrm{Re}(z)$ is not affected by such addition, $f(z) = \sqrt{z}$ or $f(z) = \ln(z)$ is affected in general (see for example Problem 11 of § 1.11; also see Problem 8 of § 1.8.10).

1.8.3 Equality, Inequality and Non-equality

Two complex numbers z_1 and z_2 are equal *iff* $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$. Accordingly, complex equations represent two real equations: one for the real part and one for the imaginary part (e.g. $3z + i2 = 4 - i$ means $3x = 4$ and $3y + 2 = -1$). As indicated earlier (see § 1.8.1), inequalities[53] between complex numbers (as such) have no meaning although they are meaningful with regard to some of their attributes, e.g. the real part of a complex number is smaller than the real part of another complex number or the modulus of a complex number is larger than the modulus of another complex number. We also note that inequalities between real numbers (which are a subset of complex numbers) are meaningful. Regarding non-equalities (i.e. relations symbolized by $\neq$), they are meaningful for complex numbers (as for real numbers) where their meaningfulness is based on the meaningfulness of complex equalities. The issue of non-equalities will be discussed further in the Problems.

Problems

1. What it means when we write, for instance, $z \neq 2 + i8$?
   Answer: It means $x \neq 2$ OR $y \neq 8$ (or both) and hence it does not mean $x \neq 2$ AND $y \neq 8$. The reason is that two complex numbers are equal *iff* their real parts are equal AND their imaginary parts are equal and this AND is violated when one of these equalities does not hold even if the other equality holds.[54] For example, we correctly write $3 - i \neq 3 + i$ because the imaginary parts are not equal even though the real parts are equal.
2. Discuss the issue of the equality and non-equality of two complex numbers from the perspective of Cartesian and polar representations.
   Answer: Referring to our previous discussion about the uniqueness of the Cartesian representation and the non-uniqueness of the polar representation of a given complex number (or rather given complex "Cartesian number"; see § 1.8.2 and Problem 3 of § 1.8.2 in particular), we can say that two complex numbers can be equal and non-equal at the same time. For example, if $z_1 = 1 + i = \sqrt{2}e^{i\pi/4}$ and $z_2 = 1 + i = \sqrt{2}e^{i9\pi/4}$ then from the Cartesian perspective $z_1 = z_2$ but from the polar perspective $z_1 \neq z_2$ since $(\sqrt{2}, \pi/4) \neq (\sqrt{2}, 9\pi/4)$. In fact, the above definition of equality $\big[$i.e. z_1 and z_2 are equal *iff* $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)\big]$ is based on the Cartesian representation (as a primary and reference form) which implies that: z_1 and z_2 are equal *iff* $|z_1| = |z_2|$ and $\mathrm{Arg}(z_1) = \mathrm{Arg}(z_2)$ where the latter condition is equivalent to $\arg(z_1) = \arg(z_2) + 2n\pi$. So, if we adopt a purely polar criterion for defining equality then we may say: z_1 and z_2 are equal *iff* $|z_1| = |z_2|$ and $\arg(z_1) = \arg(z_2)$.

1.8.4 Zero and Unity

A complex number z is zero *iff* $\mathrm{Re}(z) = \mathrm{Im}(z) = 0$. So, in Cartesian representation a zero complex number $z = x + iy$ is identified by the condition $x = y = 0$, i.e. $z = 0 + i0$ in Cartesian form or $z = (0, 0)$ in Cartesian pair form. Similarly, in polar representation a zero complex number $z = |z| e^{i \arg(z)} = re^{i\theta}$ is identified by the condition $r = 0$ (with θ undefined).

A complex number z is unity *iff* its modulus (or magnitude) is unity. So, in Cartesian form a unity complex number $z = x + iy$ is identified by the condition $|z| = \sqrt{x^2 + y^2} = 1$. Similarly, in polar form a unity complex number $z = re^{i\theta}$ is identified by the condition $r = 1$, i.e. $z = e^{i\theta}$ ($\theta \in \mathbb{R}$). However, it is worth noting that "unity" may be used specifically for the real number "1".

Problems

1. Investigate the properties of zero and unity of complex numbers and compare them.
   Answer: We note the following:
   • While zero is a single number, unity is an infinite set of numbers. However, "unity" may also be used to label the number 1 specifically (which is a single number).

[53] We remind the reader of the difference between "inequality" (i.e. $<, >, \leq, \geq$) and "non-equality" (i.e. $\neq$) which was explained in § 1.1.

[54] This is based on a simple rule of logic, that is: the negation of "A AND B" is "(not A) OR (not B)" (where OR is non-exclusive).

• Zero can be considered as a complex number (i.e. $0+i0$), as a real number (i.e. 0), and as an imaginary number (i.e. $i0$) at the same time. Also, the set of unity numbers includes (strictly) complex numbers (like $e^{i\pi/4} = 2^{-1/2} + i2^{-1/2}$), real numbers (like $e^{i\pi} = -1$) and imaginary numbers (like $e^{i\pi/2} = i$). We note that we have only two unity real numbers (i.e. ± 1) and only two unity imaginary numbers (i.e. $\pm i$), so the entire set of unity numbers are (strictly) complex except these four.[55]
• Zero is the identity element with regard to the operation of addition and the destruction element with regard to the operation of multiplication (since it annihilates its multiplicand), while unity is the identity element with regard to the operation of multiplication if we consider the modulus only (since it does not affect the magnitude of its multiplicand noting also that 1 specifically is the identity element with regard to the operation of multiplication unconditionally since it affects neither the modulus nor the argument of its multiplicand).

2. Make a graphic representation of the set of unity numbers in the (upper) quarter-plane between the lines $y = x$ and $y = -x$.
Answer: See Figure 5.

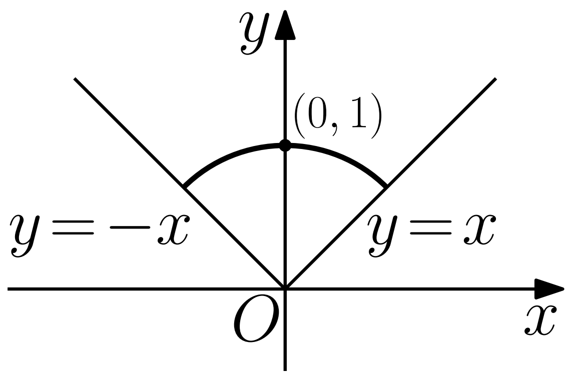

Figure 5: Graphic representation of the set of unity numbers (thick circular arc) between the lines $y = x$ and $y = -x$. These numbers can be represented in polar form as $z = e^{i(\theta_p + 2n\pi)}$ $\left(\frac{\pi}{4} \le \theta_p \le \frac{3\pi}{4}\right)$. See Problem 2 of § 1.8.4.

1.8.5 Arithmetic and Algebraic Operations

In general, the arithmetic and algebra of complex numbers follow the same rules as those of real numbers with i (i.e. the imaginary unit) being treated as a constant number (like any other number such as 1 and π) but with its special properties and significance such as $i^2 = -1$. This applies to the complex numbers as a whole (as usually represented and symbolized by z and w) and to their real and imaginary components in the Cartesian form (like $z = x + iy$ and $w = u + iv$) as well as to their modulus and argument in the polar form (like $z = re^{i\theta}$). However, we should note that since i cannot merge into the real numbers (for instance $i \times 3 = i3$ where i and 3 are kept separate in $i3$ unlike $2 \times 3 = 6$ where 2 and 3 are merged into 6 which is another number) it behaves like an algebraic symbol from this perspective and this facilitates the separation of real and imaginary parts of any complex number and variable. In the following points we present some of the general rules and regulations that govern the arithmetic and algebra of complex numbers noting that these rules and regulations are just instantiations and applications of the above general rule.
• The set of complex numbers is closed under the arithmetic and algebraic operations, i.e. the (legitimate) arithmetic and algebraic operations conducted on complex numbers produce complex numbers in general.
• The algebraic sum of two complex numbers z_1 and z_2 (i.e. $z_1 \pm z_2$) is a complex number z whose real part is the algebraic sum of their real parts and its imaginary part is the algebraic sum of their imaginary

[55] When we say "two" or "four" we consider the numbers as identified by points in the complex plane, i.e. we consider the numbers as represented uniquely by their Cartesian form. We may also consider their principal polar form.

parts, that is: $\text{Re}(z) = \text{Re}(z_1) \pm \text{Re}(z_2)$ and $\text{Im}(z) = \text{Im}(z_1) \pm \text{Im}(z_2)$. So, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then their algebraic sum is:

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2) \tag{20}$$

• It is obvious that addition and subtraction are easier in Cartesian form than in polar form. In fact, these operations are naturally conducted in Cartesian form and hence if the numbers are given in polar form they are usually converted to Cartesian form before adding or subtracting them.
• If two complex numbers z_1 and z_2 are given in Cartesian form as $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then their product is:

$$z_1 \times z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + x_1iy_2 + iy_1x_2 + iy_1iy_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \tag{21}$$

i.e. the real part of the product is the product of the real parts minus the product of the imaginary parts while the imaginary part of the product is the sum of the products of the mixed (or non-corresponding) real and imaginary parts.
• If two complex numbers z_1 and z_2 are given in polar form as $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ then their product is:

$$z_1 \times z_2 = r_1e^{i\theta_1} \times r_2e^{i\theta_2} = r_1r_2e^{i\theta_1 + i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)} \tag{22}$$

i.e. the modulus of the product is the product of the moduli while the argument of the product is the sum of the arguments. The last formula can be easily extended (by repetition and induction) to the product of n numbers, i.e. $z_1 \times z_2 \times \cdots \times z_n = r_1r_2\cdots r_ne^{i(\theta_1+\theta_2+\cdots+\theta_n)}$.
• If two complex numbers z_1 and z_2 are given in Cartesian form as $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ ($z_2 \neq 0$) then their quotient is:

$$\frac{z_1}{z_2} = \frac{z_1z_2^*}{z_2z_2^*} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} \tag{23}$$

where we multiplied the numerator and denominator by the conjugate of the denominator to make the denominator real and hence separate the real and imaginary parts of the quotient.
• If two complex numbers z_1 and z_2 are given in polar form as $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ ($z_2 \neq 0$) then their quotient is:

$$\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i\theta_1 - i\theta_2} = r_1r_2^{-1}e^{i(\theta_1-\theta_2)} \tag{24}$$

i.e. the modulus of the quotient is the quotient of the moduli while the argument of the quotient is the argument of the dividend minus the argument of the divisor.
• It is obvious that multiplication and division are easier in polar form than in Cartesian form.
• Division can be seen as multiplication of the dividend by the reciprocal of its divisor (see § 1.8.9).
• There are obvious geometric interpretations of the multiplication and division of complex numbers (as position vectors in the z plane); some of which will be investigated here (as well as later in the book). From the perspective of the polar form of the numbers involved in the multiplication operation (i.e. $z_1 \times z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$) it is obvious that this operation means scaling associated with rotation, i.e. we are scaling (up or down) the magnitude (i.e. modulus) of one of the numbers by the magnitude of the other number and rotating the scaled number by the phase angle (i.e. argument) of the other number. Similarly, from the perspective of the polar form of the numbers involved in the division operation (i.e. $z_1/z_2 = r_1r_2^{-1}e^{i(\theta_1-\theta_2)}$) it is obvious that this operation also means scaling associated with rotation, i.e. we are scaling (down or up) the magnitude of the dividend by the reciprocal of the magnitude of the divisor and rotating the (scaled) dividend by the negative of the phase angle of the divisor.
• As indicated earlier, most of the properties of arithmetic and algebraic operations on real numbers are also possessed by these operations on complex numbers. For example:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \qquad \text{(associativity of addition)} \tag{25}$$

$$(z_1 \times z_2) \times z_3 = z_1 \times (z_2 \times z_3) \qquad \text{(associativity of multiplication)} \tag{26}$$

$$z_1 + z_2 = z_2 + z_1 \qquad \text{(commutativity of addition)} \tag{27}$$

$$z_1 \times z_2 = z_2 \times z_1 \qquad \text{(commutativity of multiplication)} \tag{28}$$
$$z_1 \times (z_2 \pm z_3) = (z_1 \times z_2) \pm (z_1 \times z_3) \qquad \text{(distributivity of multiplication over addition)} \tag{29}$$
$$0 + z = z + 0 = z \qquad \text{(additive identity)} \tag{30}$$
$$1 \times z = z \times 1 = z \qquad \text{(multiplicative identity)} \tag{31}$$
$$z + (-z) = (-z) + z = 0 \qquad \text{(additive inverse)} \tag{32}$$
$$z \times (1/z) = (1/z) \times z = 1 \qquad \text{(multiplicative inverse, } z \neq 0\text{)} \tag{33}$$
$$0 \times z = z \times 0 = 0 \qquad \text{(annihilation property of 0)} \tag{34}$$

Problems

1. Complex numbers may be compared to vectors in 2D plane[56] and their arithmetic may be thought to be similar. Comment on this issue.
   Answer: Although complex numbers look like vectors in 2D plane (i.e. complex plane), their arithmetic is not the same as the arithmetic of vectors. In brief, addition and subtraction of complex numbers are similar to those of vectors, but multiplication and division are not. As we know, the multiplication of complex numbers is similar neither to the dot product of vectors nor to the cross product of vectors. Moreover, division is allowed in complex numbers but not in vectors.
   In fact, the profound difference between complex numbers and vectors is not restricted to these arithmetic operations and attributes but it extends to other mathematical operations and attributes. For example, common functions (like polynomials, exponentials, logarithmic, trigonometric, hyperbolic, etc.) can operate within the domain of complex numbers but not within the domain of vectors (i.e. in the usual sense although some extensions and generalizations may be made). So in brief, complex numbers and vectors are fundamentally different mathematical entities (where the former are essentially a type of abstract numbers while the latter are essentially geometric objects) although they share some similarities due to their 2D representation in the coordinate plane.
2. Describe in words the operation of multiplication and division of complex numbers.
   Answer: Using the polar representation of complex numbers, we can say: the product of two complex numbers is a complex number whose modulus is the product of their moduli and its argument (or phase angle) is the sum of their arguments, while the quotient of two complex numbers is a complex number whose modulus is the quotient of their moduli (i.e. the modulus of the dividend divided by the modulus of the divisor) and its argument (or phase angle) is the difference between their arguments (i.e. the argument of the dividend minus the argument of the divisor).
3. Verify the properties given by Eqs. 25-34.
   Answer:[57] These properties are trivially based on the corresponding properties in real numbers (as well as the property of i and $e^{i\theta}$ as multiplicands of real numbers), that is:

   - $$\begin{aligned}(z_1 + z_2) + z_3 &= ([x_1 + x_2] + i\,[y_1 + y_2]) + x_3 + iy_3 = x_1 + x_2 + iy_1 + iy_2 + x_3 + iy_3 \\ &= x_1 + iy_1 + ([x_2 + x_3] + i\,[y_2 + y_3]) = z_1 + (z_2 + z_3)\end{aligned}$$
   - $$\begin{aligned}(z_1 \times z_2) \times z_3 &= \left(r_1 e^{i\theta_1} \times r_2 e^{i\theta_2}\right) \times r_3 e^{i\theta_3} = \left(r_1 r_2 e^{i(\theta_1+\theta_2)}\right) \times r_3 e^{i\theta_3} = r_1 r_2 r_3 e^{i(\theta_1+\theta_2+\theta_3)} \\ &= r_1 e^{i\theta_1} \times \left(r_2 r_3 e^{i(\theta_2+\theta_3)}\right) = r_1 e^{i\theta_1} \times \left(r_2 e^{i\theta_2} \times r_3 e^{i\theta_3}\right) = z_1 \times (z_2 \times z_3)\end{aligned}$$
   - $$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2) = (x_2 + x_1) + i\,(y_2 + y_1) = z_2 + z_1$$

[56] The restriction "in 2D plane" is essential because vectors in spaces of higher dimensionality (e.g. 3D space) are totally different and hence they cannot be compared to complex numbers. This also applies to 2D non-planar surfaces.

[57] In the following verifications we use whichever more convenient form of representation (whether Cartesian or polar or both).

- $$z_1 \times z_2 = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} = r_2 r_1 e^{i(\theta_2+\theta_1)} = r_2 e^{i\theta_2} \times r_1 e^{i\theta_1} = z_2 \times z_1$$

- $$\begin{aligned} z_1 \times (z_2 \pm z_3) &= r_1 e^{i\theta_1} \times ([x_2 \pm x_3] + i\,[y_2 \pm y_3]) = r_1 e^{i\theta_1}(x_2 \pm x_3) + i r_1 e^{i\theta_1}(y_2 \pm y_3) \\ &= \left(r_1 e^{i\theta_1} x_2 \pm r_1 e^{i\theta_1} x_3\right) + i\left(r_1 e^{i\theta_1} y_2 \pm r_1 e^{i\theta_1} y_3\right) \\ &= \left(r_1 e^{i\theta_1} x_2 + i r_1 e^{i\theta_1} y_2\right) \pm \left(r_1 e^{i\theta_1} x_3 + i r_1 e^{i\theta_1} y_3\right) \\ &= \left[r_1 e^{i\theta_1} \times (x_2 + i y_2)\right] \pm \left[r_1 e^{i\theta_1} \times (x_3 + i y_3)\right] = (z_1 \times z_2) \pm (z_1 \times z_3) \end{aligned}$$

- $$\begin{aligned} 0 + z &= (0 + i0) + (x + iy) = (0 + x) + i(0 + y) = x + iy = z \\ &= x + iy = (x + 0) + i(y + 0) = (x + iy) + (0 + i0) = z + 0 \end{aligned}$$

- $$\begin{aligned} 1 \times z &= \left(1e^{i0}\right) \times \left(re^{i\theta}\right) = (1 \times r) \times e^{i(0+\theta)} = r \times e^{i\theta} = re^{i\theta} = z \\ &= re^{i\theta} = r \times e^{i\theta} = (r \times 1) \times e^{i(\theta+0)} = \left(re^{i\theta}\right) \times \left(1e^{i0}\right) = z \times 1 \end{aligned}$$

- $$\begin{aligned} z + (-z) &= (x + iy) + (-x - iy) = (x - x) + i(y - y) = 0 + i0 = 0 \\ &= 0 + i0 = (-x + x) + i(-y + y) = (-x - iy) + (x + iy) = (-z) + z \end{aligned}$$

- $$\begin{aligned} z \times \frac{1}{z} &= re^{i\theta} \times \frac{1}{re^{i\theta}} = \frac{r}{r} \times \frac{e^{i\theta}}{e^{i\theta}} = 1 \times e^{i(\theta-\theta)} = e^{i(0)} = \cos 0 + i \sin 0 = 1 + i0 = 1 \\ &= 1 + i0 = \cos 0 + i \sin 0 = e^{i(0)} = 1 \times e^{i(\theta-\theta)} = \frac{r}{r} \times \frac{e^{i\theta}}{e^{i\theta}} = \frac{1}{re^{i\theta}} \times re^{i\theta} = \frac{1}{z} \times z \end{aligned}$$

- $$\begin{aligned} 0 \times z &= (0 + i0) \times (x + iy) = (0 \times x - 0 \times y) + i(0 \times y + 0 \times x) = 0 + i0 = 0 \\ &= 0 + i0 = (x \times 0 - y \times 0) + i(x \times 0 + y \times 0) = (x + iy) \times (0 + i0) = z \times 0 \end{aligned}$$

4. If $z_1 = 5$, $z_2 = i3$, $z_3 = 11 + i2$, $z_4 = 6 - i17$, $z_5 = \sqrt{13}e^{i\pi/4}$ and $z_6 = e^{-i\pi/3}$, find the following:
   (a) $16z_1z_3 - 4z_2z_4$. **(b)** $z_3 + z_2z_5 - 3z_6$. **(c)** $z_3z_4 + z_2z_3$.
   (d) z_4/z_3. **(e)** $z_5 \times z_6$. **(f)** $z_5/(z_2z_6)$.

   Answer: We use Cartesian or polar forms (whichever is more convenient).

   (a) $$16z_1z_3 - 4z_2z_4 = 16\,[5\,(11 + i2)] - 4\,[i3\,(6 - i17)] = (880 + i160) - (204 + i72) = 676 + i88$$

   (b) $$\begin{aligned} z_3 + z_2z_5 - 3z_6 &= (11 + i2) + i3\left(\sqrt{13}e^{i\pi/4}\right) - 3\left(e^{-i\pi/3}\right) \\ &= (11 + i2) + i3\left(\sqrt{\frac{13}{2}} + i\sqrt{\frac{13}{2}}\right) - 3\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \\ &= (11 + i2) + \left(-3\sqrt{\frac{13}{2}} + i3\sqrt{\frac{13}{2}}\right) - \left(\frac{3}{2} - i\frac{3\sqrt{3}}{2}\right) \simeq 1.8515 + i12.2466 \end{aligned}$$

   (c) $$\begin{aligned} z_3z_4 + z_2z_3 &= z_3\,(z_4 + z_2) = (11 + i2)\left([6 - i17] + i3\right) = (11 + i2)\,(6 - i14) \\ &= (11 \times 6 - 2 \times [-14]) + i(11 \times [-14] + 6 \times 2) = 94 - i142 \end{aligned}$$

   (d) $$\frac{z_4}{z_3} = \frac{6 - i17}{11 + i2} = \frac{(6 - i17)\,(11 - i2)}{(11 + i2)\,(11 - i2)} = \frac{32 - i199}{125} = 0.256 - i1.592$$

(**e**) $$z_5 \times z_6 = \sqrt{13}e^{i\pi/4} \times e^{-i\pi/3} = \sqrt{13}e^{i(\pi/4-\pi/3)} = \sqrt{13}e^{-i\pi/12} \simeq 3.4827 - i0.9332$$

(**f**) $$\frac{z_5}{z_2 z_6} = \frac{\sqrt{13}e^{i\pi/4}}{i3e^{-i\pi/3}} = -i\frac{\sqrt{13}}{3}e^{i(\pi/4+\pi/3)} = e^{-i\pi/2}\frac{\sqrt{13}}{3}e^{i7\pi/12}$$
$$= \frac{\sqrt{13}}{3}e^{i(7\pi/12-\pi/2)} = \frac{\sqrt{13}}{3}e^{i\pi/12} \simeq 1.1609 + i0.3111$$

5. What is the effect of multiplying a given complex number z by i, -1, $-i$ and 1?
Answer: We note first that all these numbers are of unit length and hence they will not change the modulus of z. So, all they do is to change the argument of z, i.e. rotate z by a given angle. Accordingly:
• Multiplying z by i is equivalent to rotating it by $\pi/2$ (since $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$).
• Multiplying z by -1 is equivalent to rotating it by π (since $-1 = \cos\pi + i\sin\pi = e^{i\pi}$). This should also be concluded from the fact that $-1 = ii$ where each i rotates by $\pi/2$.
• Multiplying z by $-i$ is equivalent to rotating it by $-\pi/2$ (since $-i = \cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2} = e^{-i\pi/2}$). This can also be seen as rotating by $\frac{3\pi}{2}$ (since $-i = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = e^{i3\pi/2}$). This can also be concluded from the fact that $-i = iii$ where each i rotates by $\pi/2$.
• Multiplying z by 1 keeps it unchanged (since $1 = \cos 0 + i\sin 0 = e^{i0}$ which is consistent with the fact that 1 is unity for multiplication). This can also be seen as rotating by 2π (since $1 = \cos 2\pi + i\sin 2\pi = e^{i2\pi}$). This can also be concluded from the fact that $1 = iiii$ where each i rotates by $\pi/2$.
So in brief, i rotates by $\frac{\pi}{2}$, $-1 = ii$ rotates by $2 \times \frac{\pi}{2} = \pi$, $-i = iii$ rotates by $3 \times \frac{\pi}{2} = \frac{3\pi}{2}$, and $1 = iiii$ rotates by $4 \times \frac{\pi}{2} = 2\pi$.
Note: in the above answer we did not consider rotating by angles that include integer multiples of 2π because such angles do not produce different results to the above results since adding integer multiples of 2π to any of the above angles does not affect the real or imaginary part of the product. Nevertheless, for the sake of generality and thoroughness we may say: multiplying z by i is equivalent to rotating it by $\frac{\pi}{2} + 2n\pi$ (and similarly for the rest).
6. Given that z is a strictly complex number (i.e. not real or imaginary) in the first quadrant, represent graphically z, iz, $-z$, and $-iz$ as position vectors in Cartesian coordinates.
Answer: See Figure 6.
Comment: iz is a rotation of z by $\pi/2$. $-z$ is a rotation of z by π. $-iz$ is a rotation of z by $-\pi/2$. See Problem 5.
7. Which of the following numbers are real and which are imaginary:
(**a**) zz^*. (**b**) $z - z^*$. (**c**) $z + z^*$. (**d**) $i^{11}e^{i5\pi/2}$.
(**e**) $(\mathrm{Re}\, z)/i$. (**f**) $\mathrm{Im}(z/i)$. (**g**) $\mathrm{arccosh}(1)$. (**h**) $\pi e^{i5\pi/6}$.
Answer:
(**a**) This is real because:
$$zz^* = (x + iy)(x - iy) = (x^2 + y^2) + i(yx - xy) = x^2 + y^2$$
(**b**) This is imaginary because:
$$z - z^* = (x + iy) - (x - iy) = (x - x) + i(y + y) = i2y$$
(**c**) This is real because:
$$z + z^* = (x + iy) + (x - iy) = (x + x) + i(y - y) = 2x$$
(**d**) This is real because:
$$i^{11}e^{i5\pi/2} = -ie^{i5\pi/2} = -i\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = -i\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = -i\,(0 + i) = 1$$

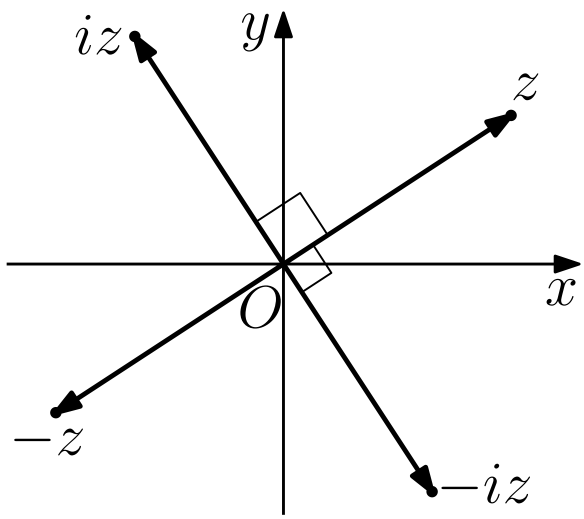

Figure 6: Graphic representation of z, iz, $-z$, and $-iz$ as position vectors in Cartesian coordinates. We note that if z has coordinates (X, Y) then iz, $-z$, and $-iz$ have coordinates $(-Y, X)$, $(-X, -Y)$ and $(Y, -X)$ respectively. Also see the comment of Problem 6 of § 1.8.5.

(**e**) This is imaginary because:

$$\frac{\operatorname{Re} z}{i} = -i \times \operatorname{Re} z = -i \times \operatorname{Re}(x + iy) = -ix$$

(**f**) This is real because the imaginary part of any number is real (by definition).
(**g**) This can be regarded as real and imaginary (as well as complex) because $\operatorname{arccosh}(1) = 0$ (considering the principal value; see Eq. 154).
(**h**) This is neither real nor imaginary because:

$$\pi e^{i5\pi/6} = \pi \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) = \pi \left(-\frac{\sqrt{3}}{2} + i\frac{1}{2} \right) = -\frac{\pi\sqrt{3}}{2} + i\frac{\pi}{2}$$

8. Identify the effects of the following mathematical operations on a complex number $z = x + iy$ (i.e. in its Cartesian form):
(**a**) Division by i. (**b**) Multiplication by i^3. (**c**) Multiplication by $-(i^*)^2$.
Answer:
(**a**) Rotation of z by $-\pi/2$ (i.e. *clockwise* rotation by $\pi/2$) because:

$$\frac{z}{i} = \frac{iz}{i^2} = \frac{iz}{-1} = -iz = -i(x + iy) = -ix - i^2 y = -ix + y = y - ix$$

Noting that $x = |z| \cos\theta$ and $y = |z| \sin\theta$ $\left[\text{as well as the trigonometric identities } \cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta \text{ and } \sin\left(\theta - \frac{\pi}{2}\right) = -\cos\theta\right]$, it is obvious that $y - ix$ is obtained by rotating $x + iy$ by $-\pi/2$. Also, see Problem 5.
(**b**) The same as part (a) because $i^3 z = i^2 \times iz = -iz = z/i$.
(**c**) Doing nothing (i.e. like multiplying z by 1) because:

$$-(i^*)^2 z = -(-i)^2 z = -(-1 \times i)^2 z = -(-1)^2 \times (i^2) z = -(i^2) z = -(-1) z = 1 \times z = z$$

Note: regarding part (b), being the same is from the Cartesian perspective (in accord with the statement of the question). Hence, it may not be the same from the polar perspective (or even from a "physical" perspective if we consider representing physical processes and operations by these mathematical operations). We should also repeat the essence of the note of Problem 5.

9. Identify the effects of the following mathematical operations on a complex number $z = re^{i\theta}$ (i.e. in its polar form):

   (**a**) Multiplication by $e^{-i\pi/2}$. (**b**) Division by $e^{-i3\pi/2}$. (**c**) Multiplication by $-e^{-i7\pi}$.

   Answer: The effects of these operations are (correspondingly) identical to the effects of the operations in Problem 8 because:

   (**a**)
   $$e^{-i\pi/2}\, z = \frac{z}{e^{i\pi/2}} = \frac{z}{\cos(\pi/2) + i\sin(\pi/2)} = \frac{z}{0+i} = \frac{z}{i}$$

   (**b**)
   $$\frac{z}{e^{-i3\pi/2}} = e^{i3\pi/2}\, z = \left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) z = (0-i)\, z = -iz = i^2 \times iz = i^3 z$$

   (**c**)
   $$-e^{-i7\pi}\, z = -\big[\cos(-7\pi) + i\sin(-7\pi)\big] z = -\big[-1 + i0\big] z = 1 \times z = z$$

   Note: the effects of the operations in this Problem are unique because the operations in this Problem are given in polar form where each argument is defined uniquely (unlike the situation in Problem 8). Hence, from this perspective the effects of the operations in this Problem are not *definitely* identical to the effects of the operations in Problem 8.

10. Show that the product of two complex numbers is zero *iff* at least one of them is zero.

    Answer: From Eq. 22 we have $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$. So, if $z_1 = 0$ or $z_2 = 0$ (or both) then $r_1 = 0$ or $r_2 = 0$ (or both) and hence $z_1 z_2 = 0$. On the other hand, if $z_1 z_2 = 0$ then either $r_1 = 0$ or $r_2 = 0$ (or both) and hence $z_1 = 0$ or $z_2 = 0$ (or both).

1.8.6 Real and Imaginary Parts

As noted earlier, the split of complex numbers to real and imaginary parts is related to the Cartesian form of representation of complex numbers (noting that both parts are real). Many of the general aspects of the real and imaginary parts of complex numbers have already been investigated. More investigation and discussion about these issues will be given in the Problems of this subsection (as well as in the upcoming parts of the book noting that some of these issues depend on other issues that are waiting investigation).

Problems

1. Represent graphically the sets of complex numbers that satisfy the following relations:

   (**a**) $\mathrm{Re}\, z - \mathrm{Im}\, z = 0$. (**b**) $\mathrm{Re}\, z + \mathrm{Im}\, z = 0$. (**c**) $\mathrm{Im}\, z/\mathrm{Re}\, z = 1 \quad (z \neq 0)$.

   (**d**) $\mathrm{Re}\, z = 6$. (**e**) $\mathrm{Im}\, z = 0$. (**f**) $(\mathrm{Re}\, z)^2 + (\mathrm{Im}\, z)^2 = 25$.

   Answer: See Figure 7.

2. Identify the mathematical operations that do the following to a complex number $z = x + iy$ (i.e. in its Cartesian form):

   (**a**) Replacing the real and imaginary parts of z by their negative, i.e. $x + iy \to -x - iy$.
   (**b**) Replacing the real part of z by its negative, i.e. $x + iy \to -x + iy$.
   (**c**) Replacing the imaginary part of z by its negative, i.e. $x + iy \to x - iy$.
   (**d**) Exchanging the real and imaginary parts of z, i.e. $x + iy \to y + ix$.
   (**e**) Replacing the real part of z by the negative of its imaginary part and vice versa, i.e. $x+iy \to -y-ix$.
   (**f**) Annihilating the real part of z, i.e. $x + iy \to iy$.
   (**g**) Annihilating the imaginary part of z, i.e. $x + iy \to x$.

   Answer:
   (**a**) Multiplication by -1, that is: $-z = -(x + iy) = -x - iy$.
   (**b**) Conjugation with negation (in whichever order), that is: $-z^* = -(x + iy)^* = -(x - iy) = -x + iy$ or $(-z)^* = (-x - iy)^* = -x + iy$.
   (**c**) Conjugation, that is: $z^* = (x + iy)^* = x - iy$.
   (**d**) Conjugation followed by multiplication by i, that is: $iz^* = i(x+iy)^* = i(x-iy) = (ix+y) = y+ix$.

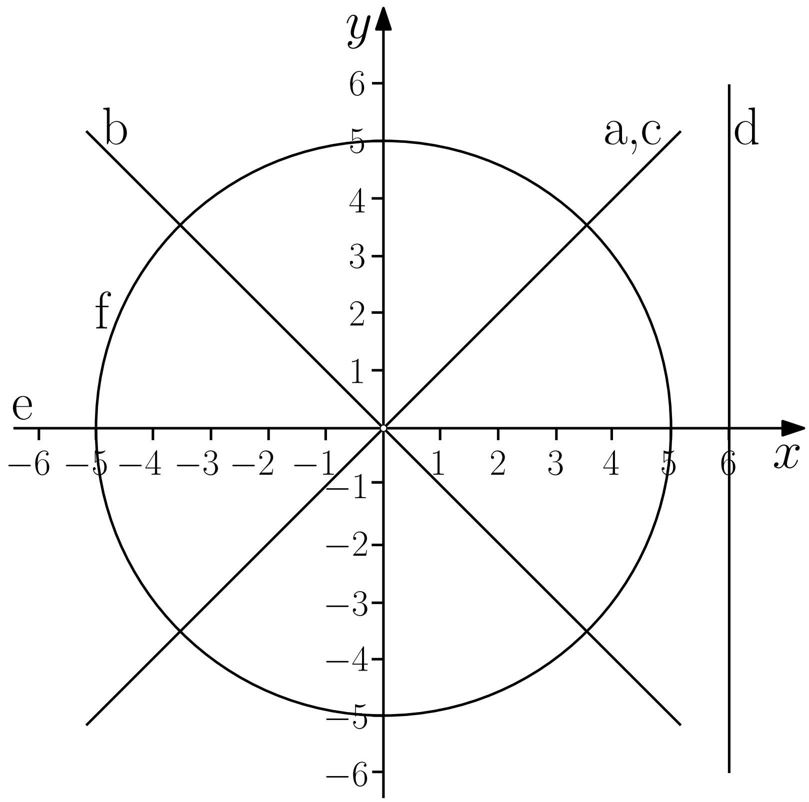

Figure 7: Graphic representation of the sets of complex numbers of Problem 1 of § 1.8.6. All straight lines goes to infinity in both directions and the puncture at the origin belongs to part (c).

It can also be achieved by multiplication by $-i$ followed by conjugation, that is: $(-iz)^* = (-ix+y)^* = (y-ix)^* = y+ix$.
(**e**) Conjugation followed by multiplication by $-i$, that is: $-i(x+iy)^* = -i(x-iy) = -(ix+y) = -y-ix$. It can also be achieved by multiplication by i followed by conjugation, that is: $(iz)^* = (ix-y)^* = (-y+ix)^* = -y-ix$.
(**f**) Subtraction of its conjugate followed by division by 2, that is: $\frac{z-z^*}{2} = \frac{(x+iy)-(x-iy)}{2} = \frac{i2y}{2} = iy$.
(**g**) Addition of its conjugate followed by division by 2, that is: $\frac{z+z^*}{2} = \frac{(x+iy)+(x-iy)}{2} = \frac{2x}{2} = x$.
Also, see Problem 8 of § 1.8.8.
Note: in the above answer we repeatedly use conjugation which may not be a basic mathematical operation like addition and subtraction. So, what are the basic mathematical operations whose combination achieves conjugation (and hence they enable us to express the above operations that involve conjugation in terms of more basic mathematical operations)? This issue is addressed in Problem 10 of § 1.8.8 where we will see that in Cartesian form (at least) there may not be more basic mathematical operations by which conjugation can be achieved and accordingly conjugation is a basic mathematical operation that is not based on (or achievable by) more basic operations.

3. Solve the following complex equations, inequalities and non-equality (for $z \in \mathbb{C}$):[58]

[58] We remind the reader that the double-bar symbol in $|\alpha|$ means modulus when α is complex and means absolute value when α is real (see § 1.1). We should also note that although the relations that are expressed by non-equalities can generally be expressed in terms of inequalities, in certain situations the non-equalities produce simpler and clearer mathematical relations and expressions and hence they are advantageous as a tool for mathematical expression and representation.

(**a**) $\operatorname{Re} z < (\operatorname{Im} z)^2 + 8$. (**b**) $\operatorname{Re}|z| \le 3$. (**c**) $\operatorname{Re}(z)\operatorname{Im}(z) = 5 \quad (0 < \arg z < \frac{\pi}{2})$.

(**d**) $|\operatorname{Im} z| = e$. (**e**) $(\operatorname{Im} z)^4 + (\operatorname{Re} z)^2 + 4 = 0$. (**f**) $\operatorname{Im} z \ne 2\operatorname{Re} z + 1$.

Answer:

(**a**) The equation $\operatorname{Re} z = (\operatorname{Im} z)^2 + 8$ represents the parabola $x = y^2 + 8$ which is symmetrical about the real axis and opens to the right with vertex at $(8, 0)$. Hence, the solution of the inequality is all the numbers in the z plane to the left of this parabola.

(**b**) The modulus $|z|$ is real and hence $\operatorname{Re}|z| = |z|$. So, what we have is $|z| \le 3$ which is an equation of the origin-centered disk of radius 3. Therefore, the solution is all the complex numbers represented by this disk.

(**c**) The equation $\operatorname{Re}(z)\operatorname{Im}(z) = 5$ represents the hyperbola $y = 5/x$ $(x \ne 0)$. However, the condition $0 < \arg z < \frac{\pi}{2}$ restricts the solution to the numbers represented by the branch of this hyperbola in the first quadrant.

(**d**) We have $|\operatorname{Im} z| = |y| = e$ and hence $y = \pm e$. So, the solution is all the complex numbers represented by the two (horizontal) lines: $y = +e$ and $y = -e$.

(**e**) This equation means $(\operatorname{Im} z)^4 + (\operatorname{Re} z)^2 = -4$ and hence it has no solution because both $\operatorname{Im} z$ and $\operatorname{Re} z$ are real and hence both $(\operatorname{Im} z)^4$ and $(\operatorname{Re} z)^2$ are non-negative, so their sum cannot be negative.

(**f**) The equation $\operatorname{Im} z = 2\operatorname{Re} z + 1$ represents the line $y = 2x + 1$. Hence, the solution of this non-equality is the entire set of complex numbers excluding those represented by this line.

4. Solve the following systems of simultaneous complex equations (for $z \in \mathbb{C}$):

(**a**) $\operatorname{Re} z = 2\left[1 - \left(\frac{\operatorname{Im} z}{5}\right)^2\right]^{1/2}$ and $\operatorname{Im} z - 6\operatorname{Re} z - 1 = 0$.

(**b**) $|\operatorname{Re} z| = 7$ and $(\operatorname{Re} z - 1)^2 + (\operatorname{Im} z + 2)^2 = 9$.

(**c**) $\operatorname{Im}(z)\operatorname{Re}(z) = 1$ and $(\operatorname{Re} z)^2 + 2(\operatorname{Im} z)^2 - 5 = 0$.

(**d**) $\operatorname{Im} z - \operatorname{Re} z - 1 = 0$ and $\operatorname{Im} z - (\operatorname{Re} z)^2 + 2\operatorname{Re} z = 2$.

Answer:[59]

(**a**) We have:

$$\operatorname{Re} z = 2\left[1 - \left(\frac{\operatorname{Im} z}{5}\right)^2\right]^{1/2}$$

$$\left(\frac{\operatorname{Re} z}{2}\right)^2 + \left(\frac{\operatorname{Im} z}{5}\right)^2 = 1 \qquad (\operatorname{Re} z > 0)$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{5}\right)^2 = 1 \qquad (x > 0)$$

which is an equation of the right half of the origin-centered ellipse with semi-axes 2 (along the x direction) and 5 (along the y direction). Similarly, the equation $\operatorname{Im} z - 6\operatorname{Re} z - 1 = 0$ represents the straight line $y = 6x + 1$. These two curves meet only at the point with $x = (20\sqrt{42} - 24)/169 \simeq 0.6249$ and $y = (120\sqrt{42} + 25)/169 \simeq 4.7496$. Hence, the solution of this system of simultaneous equations is the point $z \simeq 0.6249 + i4.7496$.

(**b**) The first equation represents the two vertical lines $x = -7$ and $x = +7$ while the second equation represents the circle with center $z_0 = 1 - i2$ and radius $R = 3$. It is obvious that these three curves do not intersect at any point in the complex plane and hence this system of simultaneous equations has no solution.

(**c**) The first equation represents the hyperbola $y = 1/x$ while the second equation represents the origin-centered ellipse with semi-axes $\sqrt{5}$ (along the x direction) and $\sqrt{5/2}$ (along the y direction).

[59] We note that the solutions (in this Problem and its alike) can be verified by inserting them into the systems.

These two curves obviously meet at four points which are:

$$z_{1,2} = \pm\left(\sqrt{\frac{5+\sqrt{17}}{2}} + i\sqrt{\frac{5-\sqrt{17}}{4}}\right) \simeq \pm(2.1358 + i0.4682)$$

$$z_{3,4} = \pm\left(\sqrt{\frac{5-\sqrt{17}}{2}} + i\sqrt{\frac{5+\sqrt{17}}{4}}\right) \simeq \pm(0.6622 + i1.5102)$$

So, these four points are the solution of this system of simultaneous equations.
(**d**) The first equation represents the straight line $y = x + 1$ while the second equation represents the parabola $y = x^2 - 2x + 2$. These two curves meet at two points which are:

$$z_1 = \frac{3-\sqrt{5}}{2} + i\frac{5-\sqrt{5}}{2} \simeq 0.3820 + i1.3820 \qquad \& \qquad z_2 = \frac{3+\sqrt{5}}{2} + i\frac{5+\sqrt{5}}{2} \simeq 2.6180 + i3.6180$$

These two points are the solution of this system of simultaneous equations.

5. What is the effect of the operation of taking the real/imaginary part on sets of complex numbers (such as curves, shapes and regions) in the complex plane? What is the effect of the operation of taking the imaginary component?
Answer: The effect of taking the real part is that the projection of the "cross section" (i.e. the cross section that faces the real axis) of the set on the real axis is mapped onto the real axis. For example, on taking the real part of the set represented by $|z| = 2$ (i.e. the origin-centered circle of radius 2) we get the segment on the real axis between the point -2 and the point 2, i.e. we compress it (or collapse it) on the real axis.
The effect of taking the imaginary part is that the projection of the "cross section" (i.e. the cross section that faces the imaginary axis) of the set on the imaginary axis is mapped onto the real axis. For example, on taking the imaginary part of the set represented by $\mathrm{Im}(z) = 5\,\mathrm{Re}(z) + 3$ with $1 \le \mathrm{Re}(z) \le 3$ (i.e. the segment of the line $y = 5x + 3$ between $z_1 = 1 + i8$ and $z_2 = 3 + i18$) we get the segment on the real axis between the point 8 and the point 18.
The effect of taking the imaginary component is that the projection of the "cross section" (i.e. the cross section that faces the imaginary axis) of the set on the imaginary axis is mapped onto the imaginary axis. For example, on taking the imaginary component of the set represented by $\mathrm{Im}(z) = 5\,\mathrm{Re}(z) + 3$ with $1 \le \mathrm{Re}(z) \le 3$ we get the segment on the imaginary axis between the point $i8$ and the point $i18$.

1.8.7 Modulus and Argument

As noted earlier, the split of complex numbers to modulus and argument is related to the polar form of representation of these numbers (noting that the modulus and argument are both real with the modulus being non-negative). Some of the general aspects of modulus and argument have already been investigated. In the following points we outline the main properties and rules of modulus and argument of complex numbers:
• The modulus (or magnitude) of a complex number z is a real number obtained by taking the (positive) square root of the product of the number by its conjugate, that is $|z| = \sqrt{zz^*}$. This can be shown (using the Cartesian representation $z = x + iy$) as follows:

$$|z| = \sqrt{x^2 + y^2} = \sqrt{(x + iy)(x - iy)} = \sqrt{zz^*} \tag{35}$$

This can also be shown (using the polar representation $z = re^{i\theta}$) as follows:

$$|z| = r = \sqrt{r^2} = \sqrt{re^{i\theta} \times re^{-i\theta}} = \sqrt{zz^*} \tag{36}$$

• If $z = re^{i\theta}$ then $\theta = \arg(z)$ where arg stands for argument. If θ is in the interval $-\pi < \theta \le \pi$ then it is commonly labeled as the principal argument and is usually symbolized with $\mathrm{Arg}(z)$.[60] As indicated

[60] As indicated earlier, $\mathrm{Arg}(z)$ may be taken in the interval $0 \le \theta < 2\pi$ (according to another convention).

earlier (see § 1.8.2), although a given complex number (as represented by a unique point in the complex plane) is unique, it has infinite number of arguments because we can add an integer multiple of 2π (i.e. $2n\pi$) to its principal argument without affecting its real or imaginary part. In other words, $\arg(z)$ has an infinite number of (distinct) values, i.e. it is an infinitely multi-valued function (like the natural logarithm function which will be investigated in § 2.2; also see § 1.11).

• The modulus of a product is the product of the moduli, that is $|z_1 z_2| = |z_1|\,|z_2|$. This can be easily shown using the polar form, that is:

$$|z_1 z_2| = \left|r_1 e^{i\theta_1} r_2 e^{i\theta_2}\right| = r_1 r_2 \left|e^{i(\theta_1+\theta_2)}\right| = r_1 r_2 = |z_1|\,|z_2| \tag{37}$$

where we used $\left|e^{i\theta}\right| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$ (and noting that r_1 and r_2 are non-negative real numbers). This can be easily generalized (by repetitive application) to more than two factors, i.e. $|z_1 \times z_2 \times \cdots \times z_n| = |z_1| \times |z_2| \times \cdots \times |z_n|$.

• The modulus of a quotient is the quotient of the moduli, that is $|z_1/z_2| = |z_1| / |z_2|$. This can also be easily shown using the polar form, that is:

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right| = \frac{r_1}{r_2}\left|e^{i(\theta_1-\theta_2)}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \tag{38}$$

• The argument of a product is the sum of the arguments of the multiplicands, that is $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. This can be easily shown using the polar form, that is:

$$\arg(z_1 z_2) = \arg(r_1 e^{i\theta_1} r_2 e^{i\theta_2}) = \arg(r_1 r_2 e^{i(\theta_1+\theta_2)}) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2) \tag{39}$$

• The argument of a quotient is the argument of the dividend minus the argument of the divisor, that is $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$. This can also be easily shown using the polar form, that is:

$$\arg\left(\frac{z_1}{z_2}\right) = \arg\left(\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right) = \arg\left(\frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2) \tag{40}$$

• The moduli of complex numbers satisfy the following semi-inequalities (among other inequalities):

$$|z_1 + z_2| \le |z_1| + |z_2| \tag{41}$$

$$|z_1 + z_2| \ge |z_1| - |z_2| \tag{42}$$

$$|z_1 - z_2| \ge |z_1| - |z_2| \tag{43}$$

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n| \tag{44}$$

The first semi-inequality becomes equality when the principal arguments of z_1 and z_2 are identical or (at least) one of these numbers is zero. The second semi-inequality becomes equality when the principal arguments of z_1 and z_2 differ by π (with the smaller in modulus being labeled z_2) or (at least) one of these numbers is zero (which should be labeled z_2 if only one is zero). The third semi-inequality becomes equality when the principal arguments of z_1 and z_2 are identical (with the smaller in modulus being labeled z_2) or (at least) one of these numbers is zero (which should be labeled z_2 if only one is zero). Similar conditions can be easily formulated for the fourth semi-inequality (to become equality) noting that it is a generalization of the first semi-inequality. We note that some or all of the above semi-inequalities (and even other similar inequalities and semi-inequalities) are commonly called the triangle inequalities (noting that this is a generic name which may be used to label any relation of the above types that compares the modulus of a sum or difference of numbers to the sum or difference of their individual moduli). We also note that a semi-inequality similar to the last semi-inequality also holds for integrals (noting that integration is essentially a summation operation over an infinite number of terms) and hence we have a "triangle inequality for integrals" (as well as for algebraic sums), that is:

$$\left|\int_\alpha^\beta f(\xi)\,d\xi\right| \le \int_\alpha^\beta |f(\xi)|\,d\xi \qquad (\alpha, \beta, \xi \in \mathbb{R} \text{ with } \alpha \le \xi \le \beta) \tag{45}$$

Problems

1. Verify the following relations using the Cartesian form of complex numbers:
   (**a**) $|z_1 z_2| = |z_1|\,|z_2|$. (**b**) $|z_1/z_2| = |z_1|\,/\,|z_2|$.
   (**c**) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. (**d**) $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$.
   Answer:
   (**a**) From Eq. 21 we have:

$$\begin{aligned}
|z_1 z_2| &= |(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)| = \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2} \\
&= \sqrt{x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_1^2y_2^2 + 2x_1x_2y_1y_2 + x_2^2y_1^2} = \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2} \\
&= \sqrt{(x_1^2 + y_1^2)\,(x_2^2 + y_2^2)} = \sqrt{x_1^2 + y_1^2}\,\sqrt{x_2^2 + y_2^2} = |z_1|\,|z_2|
\end{aligned}$$

(**b**) From Eq. 23 we have:

$$\begin{aligned}
\left|\frac{z_1}{z_2}\right| &= \left|\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right| = \sqrt{\left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right)^2 + \left(\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right)^2} \\
&= \sqrt{\frac{x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2}{\left(x_2^2 + y_2^2\right)^2} + \frac{x_2^2y_1^2 - 2x_1x_2y_1y_2 + x_1^2y_2^2}{\left(x_2^2 + y_2^2\right)^2}} = \sqrt{\frac{x_1^2x_2^2 + y_1^2y_2^2 + x_2^2y_1^2 + x_1^2y_2^2}{\left(x_2^2 + y_2^2\right)^2}} \\
&= \sqrt{\frac{(x_1^2 + y_1^2)\,(x_2^2 + y_2^2)}{\left(x_2^2 + y_2^2\right)^2}} = \sqrt{\frac{x_1^2 + y_1^2}{x_2^2 + y_2^2}} = \frac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_2^2 + y_2^2}} = \frac{|z_1|}{|z_2|}
\end{aligned}$$

(**c**) If $\theta_1 \equiv \arg(z_1)$ and $\theta_2 \equiv \arg(z_2)$ then from the trigonometric identity of the tangent of a sum of angles we get:

$$\begin{aligned}
\tan(\theta_1 + \theta_2) &= \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} = \frac{\frac{y_1}{x_1} + \frac{y_2}{x_2}}{1 - \frac{y_1}{x_1}\frac{y_2}{x_2}} = \frac{x_2y_1 + x_1y_2}{x_1x_2 - y_1y_2} \\
\theta_1 + \theta_2 \equiv \arg(z_1) + \arg(z_2) &= \arctan\left(\frac{x_1y_2 + x_2y_1}{x_1x_2 - y_1y_2}\right) \qquad (46)
\end{aligned}$$

Also, from Eq. 21 we have $z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$ and hence:

$$\arg(z_1z_2) = \arctan\left(\frac{x_1y_2 + x_2y_1}{x_1x_2 - y_1y_2}\right) \qquad (47)$$

On comparing Eq. 46 and Eq. 47 we get $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$.
(**d**) If $\theta_1 \equiv \arg(z_1)$ and $\theta_2 \equiv \arg(z_2)$ then from the trigonometric identity of the tangent of a difference of angles we get:

$$\begin{aligned}
\tan(\theta_1 - \theta_2) &= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} = \frac{\frac{y_1}{x_1} - \frac{y_2}{x_2}}{1 + \frac{y_1}{x_1}\frac{y_2}{x_2}} = \frac{x_2y_1 - x_1y_2}{x_1x_2 + y_1y_2} \\
\theta_1 - \theta_2 \equiv \arg(z_1) - \arg(z_2) &= \arctan\left(\frac{x_2y_1 - x_1y_2}{x_1x_2 + y_1y_2}\right) \qquad (48)
\end{aligned}$$

Also, from Eq. 23 we have $\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$ and hence:

$$\arg\left(\frac{z_1}{z_2}\right) = \arctan\left(\frac{x_2y_1 - x_1y_2}{x_1x_2 + y_1y_2}\right) \qquad (49)$$

On comparing Eq. 48 and Eq. 49 we get $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$.
Note: in parts (c) and (d) we consider the principal argument (although we used θ and arg). We also ignored some conditions and restrictions noting that the purpose of the Problem is to show the general validity of the formulae (i.e. being obtainable from the Cartesian form) without going through well-known and boring details.

2. Find the moduli and the (principal) arguments of the following numbers:
   (**a**) $z = 3\pi - i$. (**b**) $z = i9$. (**c**) $z = 0.5e^{2\pi}$. (**d**) $z = i55 - 55$.
   Answer:
   (**a**) $|z| = \sqrt{9\pi^2 + 1} \simeq 9.4777$ and $\text{Arg}(z) = \arctan[-1/(3\pi)] \simeq -0.1057$.
   (**b**) $|z| = 9$ and $\text{Arg}(z) = \pi/2$.
   (**c**) $|z| = 0.5e^{2\pi} \simeq 267.7458$ and $\text{Arg}(z) = 0$.
   (**d**) $|z| = \sqrt{(-55)^2 + 55^2} = 55\sqrt{2} \simeq 77.7817$ and $\text{Arg}(z) = \arctan[55/(-55)] = 3\pi/4$.
3. If $z_1 = 1 - i8$, $z_2 = 4 + i\pi$, $z_3 = i12$ and $z_4 = 7 + i$, find the moduli and the (principal) arguments of the following products and quotients:
   (**a**) $z_1 \times z_2$. (**b**) z_1/z_3. (**c**) $(z_1 \times z_2)/(z_3 \times z_4)$.
   Answer: To reduce the calculations, we convert these numbers to polar form, that is:

$$z_1 \simeq \sqrt{65}e^{-i1.4464} \qquad z_2 \simeq \sqrt{16 + \pi^2}e^{i0.6658} \qquad z_3 = 12e^{i\pi/2} \qquad z_4 \simeq \sqrt{50}e^{i0.1419}$$

(**a**)

$$z_1 \times z_2 \simeq \sqrt{65}e^{-i1.4464} \times \sqrt{16 + \pi^2}e^{i0.6658} \simeq \sqrt{65} \times \sqrt{16 + \pi^2}e^{i(-1.4464+0.6658)} \simeq 41.006393e^{-i0.7807}$$

Hence, the modulus is 41.006393 and the argument is -0.7807.
(**b**)

$$\frac{z_1}{z_3} \simeq \frac{\sqrt{65}e^{-i1.4464}}{12e^{i\pi/2}} = \frac{\sqrt{65}}{12}e^{i(-1.4464-\pi/2)} \simeq 0.6719e^{-i3.01724}$$

Hence, the modulus is 0.6719 and the argument is -3.01724.
(**c**)

$$\begin{aligned}\frac{z_1 \times z_2}{z_3 \times z_4} &\simeq \frac{\sqrt{65}e^{-i1.4464} \times \sqrt{16 + \pi^2}e^{i0.6658}}{12e^{i\pi/2} \times \sqrt{50}e^{i0.1419}} = \frac{\sqrt{65} \times \sqrt{16 + \pi^2}}{12 \times \sqrt{50}}e^{i(-1.4464+0.6658-\pi/2-0.1419)} \\ &\simeq 0.4833e^{-i2.4934}\end{aligned}$$

Hence, the modulus is 0.4833 and the argument is -2.4934.
4. Prove the following relations (with n being integer):
   (**a**) $|z^n| = |z|^n$ $(z \neq 0)$. (**b**) $2|z|^2 \geq \big(|\text{Re}\, z| + |\text{Im}\, z|\big)^2$. (**c**) $|z_1 + z_2| \leq |z_1| + |z_2|$.
   (**d**) $|z_1 + z_2| \geq |z_1| - |z_2|$. (**e**) $|z_1 - z_2| \geq |z_1| - |z_2|$. (**f**) $|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$.
   Answer:
   (**a**) If $n = 0, \pm1$ the result is obvious. If $n > 1$ then we have (noting the generalization of Eq. 37):

$$|z^n| = |z \times z \times \cdots \times z| = |z| \times |z| \times \cdots \times |z| = |z|^n$$

If $n < -1$ then $n = -|n|$ and we have (noting Eq. 38):

$$|z^n| = \left|z^{-|n|}\right| = \left|\frac{1}{z^{|n|}}\right| = \frac{1}{|z \times z \times \cdots \times z|} = \frac{1}{|z| \times |z| \times \cdots \times |z|} = \frac{1}{|z|^{|n|}} = |z|^{-|n|} = |z|^n$$

(**b**)

$$\begin{aligned}\big(|\text{Re}\, z| - |\text{Im}\, z|\big)^2 &\geq 0 \\ |\text{Re}\, z|^2 - 2|\text{Re}\, z||\text{Im}\, z| + |\text{Im}\, z|^2 &\geq 0 \\ |\text{Re}\, z|^2 + |\text{Im}\, z|^2 &\geq 2|\text{Re}\, z||\text{Im}\, z| \\ 2|\text{Re}\, z|^2 + 2|\text{Im}\, z|^2 &\geq |\text{Re}\, z|^2 + 2|\text{Re}\, z||\text{Im}\, z| + |\text{Im}\, z|^2\end{aligned}$$

$$\begin{aligned}
2\left(|\mathrm{Re}\,z|^2+|\mathrm{Im}\,z|^2\right) &\geq \left(|\mathrm{Re}\,z|+|\mathrm{Im}\,z|\right)^2\\
2\,|z|^2 &\geq \left(|\mathrm{Re}\,z|+|\mathrm{Im}\,z|\right)^2
\end{aligned}$$

(**c**)

$$\begin{aligned}
0 &\leq (x_1y_2-y_1x_2)^2\\
0 &\leq x_1^2y_2^2-2x_1x_2y_1y_2+y_1^2x_2^2\\
2x_1x_2y_1y_2 &\leq x_1^2y_2^2+y_1^2x_2^2\\
x_1^2x_2^2+2x_1x_2y_1y_2+y_1^2y_2^2 &\leq x_1^2x_2^2+x_1^2y_2^2+y_1^2x_2^2+y_1^2y_2^2\\
x_1^2x_2^2+2x_1x_2y_1y_2+y_1^2y_2^2 &\leq \left(x_1^2+y_1^2\right)\left(x_2^2+y_2^2\right)\\
(x_1x_2+y_1y_2)^2 &\leq \left(\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}\right)^2\\
x_1x_2+y_1y_2 &\leq \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}\\
2x_1x_2+2y_1y_2 &\leq 2\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}\\
x_1^2+2x_1x_2+x_2^2+y_1^2+2y_1y_2+y_2^2 &\leq \left(x_1^2+y_1^2\right)+2\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}+\left(x_2^2+y_2^2\right)\\
(x_1+x_2)^2+(y_1+y_2)^2 &\leq |z_1|^2+2\,|z_1|\,|z_2|+|z_2|^2\\
|z_1+z_2|^2 &\leq (|z_1|+|z_2|)^2\\
|z_1+z_2| &\leq |z_1|+|z_2|
\end{aligned}$$

(**d**) We start from the fifth line of the proof of part (c), that is:

$$\begin{aligned}
x_1^2x_2^2+2x_1x_2y_1y_2+y_1^2y_2^2 &\leq \left(x_1^2+y_1^2\right)\left(x_2^2+y_2^2\right)\\
(-x_1x_2-y_1y_2)^2 &\leq \left(\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}\right)^2\\
-x_1x_2-y_1y_2 &\leq \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}\\
2x_1x_2+2y_1y_2 &\geq -2\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}\\
x_1^2+2x_1x_2+x_2^2+y_1^2+2y_1y_2+y_2^2 &\geq \left(x_1^2+y_1^2\right)-2\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}+\left(x_2^2+y_2^2\right)\\
(x_1+x_2)^2+(y_1+y_2)^2 &\geq |z_1|^2-2\,|z_1|\,|z_2|+|z_2|^2\\
|z_1+z_2|^2 &\geq (|z_1|-|z_2|)^2\\
|z_1+z_2| &\geq |z_1|-|z_2|
\end{aligned}$$

Note: this result may be obtained more simply from the result of part (c), that is:

$$|z_1|=|z_1+z_2-z_2|=|(z_1+z_2)+(-z_2)|\leq|z_1+z_2|+|-z_2|=|z_1+z_2|+|z_2|$$

and hence on subtracting $|z_2|$ from both sides we get $|z_1|-|z_2|\leq|z_1+z_2|$.
(**e**) We start from the seventh line of the proof of part (c), that is:

$$\begin{aligned}
x_1x_2+y_1y_2 &\leq \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}\\
-2x_1x_2-2y_1y_2 &\geq -2\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}\\
x_1^2-2x_1x_2+x_2^2+y_1^2-2y_1y_2+y_2^2 &\geq \left(x_1^2+y_1^2\right)-2\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}+\left(x_2^2+y_2^2\right)
\end{aligned}$$

$$\begin{aligned}(x_1-x_2)^2+(y_1-y_2)^2 &\geq |z_1|^2-2|z_1||z_2|+|z_2|^2\\ |z_1-z_2|^2 &\geq (|z_1|-|z_2|)^2\\ |z_1-z_2| &\geq |z_1|-|z_2|\end{aligned}$$

Note: this result may be obtained more simply from the result of part (c), that is:

$$|z_1|=|z_1-z_2+z_2|=|(z_1-z_2)+z_2|\leq|z_1-z_2|+|z_2|$$

and hence on subtracting $|z_2|$ from both sides we get $|z_1|-|z_2|\leq|z_1-z_2|$.
(**f**) This inequality is just an extension of the inequality of part (c) and hence it can be proved by repetitive application of part (c). For example:

$$\begin{aligned}|z_1+z_2+z_3| &= |z_1+(z_2+z_3)|\\ &\leq |z_1|+|z_2+z_3|\\ &\leq |z_1|+|z_2|+|z_3|\end{aligned}$$

The generalization can be established by induction.

5. Solve the following complex equations, inequalities and non-equality (for $z\in\mathbb{C}$):

(**a**) $2|z|=7$.
(**b**) $|z|^2+\pi=0$.
(**c**) $|z|-3\,\mathrm{Re}(z)=0$.
(**d**) $|z|+\mathrm{Im}(z)=2$.
(**e**) $2\arg(z)-\frac{\pi}{2}=0$.
(**f**) $|z|-4\arg(z)=\pi$.
(**g**) $|z-2-i6|\geq 6$.
(**h**) $0\leq\arg(z)<\frac{\pi}{2}$.
(**i**) $|z+i2|\neq 10$.
(**j**) $0\leq\mathrm{Arg}(z)<\frac{\pi}{2}$.

Answer:
(**a**) We have $2|z|=7$ and hence $|z|=7/2$. So, the solution is the set of complex numbers whose modulus is $7/2$ which is represented by the origin-centered circle with radius $7/2$.
(**b**) We have $|z|^2+\pi=0$ and hence $|z|^2=-\pi$. So, the solution is the set of complex numbers whose modulus squared is $-\pi$ which is the empty set, i.e. there is no solution (noting that by definition the modulus and its square are real non-negative).
(**c**)

$$\begin{aligned}|z|-3\,\mathrm{Re}(z) &= 0\\ \sqrt{x^2+y^2}-3x &= 0\\ \sqrt{x^2+y^2} &= 3x\\ x^2+y^2 &= 9x^2 \qquad\qquad (x\geq 0)\\ y^2 &= 8x^2\\ y &= \pm\sqrt{8}x\end{aligned}$$

So, the solution is the set of complex numbers represented by the two lines $y=\sqrt{8}x$ and $y=-\sqrt{8}x$ (with x being restricted by the condition $x\geq 0$ since the modulus $|z|=\sqrt{x^2+y^2}$ is by definition non-negative as can be seen in line 3 and as indicated in line 4).
(**d**)

$$\begin{aligned}|z|+\mathrm{Im}(z) &= 2\\ \sqrt{x^2+y^2}+y &= 2\\ \sqrt{x^2+y^2} &= 2-y\\ x^2+y^2 &= 4-4y+y^2\\ x^2 &= 4-4y\\ y &= 1-\frac{x^2}{4}\end{aligned}$$

So, the solution is the set of complex numbers represented by the parabola $y = 1 - \frac{x^2}{4}$. We note that although it seems (from line 3 more obviously) that we need to impose the condition $y \le 2$, this condition is already satisfied by the parabola $y = 1 - \frac{x^2}{4}$ whose maximum is $y(0) = 1$. Therefore, we do not need to impose this condition.
(**e**)

$$\begin{aligned} 2\arg(z) - \frac{\pi}{2} &= 0 \\ 2\theta - \frac{\pi}{2} &= 0 \\ \theta &= \frac{\pi}{4} \end{aligned}$$

So, the solution is the set of complex numbers represented by the line $y = x$ (with $x > 0$ since these numbers are in the first quadrant).
(**f**)

$$\begin{aligned} |z| - 4\arg(z) &= \pi \\ r - 4\theta &= \pi \\ r &= 4\theta + \pi \qquad (\theta \ge -\frac{\pi}{4}) \end{aligned}$$

So, the solution is the set of complex numbers represented by the spiral $\rho = 4\phi + \pi$ in polar coordinates (considering a suitable range for ϕ corresponding to the condition $\theta \ge -\frac{\pi}{4}$ which is imposed above by the fact that $r \ge 0$).
(**g**) The equation $|z - 2 - i6| = 6$ represents a circle with center $2 + i6$ and radius 6. Hence, the solution of the semi-inequality is all the numbers in the z plane excluding those inside this circle.
(**h**) The solution of this double inequality is the complex numbers in the first quadrant (of Cartesian complex plane) including the (positive) real axis and excluding the (positive) imaginary axis.
(**i**) The equation $|z + i2| = 10$ represents the circle with center $-i2$ and radius 10. Hence, the solution of this non-equality is the entire set of complex numbers excluding those represented by this circle. We note that the essence of this non-equality may also be expressed (less compactly, comprehensibly and elegantly) by the relation $\big(|z + i2| < 10\big) \cup \big(|z + i2| > 10\big)$.
(**j**) From a Cartesian viewpoint, the solution is the same as the solution of part (h). However, the solution can also be given (from a polar viewpoint) as: $2n\pi \le \arg(z) < (2n\pi + \frac{\pi}{2})$ which represents all the complex numbers whose arguments satisfy this double inequality (which is more extensive than the double inequality of part h).

6. Solve the following systems of simultaneous complex equations (for $z \in \mathbb{C}$):
   (**a**) $|z| = 3$ and $z - \mathrm{Re}(z) + i5\mathrm{Re}(z) = i3$.
   (**b**) $|z| = 1$ and $z + \mathrm{Re}(z) + \mathrm{Im}(z) - i\mathrm{Im}(z) = 0.5$.
   (**c**) $|z| = 5$ and $\arg(z) = \pi/3$.
   (**d**) $\mathrm{Re}(z) = 3$ and $\arg(z) = -\pi/5$.
   Answer:
   (**a**) From the first equation we have $|z| = \sqrt{x^2 + y^2} = 3$ (and hence $x^2 + y^2 = 9$), while from the second equation we have:

$$\begin{aligned} (x + iy) - x + i5x &= i3 \\ i\,(y + 5x) &= i3 \\ y &= 3 - 5x \end{aligned}$$

On substituting from the last equation into $x^2 + y^2 = 9$ (which we obtained earlier) we get:

$$\begin{aligned} x^2 + (3 - 5x)^2 &= 9 \\ x^2 + 9 - 30x + 25x^2 &= 9 \end{aligned}$$

$$\begin{aligned} 26x^2 - 30x &= 0 \\ x(26x - 30) &= 0 \end{aligned}$$

So, either $x = 0$ and hence $y = 3$ or $x = 15/13$ and hence $y = -36/13$. Therefore, we have two solutions: $z = i3$ and $z = \frac{15-i36}{13}$.
(**b**) From the first equation we have $|z| = \sqrt{x^2 + y^2} = 1$ (and hence $x^2 + y^2 = 1$), while from the second equation we have:

$$\begin{aligned} (x + iy) + x + y - iy &= 0.5 \\ 2x + y &= 0.5 \\ y &= 0.5 - 2x \end{aligned}$$

On substituting from the last equation into $x^2 + y^2 = 1$ we get:

$$\begin{aligned} x^2 + (0.5 - 2x)^2 &= 1 \\ x^2 + 0.25 - 2x + 4x^2 &= 1 \\ 5x^2 - 2x &= 0.75 \\ x^2 - 0.4x &= 0.15 \\ (x - 0.2)^2 &= 0.15 + 0.04 \\ x &= 0.2 \pm \sqrt{0.19} \end{aligned}$$

So, either $x = 0.2 + \sqrt{0.19}$ and hence $y = 0.1 - 2\sqrt{0.19}$ or $x = 0.2 - \sqrt{0.19}$ and hence $y = 0.1 + 2\sqrt{0.19}$. Therefore, we have two solutions: $z = \left(0.2 \pm \sqrt{0.19}\right) + i\left(0.1 \mp 2\sqrt{0.19}\right)$.
(**c**) The equation $|z| = 5$ represents the origin-centered circle with radius 5, while the equation $\arg(z) = \pi/3$ represents the line $y = \left(\tan\frac{\pi}{3}\right)x = \sqrt{3}\,x$ (with $x > 0$). So, the solution is the point where these curves meet, i.e. $z = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{5}{2} + i\frac{5\sqrt{3}}{2}$.
(**d**) The equation $\mathrm{Re}(z) = 3$ represents the (vertical) line $x = 3$, while the equation $\arg(z) = -\pi/5$ represents the line $y = -\left(\tan\frac{\pi}{5}\right)x \simeq -0.7265\,x$ (with $x > 0$). So, the solution is the point where these lines meet, i.e. $z = 3 - i3\tan\frac{\pi}{5} \simeq 3 - i2.1796$.

7. What is the effect of the operation of taking the modulus/argument of complex numbers on the complex plane and how it shifts curves and regions in this plane?
Answer: Taking the modulus is equivalent to rotating a complex number $z = re^{i\theta}$ by $-\theta$ and hence projecting or mapping it (rotationally) onto the positive real axis. So, the effect (on the complex plane) of taking the modulus is to fold the upper and lower halves of the complex plane onto the positive real axis (similar to folding a handheld folding fan). Accordingly, lines and regions in the complex plane are projected rotationally onto the positive real axis in such a way that their part in the upper/lower half of the complex plane is rotated clockwise/anticlockwise to be mapped onto its "rotational cross section" on the positive real axis.
Taking the principal argument is equivalent to mapping the value of the argument onto the real axis (whether positive or negative or zero). So, the effect (on the complex plane) of taking the principal argument is to map the complex plane onto the real interval $-\pi < x \leq \pi$.[61] Accordingly, curves and regions in the complex plane are mapped onto this real interval (whether on the entire interval or on part of it depending on the range of the argument which could be a single number as it is the case if the curve for instance is an origin-stemming line segment in the upper or lower half of the complex plane).
Note: if we consider the argument in general (i.e. not restricted to be principal) then the effect of taking the argument is to map the complex plane(s) onto the entire real axis.
8. What are the images (in the w plane) of the following sets of complex numbers (in the z plane) under the effect of taking the modulus:

[61] The use of x here (and in similar instances) is for simplicity and to avoid distraction.

(**a**) $\frac{(\operatorname{Re} z)^2}{4} + \frac{(\operatorname{Im} z)^2}{9} - 1 = 0$. (**b**) $\operatorname{Im} z - 3\operatorname{Re} z - 2 = 0 \quad (0 \le \operatorname{Re} z \le 2)$.

Answer:
(**a**) This set is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Noting that this ellipse is contained between the origin-centered circle of radius 2 and the origin-centered circle of radius 3, it is obvious that the moduli of the points on this ellipse are in the range $2 \le |z| \le 3$. Accordingly, the map is the line segment on the real axis (i.e. the u axis of the w plane) between 2 and 3, i.e. $2 \le u \le 3$.
(**b**) This set is the line segment $y = 3x + 2$ $(0 \le x \le 2)$. Noting that the end points of this segment (i.e. $z_1 = i2$ and $z_2 = 2 + i8$) have moduli $|z_1| = 2$ and $|z_2| = \sqrt{68}$ and the points in-between have moduli between these values, it is obvious that the moduli of the points on this segment are in the range $2 \le |z| \le \sqrt{68}$. Accordingly, the map is the line segment on the real axis (i.e. the u axis of the w plane) between 2 and $\sqrt{68}$, i.e. $2 \le u \le \sqrt{68}$.

9. What are the images (in the w plane) of the following sets of complex numbers (in the z plane) under the effect of taking the (principal) argument:

(**a**) $\operatorname{Im} z = |\operatorname{Re} z|$. (**b**) $\operatorname{Im} z = -(\operatorname{Re} z)^2 + 9 \ \ (-3 \le \operatorname{Re} z \le 3)$.

Answer:
(**a**) This set is the two semi-lines: $y = x$ in the first quadrant and $y = -x$ in the second quadrant. The (principal) argument of all the numbers on the first semi-line is $\theta_p = \pi/4$ and the argument is $\theta = \theta_p + 2n\pi = \pi/4 + 2n\pi$, while the principal argument of all the numbers on the second semi-line is $\theta_p = 3\pi/4$ and the argument is $\theta = \theta_p + 2n\pi = 3\pi/4 + 2n\pi$. Accordingly, the map is these points on the real axis (i.e. the u axis), i.e. the two points $u = \pi/4$ and $u = 3\pi/4$ if we consider the principal argument only or the infinitely-many points $u = \pi/4 + 2n\pi$ and $u = 3\pi/4 + 2n\pi$ if we ignore the "principal" restriction.
(**b**) This set is the parabola $y = -x^2 + 9 \ \ (-3 \le x \le 3)$. It is obvious that the principal argument of all the numbers on this curve is $0 \le \theta_p \le \pi$ and the argument is $2n\pi \le \theta \le (1 + 2n)\,\pi$. Accordingly, the map is the line segment(s) on the real axis (i.e. the u axis), i.e. the line segment $0 \le u \le \pi$ if we consider the principal argument only or the infinitely-many line segments $2n\pi \le u \le (1 + 2n)\,\pi$ if we ignore the "principal" restriction.

10. Given a specific complex number in a particular form, how to determine if its argument is unique or not and if it is principal or not?
Answer: Let z be a complex number and its argument is $\theta \equiv \arg z$ and the principal value of the argument is $\theta_p \equiv \operatorname{Arg} z$. Now, if z is given in polar form (i.e. $z = re^{i\theta}$) then its argument θ is unique (i.e. as it is given) and could be principal (if it is in the range $-\pi < \theta \le \pi$) or not (if it is not in the range $-\pi < \theta \le \pi$). But if z is given in Cartesian form (i.e. $z = x + iy$) then its argument θ is not unique and hence we write $\theta = \theta_p + 2n\pi$ or $\arg z = \operatorname{Arg} z + 2n\pi$ where $2n\pi$ accounts for the periodicity.

11. Let define the argument function as $\arg(z) = \theta_p + 2n\pi = \theta$ (where $z = x + iy$ and $-\pi < \theta_p \le \pi$). Investigate this function and its principal branch from the viewpoint of single-valuedness and multi-valuedness.
Answer: It is obvious that $\arg(z) = \theta_p + 2n\pi$ is multi-valued (and in fact it is infinitely multi-valued) because of $n = 0, \pm 1, \pm 2, \cdots$ (noting that $x = r\cos\theta$ and $y = r\sin\theta$ plus the 2π-periodicity of the trigonometric cosine and sine functions). However, its principal branch $\operatorname{Arg}(z) = \theta_p$ is single-valued although we need (for having a continuous function) to remove the negative real axis (i.e. the branch cut) from its (original) domain (which is the origin-punctured complex plane) to make $\operatorname{Arg}(z)$ continuous on its (reduced) domain (noting that the function has a 2π jump on the negative real axis and it is not defined at the branch point $z = 0$).

1.8.8 Conjugate

As noted earlier, the conjugate of a complex number is a mirror image of that number (as a point in the complex plane) in the real axis.[62] Some of the general aspects of the conjugates of complex numbers have

[62] In fact, this is related to the Cartesian view of complex numbers. So to be more precise, we should say: from the Cartesian perspective the conjugate of a complex number is the mirror image of that number in the real axis, while from

already been discussed or hinted or implicated in practical contexts. In the following points we summarize the main properties and rules of conjugation of complex numbers:
• The conjugate of a complex number in the Cartesian form is obtained by taking the same real part and the negative of the imaginary part. So, if $z = x + iy$ then its conjugate is $z^* = x - iy$.
• The conjugate of a complex number in the polar form is obtained by taking the same modulus and the negative of the argument. So, if $z = re^{i\theta}$ then $z^* = re^{-i\theta}$.
• The conjugate of the conjugate of a complex number is the number itself, that is: $(z^*)^* = z$ (i.e. conjugation undo itself and hence it is an involution).[63] This is obvious because the negative of the negative of a number (i.e. the imaginary part in Cartesian form or the argument in polar form) is the number itself.
• The conjugate of an algebraic sum of complex numbers is the algebraic sum of their conjugates, that is:

$$(z_1 \pm z_2)^* = \big([x_1 \pm x_2] + i[y_1 \pm y_2]\big)^* = [x_1 \pm x_2] - i[y_1 \pm y_2] = (x_1 - iy_1) \pm (x_2 - iy_2) = z_1^* \pm z_2^* \tag{50}$$

This applies (by iteration and induction) to more than two numbers, e.g.

$$(z_1 + z_2 - z_3)^* = (z_1 + \{z_2 - z_3\})^* = z_1^* + (z_2 - z_3)^* = z_1^* + z_2^* - z_3^* \tag{51}$$

• The conjugate of a product of complex numbers is the product of their conjugates, that is:

$$(z_1 \times z_2)^* = (r_1 r_2 e^{i(\theta_1+\theta_2)})^* = r_1 r_2 e^{-i(\theta_1+\theta_2)} = r_1 r_2 e^{-i\theta_1} e^{-i\theta_2} = r_1 e^{-i\theta_1} \times r_2 e^{-i\theta_2} = z_1^* \times z_2^* \tag{52}$$

This applies (by iteration and induction) to more than two numbers, e.g.

$$(z_1 \times z_2 \times z_3)^* = (z_1 \times \{z_2 \times z_3\})^* = z_1^* \times (z_2 \times z_3)^* = z_1^* \times (z_2^* \times z_3^*) = z_1^* \times z_2^* \times z_3^* \tag{53}$$

• The conjugate of a quotient of complex numbers is the quotient of their conjugates, that is:

$$\left(\frac{z_1}{z_2}\right)^* = \left(\frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}\right)^* = \frac{r_1}{r_2} e^{-i(\theta_1-\theta_2)} = \frac{r_1}{r_2} e^{-i\theta_1} e^{i\theta_2} = \frac{r_1 e^{-i\theta_1}}{r_2 e^{-i\theta_2}} = \frac{z_1^*}{z_2^*} \tag{54}$$

• The conjugate of a real number is the number itself, that is $z^* = z$ (where z is real). This is obvious because $z = x + i0 = x - i0 = z^*$.
• The sum of a number and its conjugate is equal to twice its real part, that is:

$$z + z^* = (x + iy) + (x - iy) = 2x \qquad \text{and hence} \qquad \text{Re}(z) = \frac{z + z^*}{2} \tag{55}$$

where we divided the first equation by 2 to obtain the second equation $\big[$noting that $x \equiv \text{Re}(z)\big]$.
• The difference of a number and its conjugate is equal to i times twice its imaginary part, that is:

$$z - z^* = (x + iy) - (x - iy) = i2y \qquad \text{and hence} \qquad \text{Im}(z) = \frac{z - z^*}{i2} \tag{56}$$

where we divided the first equation by $i2$ to obtain the second equation $\big[$noting that $y \equiv \text{Im}(z)\big]$.
• The product of a number and its conjugate is equal to the square of its modulus, that is:

$$z \times z^* = re^{i\theta} \times re^{-i\theta} = r^2 e^{i(\theta-\theta)} = r^2 \qquad \text{(polar)} \tag{57}$$

$$\text{or} \qquad z \times z^* = (x + iy) \times (x - iy) = x^2 - ixy + iyx - i^2 y^2 = x^2 + y^2 = |z|^2 \qquad \text{(Cartesian)} \tag{58}$$

• The quotient of a number by its conjugate is unity (with twice the argument of the number), that is:

$$\frac{z}{z^*} = \frac{re^{i\theta}}{re^{-i\theta}} = \frac{e^{i\theta}}{e^{-i\theta}} = e^{i\theta} e^{i\theta} = e^{i2\theta} \qquad \text{(polar)} \tag{59}$$

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the polar perspective it is the number with the same modulus and opposite argument.
[63] "Involution" in mathematics means an operation or a function that is its own inverse and hence it undo itself such as multiplication by -1 and reciprocation (see § 1.8.9).

$$\text{or} \qquad \frac{z}{z^*} = \frac{x+iy}{x-iy} = \frac{(x+iy)\times(x+iy)}{(x-iy)\times(x+iy)} = \frac{(x+iy)^2}{x^2+y^2} \qquad \text{(Cartesian)} \qquad (60)$$

• In general, the conjugate of any complex expression can be easily obtained by replacing each i (whether explicit or implicit) by $-i$.[64] So, to obtain the conjugate of w^{-iz} for example (assuming that $z = x+iy$ and $w = u+iv$) we need[65] first to write this expression in explicit form to show all the i's in the expression, i.e. $(u+iv)^{-i(x+iy)}$. We then replace every i in this explicit form by $-i$ to obtain the conjugate of w^{-iz}, i.e.

$$\left(w^{-iz}\right)^* = \left[(u+iv)^{-i(x+iy)}\right]^* = (u-iv)^{i(x-iy)} = (w^*)^{iz^*} \qquad (61)$$

The justification of this rule is the "bypassing" nature of the conjugation operation (as represented by the above rules such as: the conjugate of sum is the sum of conjugates, the conjugate of product is the product of conjugates, etc.) which makes any complex expression permeable and transparent to conjugation, and this enables conjugation to propagate and permeate inside any complex expression reaching every i and converting it to $-i$.[66]

Problems

1. Give some of the properties and rules of the conjugate z^* of a complex number z.
   Answer: We can say for example:
   • z^* has the same real part as z but opposite imaginary part (i.e. $z^* = x - iy$ if $z = x + iy$).
   • z^* has the same modulus as z but opposite argument (i.e. $z^* = re^{-i\theta}$ if $z = re^{i\theta}$).
   • z^* is a mirror reflection of z in the real axis (with the real axis being mapped on itself, i.e. each real number is mapped by conjugation on itself).
   • The product z^*z is a real number which equals the square of the modulus of z^* (as well as the square of the modulus of z).
   • The conjugate of a conjugate of a number is the number itself, i.e. $(z^*)^* = z$.
   • The conjugate of a real number is the number itself (i.e. $z^* = z$ *iff* z is real).
2. Verify the following relations using the Cartesian form of complex numbers:
   (**a**) $|z^*| = |z|$. (**b**) $\operatorname{Arg} z^* = -\operatorname{Arg} z$. (**c**) $(z_1z_2)^* = z_1^*z_2^*$. (**d**) $(z_1/z_2)^* = z_1^*/z_2^*$.
   Answer:
   (**a**) We have:
   $$|z^*| = |x-iy| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2} = |x+iy| = |z|$$
   (**b**) We have:
   $$\operatorname{Arg} z^* = \arctan\left(\frac{-y}{x}\right) = -\arctan\left(\frac{y}{x}\right) = -\operatorname{Arg} z$$
   (**c**) From Eq. 21 we have:
   $$(z_1z_2)^* = (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1) = (x_1 - iy_1)(x_2 - iy_2) = z_1^*z_2^*$$
   (**d**) From Eq. 23 we have:
   $$\left(\frac{z_1}{z_2}\right)^* = \frac{(x_1x_2+y_1y_2) - i\,(x_2y_1 - x_1y_2)}{x_2^2+y_2^2} = \frac{(x_1-iy_1)(x_2+iy_2)}{(x_2-iy_2)(x_2+iy_2)} = \frac{x_1-iy_1}{x_2-iy_2} = \frac{z_1^*}{z_2^*}$$
3. The conjugation of a complex number is achieved by negating its imaginary part (in Cartesian form) and by negating its argument (in polar form). Show that these two operations (or negations) are equivalent.

[64] In other words, we need to replace each number and variable in that expression by its conjugate.
[65] This "need" is for demonstration.
[66] In other words, reaching every number and variable in that expression and converting it to its conjugate (noting that the conjugate of a real number is the number itself).

Answer: The answer of this Problem is provided by the answer (of parts a and b) of Problem 2 because:

$$z^* = x - iy = |x - iy|\, e^{i\arctan(-y/x)} = |z^*|\, e^{i\mathrm{Arg}\, z^*} = |z|\, e^{-i\mathrm{Arg}\, z} = re^{-i\mathrm{Arg}\, z} = z^*$$

where reading this equation from left to right means starting from the Cartesian negation and ending with the polar negation, while reading it from right to left means starting from the polar negation and ending with the Cartesian negation.

4. Solve the following complex equations (for $z \in \mathbb{C}$):

(**a**) $z + z^* = 0$. (**b**) $z - z^* = 0$. (**c**) $4z + 2z^* = 30 - i4$.

(**d**) $z = 5/z^*$. (**e**) $z - z^* = i8$.

Answer:

(**a**)

$$\begin{aligned} (x + iy) + (x - iy) &= 0 \\ 2x &= 0 \\ x &= 0 \end{aligned}$$

So, the solution is the set of imaginary numbers which is represented by the imaginary axis.

(**b**)

$$\begin{aligned} (x + iy) - (x - iy) &= 0 \\ i2y &= 0 \\ y &= 0 \end{aligned}$$

So, the solution is the set of real numbers which is represented by the real axis.

(**c**)

$$\begin{aligned} 4(x + iy) + 2(x - iy) &= 30 - i4 \\ 6x + i2y &= 30 - i4 \end{aligned}$$

On equating the real and imaginary parts on the two sides we get $6x = 30$ (and hence $x = 5$) and $2y = -4$ (and hence $y = -2$). So, the solution is $z = 5 - i2$ which is represented by the point $(5, -2)$ in the Cartesian plane.

(**d**)

$$\begin{aligned} zz^* &= 5 \\ (x + iy)(x - iy) &= 5 \\ x^2 + y^2 &= 5 \\ |z|^2 &= 5 \\ |z| &= \sqrt{5} \end{aligned}$$

So, the solution is the set of complex numbers whose modulus is $\sqrt{5}$ which is represented by the origin-centered circle with radius $\sqrt{5}$.

(**e**)

$$\begin{aligned} (x + iy) - (x - iy) &= i8 \\ i2y &= i8 \\ y &= 4 \end{aligned}$$

So, the solution is the set of complex numbers whose imaginary part is 4 which is represented by the horizontal line $y = 4$.

5. Solve the following systems of simultaneous complex equations (for $z \in \mathbb{C}$):

(**a**) $z + z^* = 2.3$ and $3z - 3z^* = i6$.

(**b**) $z - 2z^* = -1 - i15$ and $9z + z^* = 10 - i40$.

(**c**) $2\,\text{Im}(z) - 3(z + z^*) = 2$ and $z - z^* - i6\text{Re}(z) - i2 = 0$.
(**d**) $3z + z^* - 5\text{Re}(z) = -3 + i2$ and $2z^* - 7z + i\,\text{Im}(z) = -15 - i8$.
(**e**) $z - i3z - z^* - i3z^* + i4 = 0$ and $2z - 2z^* - i12\text{Re}\,z^* + i20 = 0$.
Answer:
(**a**) On adding 3 times the first equation to the second equation side by side we get:

$$\begin{aligned} 3\,(z + z^*) + (3z - 3z^*) &= 3(2.3) + i6 \\ 6z &= 6.9 + i6 \\ z &= 1.15 + i \end{aligned}$$

So, we have only one solution, i.e. $z = 1.15 + i$.
(**b**) On adding twice the second equation to the first equation side by side we get:

$$\begin{aligned} z - 2z^* + 2\,(9z + z^*) &= -1 - i15 + 2\,(10 - i40) \\ 19z &= 19 - i95 \\ z &= 1 - i5 \end{aligned}$$

So, we have only one solution, i.e. $z = 1 - i5$.
(**c**) From the first equation we get $2y - 3(2x) = 2$ and hence $y = 3x + 1$. Also, from the second equation we get $i2y - i6x - i2 = 0$ and hence $y = 3x + 1$ again. So, we have infinite number of solutions, i.e. the set of complex numbers $z = x + i(3x + 1)$ represented by the line $y = 3x + 1$.
(**d**) From the first equation we get:

$$\begin{aligned} 3(x + iy) + (x - iy) - 5x &= -3 + i2 \\ -x + i2y &= -3 + i2 \end{aligned}$$

and hence $x = 3$ and $y = 1$. Similarly, from the second equation we get:

$$\begin{aligned} 2(x - iy) - 7(x + iy) + iy &= -15 - i8 \\ -5x - i8y &= -15 - i8 \end{aligned}$$

and hence $x = 3$ and $y = 1$ again. So, the two equations are consistent and hence the solution is $z = 3 + i$.
(**e**) From the first equation we get:

$$\begin{aligned} (z - z^*) - i3(z + z^*) &= -i4 \\ i2y - i6x &= -i4 \\ y &= 3x - 2 \end{aligned}$$

Similarly, from the second equation we get:

$$\begin{aligned} (2z - 2z^*) - i12\text{Re}\,z^* &= -i20 \\ i4y - i12x &= -i20 \\ y &= 3x - 5 \end{aligned}$$

These equations represent two parallel lines and hence there is no solution.

6. Express the following expressions in terms of $|z_1|$ and $|z_2|$:
(**a**) $|z_1 - z_2|^2 + z_1 z_2^* + z_1^* z_2$. (**b**) $|z_1 + z_2|^2 + |z_1 - z_2|^2$.
Answer:
(**a**)

$$|z_1 - z_2|^2 + z_1 z_2^* + z_1^* z_2 = (z_1 - z_2)(z_1 - z_2)^* + z_1 z_2^* + z_1^* z_2$$

$$\begin{aligned} &= (z_1 - z_2)(z_1^* - z_2^*) + z_1 z_2^* + z_1^* z_2 \\ &= z_1 z_1^* - z_1 z_2^* - z_2 z_1^* + z_2 z_2^* + z_1 z_2^* + z_1^* z_2 \\ &= z_1 z_1^* + z_2 z_2^* \\ &= |z_1|^2 + |z_2|^2 \end{aligned}$$

(**b**)

$$\begin{aligned} |z_1 + z_2|^2 + |z_1 - z_2|^2 &= (z_1 + z_2)(z_1 + z_2)^* + (z_1 - z_2)(z_1 - z_2)^* \\ &= (z_1 + z_2)(z_1^* + z_2^*) + (z_1 - z_2)(z_1^* - z_2^*) \\ &= z_1 z_1^* + z_1 z_2^* + z_2 z_1^* + z_2 z_2^* + z_1 z_1^* - z_1 z_2^* - z_2 z_1^* + z_2 z_2^* \\ &= 2 z_1 z_1^* + 2 z_2 z_2^* \\ &= 2|z_1|^2 + 2|z_2|^2 \end{aligned}$$

7. Given that z is a strictly complex number (i.e. not real or imaginary) in the first quadrant, represent graphically z, z^*, $-z$ and $-z^*$ as position vectors in Cartesian coordinates and comment.
Answer: See Figure 8.
Comment: z^* is a mirror reflection of z in the x axis. $-z^*$ is a mirror reflection of z in the y axis. $-z$ is a mirror reflection of $-z^*$ in the x axis. $-z$ is a mirror reflection of z^* in the y axis. $-z$ is a reflection of z in the origin (or a rotation of z by $2n\pi + \pi$ around the origin). $-z^*$ is a reflection of z^* in the origin (or a rotation of z^* by $2n\pi + \pi$ around the origin).

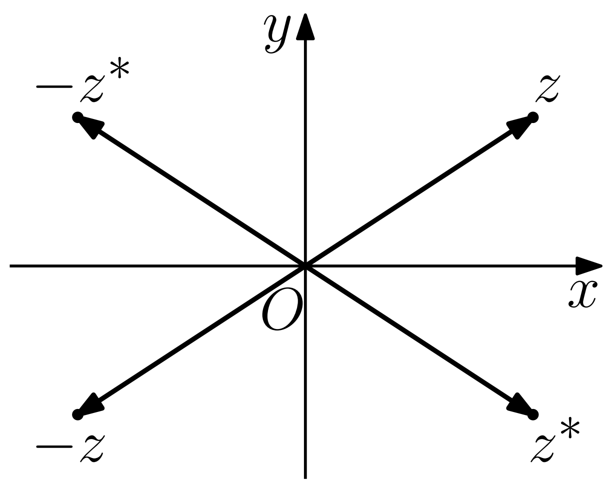

Figure 8: Graphic representation of z, z^*, $-z$ and $-z^*$ as position vectors in Cartesian coordinates. We note that if z has coordinates (X, Y) then z^*, $-z$ and $-z^*$ have coordinates $(X, -Y)$, $(-X, -Y)$ and $(-X, Y)$ respectively. Also see the comment of Problem 7 of § 1.8.8.

8. Re-solve Problem 2 of § 1.8.6 using this time the polar form of z, i.e. $z = re^{i\theta}$.
Answer: The operations here are essentially the same as those in Problem 2 of § 1.8.6 but the operators and symbolism are different in general. So, in the following we show the initial form and the final result.
(**a**) $re^{i\theta} \to -re^{i\theta} = re^{i(\theta+\pi)}$.
(**b**) $re^{i\theta} \to -re^{-i\theta} = re^{i(-\theta+\pi)}$.
(**c**) $re^{i\theta} \to re^{-i\theta}$.
(**d**) $re^{i\theta} \to ire^{-i\theta} = re^{i(-\theta+\pi/2)}$.
(**e**) $re^{i\theta} \to -ire^{-i\theta} = re^{i(-\theta-\pi/2)}$.
(**f**) $re^{i\theta} \to \frac{re^{i\theta} - re^{-i\theta}}{2} = i\,r\frac{e^{i\theta} - e^{-i\theta}}{i2} = i\,r\sin\theta$.

(**g**) $re^{i\theta} \to \frac{re^{i\theta}+re^{-i\theta}}{2} = r\frac{e^{i\theta}+e^{-i\theta}}{2} = r\cos\theta$.

9. Make a graphic illustration of the effect of conjugation of complex numbers on the complex plane and how it shifts lines and regions in this plane.
Answer: See Figure 9. As we see, conjugation mirror-reflects the upper half of the complex plane onto the lower half and vice versa and maps the real axis on itself (i.e. each number on the real axis is mapped on itself).

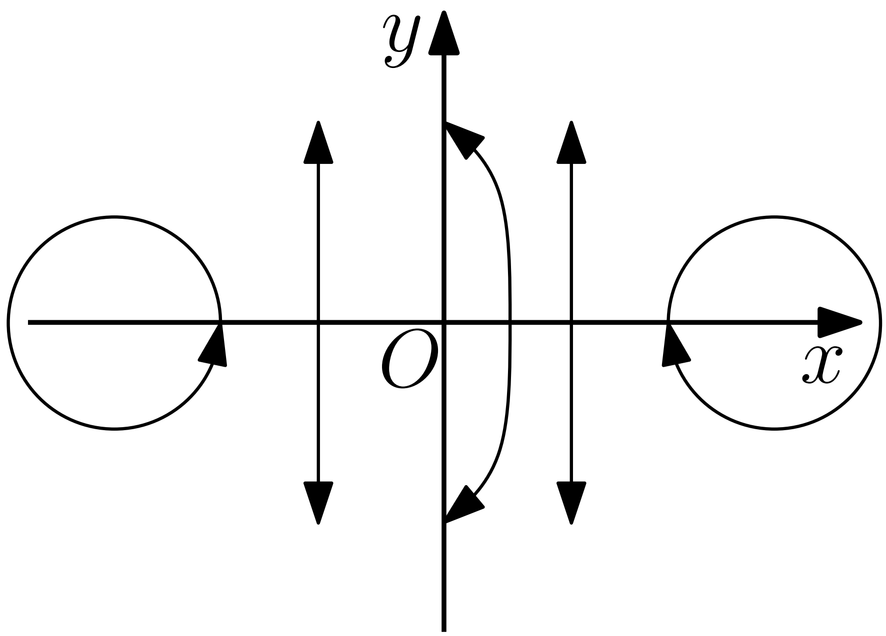

Figure 9: Graphic illustration of the effect of conjugation of complex numbers on the complex plane and how it shifts lines and regions in this plane. The double-arrow lines and curves indicate which lines and regions are exchanged by the action of conjugation. See Problem 9 of § 1.8.8.

10. Identify the basic mathematical operation(s) that achieves the conjugation of a complex number $z = re^{i\theta}$.
Answer: It is the multiplication by $e^{-i2\theta}$ because $ze^{-i2\theta} = re^{i\theta}e^{-i2\theta} = re^{i(\theta-2\theta)} = re^{-i\theta}$ which is the conjugate of z.
Note: if we consider the Cartesian form of the complex number (i.e. $z = x+iy$) then the mathematical operation that achieves conjugation is subtracting z from twice its real part, i.e. $z^* = 2\mathrm{Re}(z) - z$. However, considering the operation of taking the real part more fundamental than the conjugation itself is questionable. Accordingly, the conjugation of a complex number in Cartesian form may not be achievable by more basic mathematical operations than the conjugation itself.[67]

1.8.9 Reciprocal

If z is a complex number (not equal zero) then its reciprocal is:

$$z^{-1} = \frac{1}{z} = \frac{z^*}{zz^*} = \frac{z^*}{|z|^2} \tag{62}$$

i.e. the reciprocal is obtained by dividing the conjugate of the number by its modulus squared. Accordingly, in the Cartesian form (where $z = x + iy \neq 0$) we have $z^{-1} = \frac{x-iy}{x^2+y^2}$, while in the polar form

[67] In fact, even considering the multiplication by $e^{-i2\theta}$ as more basic than the conjugation itself may be questioned.

(where $z = re^{i\theta} \neq 0$) we have $z^{-1} = \frac{re^{-i\theta}}{r^2} = r^{-1}e^{-i\theta}$ (i.e. by taking the reciprocal of the modulus and the negative of the argument). In the following points we outline some of the observations about the reciprocals of complex numbers (as well as the reciprocals of the subsets of complex numbers, i.e. the real and imaginary numbers):

• The reciprocals of the real numbers ($\neq 0$) are real numbers and they have the same sign as the reciprocated numbers. Hence, the numbers on the positive/negative real axis remain on the positive/negative real axis after reciprocation (i.e. taking reciprocal or inversion). Regarding the magnitude, those with larger magnitude than 1 shrink to become less than 1 in magnitude, and those with smaller magnitude than 1 expands to become greater than 1 in magnitude, while ± 1 remain where they are since they are their own reciprocals.

• The reciprocals of the imaginary numbers ($\neq 0$) are imaginary numbers and they have an opposite sign to the sign of the reciprocated numbers. Hence, the numbers on the positive/negative imaginary axis become on the negative/positive imaginary axis after reciprocation. The reversal of sign is due to the fact that $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$. Regarding the magnitude, it follows the same rules as those of real numbers (as explained in the previous point noting that $\pm i$ exchange their positions since they are the reciprocals of each other).

• The reciprocals of the complex numbers in the first/fourth quadrant are in the fourth/first quadrant, while the reciprocals of the complex numbers in the second/third quadrant are in the third/second quadrant. This is because the argument of the reciprocal is the negative of the argument of the reciprocated number (see Eq. 62 and the related text). Regarding the magnitude, it follows the same rules as those of real numbers (as explained earlier).

• Every complex number has a reciprocal except zero. This is because the function $1/z$ (which is the "reciprocal function") is defined over the entire z plane excluding the origin (which corresponds to zero).

• The reciprocal of any complex number (excluding zero) is unique because $1/z$ is a one-to-one function.

• The reciprocal of the reciprocal of a complex number is the number itself, that is: $\left(z^{-1}\right)^{-1} = z$ (i.e. reciprocation undo itself and hence it is an involutory operation).

Problems

1. Find the reciprocals of the following numbers:

   (**a**) $z = \pi - i\pi$. (**b**) $z = 5e^{-\pi/2}$. (**c**) $z = \sqrt{17}e^{i\pi/5}$. (**d**) $z = i3e^4$.

   Answer:

   (**a**) $$\frac{1}{z} = \frac{1}{\pi - i\pi} = \frac{\pi + i\pi}{\pi^2 + \pi^2} = \frac{\pi + i\pi}{2\pi^2} = \frac{1+i}{2\pi} \simeq 0.1592 + i0.1592$$

   (**b**) $$\frac{1}{z} = \frac{1}{5e^{-\pi/2}} = \frac{e^{\pi/2}}{5} \simeq 0.9621$$

   (**c**) $$\frac{1}{z} = \frac{1}{\sqrt{17}e^{i\pi/5}} = \frac{e^{-i\pi/5}}{\sqrt{17}} = \frac{1}{\sqrt{17}}\left(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right) \simeq 0.1962 - i0.1426$$

   (**d**) $$\frac{1}{z} = \frac{1}{i3e^4} = -\frac{i}{3e^4} \simeq -i0.006105$$

2. Make a graphic illustration of the effect of reciprocation of complex numbers on the complex plane and how it shifts lines, curves and regions in this plane.

   Answer: See Figure 10.

3. Identify a curve in the z plane on which conjugation and reciprocation have the same effect, i.e. $z^* = z^{-1}$.

   Answer: It is the origin-centered unit circle $e^{i\theta}$ $(-\pi < \theta \leq \pi)$ because:

   $$z^* = \left(e^{i\theta}\right)^* = e^{-i\theta} = \left(e^{i\theta}\right)^{-1} = z^{-1}$$

4. What is the effect of reciprocation on the following curves and lines:

   (**a**) Origin-centered circles. (**b**) Straight lines passing through the (punctured) origin.

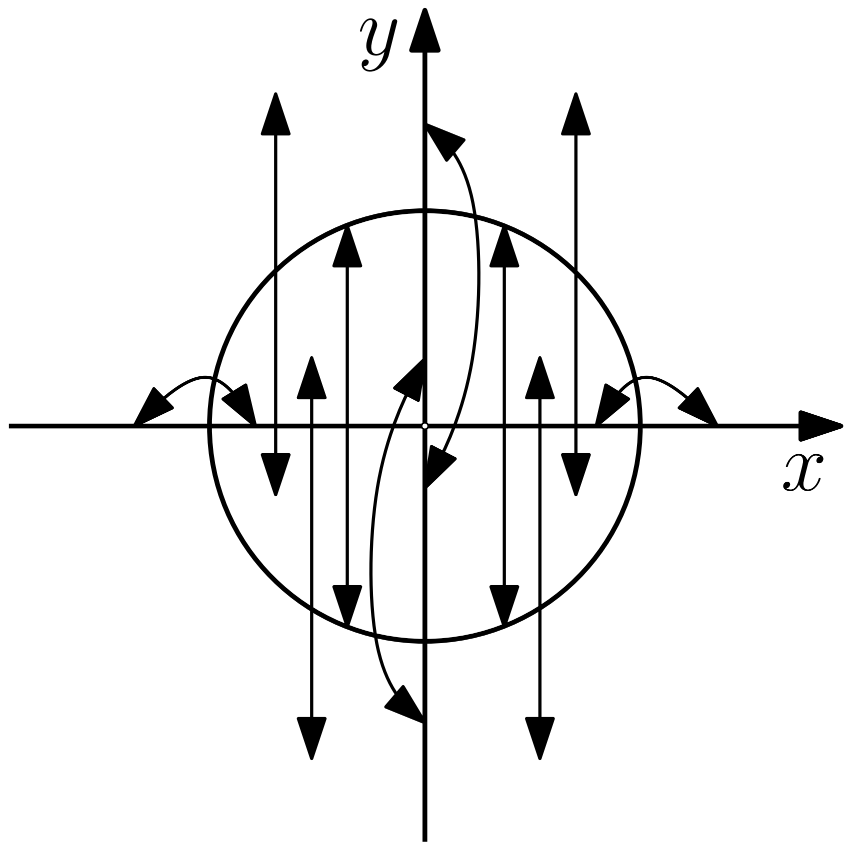

Figure 10: Graphic illustration of the effect of reciprocation of complex numbers on the complex plane and how it shifts lines, curves and regions in this plane. The circle is the origin-centered unit circle. The double-arrow lines and curves indicate which lines, curves and regions are exchanged by the action of reciprocation. The puncture at the origin indicates the exclusion of zero. See Problem 2 of § 1.8.9.

Answer:
(**a**) Referring to Eq. 62, we can see that the effect of reciprocation on a complex number z is a combination of conjugation and scaling (up or down) by the reciprocal of its modulus squared $1/|z|^2$. As we know, the effect of conjugation is a reflection in the real axis and hence it maps an origin-centered circle C onto an origin-centered circle C^* of the same radius. Now, since all the points on an origin-centered circle have the same modulus, scaling C^* by $1/|z|^2$ means that all the points on C^* are scaled by the same factor, i.e. the scaling will keep the circular shape of C^*. Finally, the scaling by $1/|z|^2$ means that the circles with radius smaller/greater than 1 will be scaled up/down to become with radius greater/smaller than 1 while the unit circle will keep its size. So to sum up, the effect of reciprocation on origin-centered circles is to map the origin-centered circles inside/outside the origin-centered unit circle onto the origin-centered circles outside/inside the origin-centered unit circle while mapping the origin-centered unit circle onto itself (with just reflecting it in the real axis).[68] Also, see Problems 4 and 6 of § 6.2.
(**b**) Referring again to Eq. 62, we can see that a line $y = ax$ ($a \in \mathbb{R}$, $x \neq 0$) will be mapped by conjugation onto a line $y = -ax$ (noting that this sort of mapping also applies to the real and imaginary axes as special cases). Each point on the line $y = -ax$ is then scaled by $1/|z|^2$ and hence it remains

[68] We note that the above explanation and justification may be clearer from some aspects if we use the polar form of reciprocal (i.e. $z^{-1} = r^{-1}e^{-i\theta}$) since the conjugation is clearly indicated by the reversal of the sign of θ while the shrinking/expanding to inside/outside the unit circle is clearly indicated by taking the reciprocal of r.

after scaling on this line (since $1/|z|^2$ is real). Accordingly, the semi-line $y = ax$ in quadrant 1,2,3,4 is mapped onto the semi-line $y = -ax$ in quadrant 4,3,2,1 (respectively) where the part of $y = ax$ inside/outside/on the origin-centered unit circle is mapped onto the part of $y = -ax$ outside/inside/on the origin-centered unit circle. As indicated earlier, this type of mapping also applies to the real and imaginary axes with some minor changes in the above phrasing due to their status as special cases (see the text about the reciprocals of real and imaginary numbers as well as Figure 10). Also, see Problems 4 and 6 of § 6.2.

1.8.10 Powers of Complex Numbers

Exponentiation (or raising to power) of complex numbers can be seen as a function $f(z, p) = z^p$ that maps a pair of complex numbers (i.e. z and p) onto another complex number (i.e. z^p). In fact, this is just a generalization of exponentiation of real numbers (i.e. x^y where $x, y \in \mathbb{R}$) which in its turn is a generalization of multiplication (i.e. when y is a positive integer).[69] Exponentiation of complex numbers is easier to deal with and manipulate in polar form where raising z to power p is done by raising the modulus to power p and multiplying the argument by p, that is:

$$z^p = \left(re^{i\theta}\right)^p = r^p e^{ip\theta} \tag{63}$$

where p is complex in general. Most of the properties of exponentiation in real numbers also apply to complex numbers. For example:

$$z^0 = 1 \qquad (z \neq 0) \tag{64}$$

$$(z_1 z_2)^p = z_1^p z_2^p \tag{65}$$

$$\left(\frac{z_1}{z_2}\right)^p = \frac{z_1^p}{z_2^p} \tag{66}$$

$$\frac{1}{z^p} = z^{-p} \tag{67}$$

$$z^{p_1+p_2} = z^{p_1} z^{p_2} \tag{68}$$

where z, z_1, z_2, p, p_1, p_2 are complex numbers (in general) and the denominators must not vanish. In this context, it is important to be aware of De Moivre's formula which is given by:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \tag{69}$$

where θ is real and n is an integer (noting that there are generalizations of De Moivre's formula to relax these conditions). This formula (which will be verified in Problem 3) is very useful by itself (as a trigonometric identity) as well as in reducing the required algebra in manipulating the powers[70] of complex numbers in polar form. Some of the applications of this formula will be investigated in the Problems.

Problems

1. Is the exponentiation function $f(z, p) = z^p$ single-valued or multi-valued?
   Answer: This issue is investigated (rather thoroughly and with regard to the distinction of values considering the nature of the exponent) in Problem 7 of § 2.2 (where it is postponed to § 2.2 due to its dependency on the exponential and natural logarithm functions which are investigated there). However, from a different perspective we may differentiate between the Cartesian representation of z and p and the polar representation of z and p where z^p should be multi-valued (in general sense regardless of the distinction of values) according to the former and single-valued according to the latter. The reason is that if z and p are given by a definite polar representation then both their modulus and argument

[69] Those who are interested in the technicalities of these generalizations should refer to the literature.

[70] "Power" here (and in similar contexts) means the entire exponentiation outcome (i.e. z^p) although it originally means the index (i.e. p). The context should be sufficient to identify the meaning and remove any potential ambiguity.

are definite and hence it should be single-valued. Yes, the Cartesian representation corresponds to infinitely-many polar representations (where the arguments of these polar representations differ by $2n\pi$) and hence it should be multi-valued (i.e. in general).

2. Find the following powers in Cartesian form:[71]
   (**a**) $(4e^{-i\pi/3})^4$. (**b**) $(1+i)^3$. (**c**) $(1-i\sqrt{3})^6$. (**d**) $(7-i\sqrt{3})^{1.3}$.
   (**e**) 8^{2-i}. (**f**) $(5-i12)^i$. (**g**) $(\sqrt{3}+i\sqrt{3})^{6+i2}$.
   Answer: We convert the base to its polar form (if it is given in Cartesian form) before conducting the operation.[72] This is followed by converting the result to its final Cartesian form (if required).
   (**a**)

$$\begin{aligned}\left(4e^{-i\pi/3}\right)^4 &= 4^4e^{-i4\pi/3} = 256\left(\cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}\right) = 256\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -128 + i128\sqrt{3} \\ &\simeq -128 + i221.7025\end{aligned}$$

(**b**)

$$(1+i)^3 = \left(\sqrt{2}e^{i\pi/4}\right)^3 = 2^{3/2}e^{i3\pi/4} = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = -2 + i2$$

(**c**)

$$\left(1 - i\sqrt{3}\right)^6 = \left(2e^{-i\pi/3}\right)^6 = 2^6e^{-i2\pi} = 64\left(\cos 2\pi - i\sin 2\pi\right) = 64\left(1 - i0\right) = 64$$

(**d**)

$$\begin{aligned}\left(7 - i\sqrt{3}\right)^{1.3} &\simeq \left(\sqrt{52}e^{-i0.2426}\right)^{1.3} \simeq 13.0437e^{-i0.3153} = 13.0437\left(\cos 0.3153 - i\sin 0.3153\right) \\ &\simeq 12.4006 - i4.0453\end{aligned}$$

(**e**)

$$\begin{aligned}8^{2-i} &= 8^2 \times 8^{-i} = 64 \times \left(e^{\log_e 8}\right)^{-i} = 64e^{-i\log_e 8} = 64\big[\cos\left(\log_e 8\right) - i\sin\left(\log_e 8\right)\big] \\ &\simeq -31.1676 - i55.8979\end{aligned}$$

(**f**)

$$\begin{aligned}(5 - i12)^i &\simeq \left(13e^{-i1.1760}\right)^i = 13^ie^{1.1760} = e^{1.1760}(e^{\log_e 13})^i = e^{1.1760}e^{i\log_e 13} \\ &= e^{1.1760}\big[\cos\left(\log_e 13\right) + i\sin\left(\log_e 13\right)\big] \simeq -2.7173 + i1.7673\end{aligned}$$

(**g**)

$$\begin{aligned}\left(\sqrt{3} + i\sqrt{3}\right)^{6+i2} &= \left(\sqrt{6}e^{i\pi/4}\right)^{6+i2} = \left(\sqrt{6}e^{i\pi/4}\right)^6\left(\sqrt{6}e^{i\pi/4}\right)^{i2} = \left(216e^{i3\pi/2}\right)\left(6^ie^{-\pi/2}\right) \\ &= \left(216e^{-\pi/2}\right)\left(e^{i3\pi/2}6^i\right) = \left(216e^{-\pi/2}\right)\left(e^{i3\pi/2}e^{i\log_e 6}\right) = 216e^{-\pi/2}e^{i(3\pi/2+\log_e 6)} \\ &\simeq 44.9020e^{i6.5041} = 44.9020\left(\cos 6.5041 + i\sin 6.5041\right) \simeq 43.8103 + i9.8411\end{aligned}$$

3. Verify De Moivre's formula.
   Answer: We have:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{(see Eq. 8)}$$

[71] Calling some of these "powers" may not be conventional.
[72] In fact, we convert the base to its *principal* polar form so that we get a unique solution in all cases (see Problem 1).

$$\left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n$$
$$e^{in\theta} = (\cos\theta + i\sin\theta)^n$$
$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n \qquad \text{(see Eq. 8)}$$

Note: although n suggests an integer power (and possibly even a positive integer which could be based on historical reasons), the formula should be general according to the above verification. In fact, the generalization should also apply to θ.

4. Explain how De Moivre's formula can be used to obtain trigonometric expressions for $\cos n\theta$ and $\sin n\theta$ in terms of $\cos\theta$ and $\sin\theta$ (considering, for simplicity, only integer $n > 1$ and real θ).
Answer: If we expand the left hand side of Eq. 69 using the binomial theorem and equate its real part to $\cos n\theta$ (which is the real part on the right hand side of Eq. 69) and equate its imaginary part to $\sin n\theta$ (which is the imaginary part on the right hand side of Eq. 69) then we obtain these trigonometric expressions. For example:

$$\begin{aligned}
(\cos\theta + i\sin\theta)^2 &= \cos 2\theta + i\sin 2\theta \\
\cos^2\theta + i2\cos\theta\sin\theta - \sin^2\theta &= \cos 2\theta + i\sin 2\theta \\
\left(\cos^2\theta - \sin^2\theta\right) + i2\cos\theta\sin\theta &= \cos 2\theta + i\sin 2\theta
\end{aligned}$$

Hence, $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\cos\theta\sin\theta$. Another example:

$$\begin{aligned}
(\cos\theta + i\sin\theta)^4 &= \cos 4\theta + i\sin 4\theta \\
\cos^4\theta + 4\cos^3\theta\ i\sin\theta + 6\cos^2\theta\ i^2\sin^2\theta + 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta &= \cos 4\theta + i\sin 4\theta \\
\left(\cos^4\theta - 6\cos^2\theta\ \sin^2\theta + \sin^4\theta\right) + i\left(4\cos^3\theta\ \sin\theta - 4\cos\theta\ \sin^3\theta\right) &= \cos 4\theta + i\sin 4\theta
\end{aligned}$$

Hence, $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\ \sin^2\theta + \sin^4\theta$ and $\sin 4\theta = 4\cos^3\theta\ \sin\theta - 4\cos\theta\ \sin^3\theta$.

5. Verify the properties of exponentiation which are given by Eqs. 64-68.
Answer:[73] These properties are generally based on the corresponding properties in real numbers (as well as the properties of i and $e^{i\theta}$), that is:

$$\begin{aligned}
z^0 &= \left(re^{i\theta}\right)^0 = r^0 e^{i0} = r^0 e^0 = 1\times 1 = 1 \\
(z_1 z_2)^p &= \left(r_1 e^{i\theta_1} r_2 e^{i\theta_2}\right)^p = r_1^p e^{ip\theta_1} r_2^p e^{ip\theta_2} = \left(r_1^p e^{ip\theta_1}\right)\left(r_2^p e^{ip\theta_2}\right) = \left(r_1 e^{i\theta_1}\right)^p \left(r_2 e^{i\theta_2}\right)^p = z_1^p z_2^p \\
\left(\frac{z_1}{z_2}\right)^p &= \left(\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}\right)^p = \frac{r_1^p e^{ip\theta_1}}{r_2^p e^{ip\theta_2}} = \frac{\left(r_1 e^{i\theta_1}\right)^p}{\left(r_2 e^{i\theta_2}\right)^p} = \frac{z_1^p}{z_2^p} \\
\frac{1}{z^p} &= \frac{1}{\left(re^{i\theta}\right)^p} = \frac{1}{r^p e^{ip\theta}} = r^{-p} e^{-ip\theta} = \left(re^{i\theta}\right)^{-p} = z^{-p} \\
z^{p_1+p_2} &= \left(re^{i\theta}\right)^{p_1+p_2} = r^{p_1+p_2} e^{i\theta(p_1+p_2)} = r^{p_1} r^{p_2} e^{ip_1\theta} e^{ip_2\theta} = \left(r^{p_1} e^{ip_1\theta}\right)\left(r^{p_2} e^{ip_2\theta}\right) \\
&= \left(re^{i\theta}\right)^{p_1}\left(re^{i\theta}\right)^{p_2} = z^{p_1} z^{p_2}
\end{aligned}$$

Note: in the above verifications we assumed that $(z^{p_1})^{p_2} = z^{p_1 p_2}$. This can be seen as a generalization of the same rule with p_2 being an integer which may be established from the definition of raising to integer power. Also, see Eq. 63 and the related text.

6. Define the "logarithm" of a given number (which is complex in general) and hence identify the relation between the operations (or functions) of exponentiation (or raising to power) and taking logarithm.
Answer: The logarithm of a given number α to a given base number β (i.e. $\log_\beta \alpha$) is the exponent γ

[73] It should be noted that the purpose of the following verifications is to demonstrate consistency rather than to provide rigorous proofs.

to which the base β must be raised to produce the number α (i.e. $\beta^\gamma = \alpha$). Accordingly, we can write $\alpha = \beta^\gamma = \beta^{\log_\beta \alpha}$ which means that exponentiation and taking logarithm (i.e. to the same base) are inverse operations (or functions), i.e. $\beta^{\log_\beta \alpha} = \log_\beta \beta^\alpha = \alpha$.

7. Using the result of Problem 6 plus the properties of exponentiation (which are given by Eqs. 64-68 and verified in Problem 5), verify the following rules of logarithm:[74]

(a) $\log_\gamma(\alpha\beta) = \log_\gamma \alpha + \log_\gamma \beta$. **(b)** $\log_\gamma(\alpha/\beta) = \log_\gamma \alpha - \log_\gamma \beta$. **(c)** $\log_\gamma \alpha^\beta = \beta \log_\gamma \alpha$.

(d) $\log_\gamma \sqrt[\beta]{\alpha} = \beta^{-1} \log_\gamma \alpha$. **(e)** $\log_\alpha 1 = 0$. **(f)** $\log_\alpha \alpha = 1$.

(g) $\log_\beta \alpha = \log_\gamma \alpha / \log_\gamma \beta$. **(h)** $\log_\beta(1/\alpha) = -\log_\beta \alpha$. **(i)** $\log_\beta \alpha = 1/\log_\alpha \beta$.

Answer:[75] We use the abbreviation "log." for "taking logarithm" and the abbreviation "exp." for "exponentiation".

(a)

$$\begin{aligned}
\alpha\beta &= \alpha \times \beta \\
\gamma^{\log_\gamma(\alpha\beta)} &= \gamma^{\log_\gamma \alpha} \times \gamma^{\log_\gamma \beta} && \text{(log. and exp. are inverses)} \\
\gamma^{\log_\gamma(\alpha\beta)} &= \gamma^{\log_\gamma \alpha + \log_\gamma \beta} && \text{(see Eq. 68)} \\
\log_\gamma(\alpha\beta) &= \log_\gamma \alpha + \log_\gamma \beta
\end{aligned}$$

where in the last step we took the logarithm of both sides to the base γ and used the fact that exponentiation and taking logarithm are inverses (or we just compared the exponents on the two sides).

(b)

$$\begin{aligned}
\frac{\alpha}{\beta} &= \alpha \times \beta^{-1} \\
\gamma^{\log_\gamma(\alpha/\beta)} &= \gamma^{\log_\gamma \alpha} \times \left(\gamma^{\log_\gamma \beta}\right)^{-1} && \text{(log. and exp. are inverses)} \\
\gamma^{\log_\gamma(\alpha/\beta)} &= \gamma^{\log_\gamma \alpha} \times \gamma^{-\log_\gamma \beta} && \text{(see the note of Problem 5)} \\
\gamma^{\log_\gamma(\alpha/\beta)} &= \gamma^{\log_\gamma \alpha - \log_\gamma \beta} && \text{(see Eq. 68)} \\
\log_\gamma\left(\frac{\alpha}{\beta}\right) &= \log_\gamma \alpha - \log_\gamma \beta
\end{aligned}$$

where in the last step we did what we did in the last step of part (a).

(c)

$$\begin{aligned}
\log_\gamma \alpha^\beta &= \log_\gamma (\alpha)^\beta \\
&= \log_\gamma \left(\gamma^{\log_\gamma \alpha}\right)^\beta && \text{(log. and exp. are inverses)} \\
&= \log_\gamma \left(\gamma^{\beta \log_\gamma \alpha}\right) && \text{(see the note of Problem 5)} \\
&= \beta \log_\gamma \alpha && \text{(log. and exp. are inverses)}
\end{aligned}$$

(d) This is an instance of the result of part (c) noting that $\sqrt[\beta]{\alpha} \equiv \alpha^{1/\beta} = \alpha^{\beta^{-1}}$.

(e) Using the fact that $1 = 1^0$ plus the result of part (c) we have:

$$\log_\alpha 1 = \log_\alpha 1^0 = 0 \times \log_\alpha 1 = 0$$

[74] In principle, α, β, γ in these rules are complex in general although some restrictions may be imposed, e.g. the base is usually required to be real, positive and $\neq 1$. In fact, there are many details about the application and validity of the individual formulae and their conditions and restrictions, but we do not investigate these details due to limitations on space, scope and objectives.

[75] As indicated earlier, in Problems like this what we actually do is to "verify" rather than "prove", and hence it is mainly an exercise of testing and checking consistency and coherence.

(**f**)

$$\begin{aligned} \alpha &= \alpha^1 \\ \alpha^{\log_\alpha \alpha} &= \alpha^1 && \text{(log. and exp. are inverses)} \\ \log_\alpha \alpha &= 1 \end{aligned}$$

where in the last step we did what we did in the last step of part (a).
(**g**)

$$\begin{aligned} \alpha &= \beta^{\log_\beta \alpha} && \text{(log. and exp. are inverses)} \\ \alpha &= \left(\gamma^{\log_\gamma \beta}\right)^{\log_\beta \alpha} && \text{(log. and exp. are inverses)} \\ \alpha &= \gamma^{\log_\gamma \beta \times \log_\beta \alpha} && \text{(see the note of Problem 5)} \\ \log_\gamma \alpha &= \log_\gamma \gamma^{\log_\gamma \beta \times \log_\beta \alpha} && \text{(taking logarithm to base } \gamma) \\ \log_\gamma \alpha &= \log_\gamma \beta \times \log_\beta \alpha && \text{(log. and exp. are inverses)} \\ \log_\beta \alpha &= \frac{\log_\gamma \alpha}{\log_\gamma \beta} \end{aligned}$$

(**h**) This is an instance of the result of part (c) noting that $1/\alpha = \alpha^{-1}$. It can also be obtained from the results of parts (b) and (e), i.e.

$$\log_\beta(1/\alpha) = \log_\beta 1 - \log_\beta \alpha = 0 - \log_\beta \alpha = -\log_\beta \alpha$$

(**i**)

$$\begin{aligned} \log_\beta \alpha \times \log_\gamma \beta &= \log_\gamma \alpha && \text{(the result of part g)} \\ \log_\beta \alpha \times \log_\alpha \beta &= \log_\alpha \alpha && (\gamma = \alpha) \\ \log_\beta \alpha \times \log_\alpha \beta &= 1 && \text{(the result of part f)} \\ \log_\beta \alpha &= \frac{1}{\log_\alpha \beta} \end{aligned}$$

8. Investigate the effects of the following operations on the given curves and lines:
(**a**) The effect of squaring (i.e. raising to power 2) on the upper half of the origin-centered circles.
(**b**) The effect of cubing (i.e. raising to power 3) on the semi-line $\mathrm{Re}\, z = \sqrt{3}\, \mathrm{Im}\, z \quad (\mathrm{Re}\, z > 0)$.
(**c**) The effect of exponentiation by i on the origin-centered unit circle.
Answer: We should remark first that since exponentiation generally involves manipulation of the argument (e.g. by doubling it in the case of squaring), the answer generally depends on the parameterization of the curves and lines and hence it is not unique. So, to have a unique answer we restrict the question and answer to the "principal parameterization", i.e. using the principal value of the argument (as defined by the range $-\pi < \theta_p \le \pi$ according to our convention) in the parameterization (see Problem 4 of § 1.8.2).
(**a**) The upper half of an origin-centered circle of radius r is given by $re^{i\theta}$ ($r > 0$ and $0 \le \theta \le \pi$). On squaring this we get $r^2 e^{i2\theta}$. So, the effect of squaring is to scale the radius by a factor of r and rotate a point with argument θ by θ (i.e. double its argument). Accordingly, the origin-centered semi-circles will be mapped onto origin-centered full circles where the semi-circles with radius $0 < r < 1$ will shrink in radius (and hence they stay inside the origin-centered unit circle but become smaller) and the semi-circles with radius $r > 1$ will expand in radius (and hence they stay outside the origin-centered unit circle but become bigger) while the semi-circle with radius $r = 1$ will keep its radius.
(**b**) This semi-line is given by $re^{i\pi/6}$ ($r > 0$). On cubing this we get $r^3 e^{i\pi/2}$. So, the effect of cubing is to scale the modulus r of a point on this semi-line by a factor of r^2 and rotate the point by $\pi/3$ (i.e. project it onto the positive imaginary axis). Accordingly, this semi-line will be mapped onto the

positive imaginary axis where the points with $0 < r < 1$ will shrink in modulus and the points with $r > 1$ will expand in modulus while the point with $r = 1$ will keep its modulus.
(**c**) This circle is given by $e^{i\theta}$ $(-\pi < \theta \leq \pi)$. On exponentiating this by i we get $\left(e^{i\theta}\right)^i = e^{-\theta}$ $(-\pi < \theta \leq \pi)$. So, the effect of exponentiation by i is to project this circle on the real axis. Accordingly, the origin-centered unit circle will be mapped onto the line segment on the positive real axis that is defined by the real interval $e^{\pi} > x \geq e^{-\pi}$.

1.8.11 Roots of Complex Numbers

A complex number w is an n^{th} root of a complex number z if $z = w^n$ (where n is an integer greater than 1). Taking roots can be seen as raising to (real) fractional powers and hence rooting (i.e. taking root) generally follows the rules of exponentiation. If $z = re^{i\theta}$ $(r > 0)$ then the n^{th} roots of z are:[76]

$$z^{1/n} = r^{1/n} e^{i(\theta+2m\pi)/n} \qquad (m = 0, 1, \cdots, n-1) \tag{70}$$

The root corresponding to $m = 0$ is labeled as the principal n^{th} root of z (assuming θ to be the principal argument). From Eq. 70 we can see that $z^{1/n}$ has n distinct values, i.e. a (non-zero) complex number has n distinct n^{th} roots (see Problem 5). Hence, the square root $z^{1/2}$ has two distinct values, the cubic root $z^{1/3}$ has three distinct values, and so on. Accordingly, the function of taking the n^{th} root of complex numbers is a finitely multi-valued function (see § 1.5).[77]

It should be noted that in Eq. 70 the meaning of $1/n$ in $r^{1/n}$ is rather different from its meaning in $z^{1/n}$ because in $r^{1/n}$ it has its usual *real* meaning (i.e. the *positive real* root of r noting that $r^{1/n}$ is the modulus of $z^{1/n}$ and hence it must be real non-negative) while in $z^{1/n}$ it has a complex meaning (as revealed in the right side of Eq. 70). This difference becomes apparent when z is a positive real number and hence the distinction between $1/n$ in $r^{1/n}$ and $1/n$ in $z^{1/n}$ is revealed, as will be seen in the Problems (refer for example to part a of Problem 1 where we used the modulus sign in $|1|$ to indicate this difference and remove any potential inconsistency).[78]

Problems

1. Find the following roots (where a in part h is a non-zero real number and n is an integer > 1):

(**a**) $1^{1/2}$. (**b**) $1^{1/3}$. (**c**) $5^{1/2}$. (**d**) $(-\pi)^{1/2}$.

(**e**) $(-i\pi)^{-1/4}$. (**f**) $(1-i)^{1/5}$. (**g**) $(3-i3)^{-1/3}$. (**h**) $a^{1/n}$.

Answer: We should note that in the following we use the principal value of the argument of the base (or radicand), i.e. θ in Eq. 70 represents θ_p. We should also note that part (e) and its alike (where the power is $-1/n$ with n being an integer greater than 1) should be interpreted as taking the n^{th} root of the reciprocal.
(**a**)

$$1^{1/2} = \left(1 \times e^{i(0+2m\pi)}\right)^{1/2} = |1|^{1/2} e^{i(2m\pi)/2} = 1e^{im\pi} = e^{im\pi} \qquad (m = 0, 1)$$

So, we have two square roots of 1:

$$z_0 = e^{i0} = \cos 0 + i \sin 0 = 1 + i0 = 1$$

[76] This formula may seem more appropriate for the convention $0 \leq \mathrm{Arg}(z) < 2\pi$, but this is not necessarily the case. In fact, this is to ease the calculations. Anyway, this is a trivial matter and hence it should not be of any concern. We should also note that θ here is the principal argument (as indicated in the reference to the convention) although the use of non-principal values should not affect the actual values of the roots due to the periodicity (see Problem 11 of § 1.11) which can also be confirmed by the fact that $w^n - z$ (for a fixed $z \neq 0$) is a polynomial (of order n) in w and hence it should have exactly n complex zeros (or roots) as will be shown in Problem 5.

[77] In fact, it is infinitely multi-valued but because of the periodicity, the m distinct values are repeated and hence it is finitely multi-valued by considering only these m distinct values.

[78] We note that we use the modulus sign only in some cases to draw the attention to this point. Hence, we generally rely on the position and context to distinguish between the real root and the complex root. We should also note that the modulus sign is used as an indication to this difference (noting that it may be appropriate to be applied to the root itself as well for more clarity).

$$z_1 = e^{i\pi} = \cos\pi + i\sin\pi = -1 + i0 = -1$$

As explained in the text, $1^{1/2}$ represents the square root of 1 as a complex number (i.e. $1 + i0$) while $|1|^{1/2}$ represents the positive square root of 1 as a real number (and hence there is no inconsistency).

(**b**)

$$1^{1/3} = \left(1 \times e^{i(0+2m\pi)}\right)^{1/3} = 1^{1/3}e^{i(2m\pi)/3} = 1e^{i(2m\pi)/3} = e^{i(2m\pi)/3} \qquad (m = 0, 1, 2)$$

So, we have three cubic roots of 1 (as a complex number):

$$\begin{aligned} z_0 &= e^{i0} = 1 + i0 = 1 \\ z_1 &= e^{i2\pi/3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \simeq -0.5 + i0.8660 \\ z_2 &= e^{i4\pi/3} = e^{-i2\pi/3} = \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \simeq -0.5 - i0.8660 \end{aligned}$$

(**c**)

$$5^{1/2} = (5 \times 1)^{1/2} = \sqrt{5} \times 1^{1/2} = \sqrt{5}e^{im\pi} \qquad (m = 0, 1)$$

where we used the result of part (a).[79] So, we have two square roots of 5:

$$\begin{aligned} z_0 &= \sqrt{5}e^{i0} = \sqrt{5}(\cos 0 + i\sin 0) = \sqrt{5}(1 + i0) = \sqrt{5} \simeq 2.2361 \\ z_1 &= \sqrt{5}e^{i\pi} = \sqrt{5}(\cos\pi + i\sin\pi) = \sqrt{5}(-1 + i0) = -\sqrt{5} \simeq -2.2361 \end{aligned}$$

(**d**)

$$(-\pi)^{1/2} = \left(\pi e^{i(\pi+2m\pi)}\right)^{1/2} = \pi^{1/2}e^{i(\pi+2m\pi)/2} = \sqrt{\pi}e^{i(\pi+2m\pi)/2} \qquad (m = 0, 1)$$

So, we have two square roots of $-\pi$:

$$\begin{aligned} z_0 &= \sqrt{\pi}e^{i\pi/2} = i\sqrt{\pi} \simeq i1.7725 \\ z_1 &= \sqrt{\pi}e^{i(\pi+2\pi)/2} = \sqrt{\pi}e^{i3\pi/2} = \sqrt{\pi}e^{-i\pi/2} = -i\sqrt{\pi} \simeq -i1.7725 \end{aligned}$$

(**e**)

$$(-i\pi)^{-1/4} = \left(\pi e^{i(-\pi/2+2m\pi)}\right)^{-1/4} = \pi^{-1/4}e^{i(\pi/8-m\pi/2)} \qquad (m = 0, 1, 2, 3)$$

So, we have four roots:

$$\begin{aligned} z_0 &= \pi^{-1/4}e^{i\pi/8} \simeq 0.6939 + i0.2874 \\ z_1 &= \pi^{-1/4}e^{i(\pi/8-\pi/2)} = \pi^{-1/4}e^{-i3\pi/8} \simeq 0.2874 - i0.6939 \\ z_2 &= \pi^{-1/4}e^{i(\pi/8-\pi)} = \pi^{-1/4}e^{-i7\pi/8} \simeq -0.6939 - i0.2874 \\ z_3 &= \pi^{-1/4}e^{i(\pi/8-3\pi/2)} = \pi^{-1/4}e^{-i11\pi/8} = \pi^{-1/4}e^{i5\pi/8} \simeq -0.2874 + i0.6939 \end{aligned}$$

(**f**)

$$(1-i)^{1/5} = \left(2^{1/2}e^{i(-\pi/4+2m\pi)}\right)^{1/5} = 2^{1/10}e^{i(-\pi/4+2m\pi)/5} \qquad (m = 0, 1, 2, 3, 4)$$

So, we have five roots:

$$\begin{aligned} z_0 &= 2^{1/10}e^{-i\pi/20} \simeq 1.0586 - i0.1677 \\ z_1 &= 2^{1/10}e^{i(-\pi/4+2\pi)/5} = 2^{1/10}e^{i7\pi/20} \simeq 0.4866 + i0.9550 \end{aligned}$$

[79] The point in this sort of manipulation is to split the complex number to a product of a real positive number a times a complex number b and hence the (real positive) root of a becomes a scaling factor for the complex root of b (and accordingly we can exploit our previous results about b). So, in our example (the real positive number) $\sqrt{5}$ is a scaling factor to (the complex number) $1^{1/2}$.

$$\begin{aligned} z_2 &= 2^{1/10}e^{i(-\pi/4+4\pi)/5} = 2^{1/10}e^{i3\pi/4} \simeq -0.7579 + i0.7579 \\ z_3 &= 2^{1/10}e^{i(-\pi/4+6\pi)/5} = 2^{1/10}e^{i23\pi/20} = 2^{1/10}e^{-i17\pi/20} \simeq -0.9550 - i0.4866 \\ z_4 &= 2^{1/10}e^{i(-\pi/4+8\pi)/5} = 2^{1/10}e^{i31\pi/20} = 2^{1/10}e^{-i9\pi/20} \simeq 0.1677 - i1.0586 \end{aligned}$$

(**g**)

$$(3 - i3)^{-1/3} = \left(18^{1/2}e^{i(-\pi/4+2m\pi)}\right)^{-1/3} = 18^{-1/6}e^{i(\pi/4-2m\pi)/3} \qquad (m = 0, 1, 2)$$

So, we have three roots:

$$\begin{aligned} z_0 &= 18^{-1/6}e^{i\pi/12} \simeq 0.5967 + i0.1599 \\ z_1 &= 18^{-1/6}e^{i(\pi/4-2\pi)/3} = 18^{-1/6}e^{-i7\pi/12} \simeq -0.1599 - i0.5967 \\ z_2 &= 18^{-1/6}e^{i(\pi/4-4\pi)/3} = 18^{-1/6}e^{-i5\pi/4} = 18^{-1/6}e^{i3\pi/4} \simeq -0.4368 + i0.4368 \end{aligned}$$

(**h**) If $a < 0$ then we have:

$$a^{1/n} = |a|^{1/n}\, e^{i(\pi+2m\pi)/n} = |a|^{1/n}\, e^{i\pi(1+2m)/n} \qquad (m = 0, 1, \cdots, n-1)$$

If $a > 0$ then we have:

$$a^{1/n} = |a|^{1/n}\, e^{i(0+2m\pi)/n} = |a|^{1/n}\, e^{i(2m\pi/n)} \qquad (m = 0, 1, \cdots, n-1)$$

where we use $|a|$ here to distinguish the positive real from the complex (as noted earlier in part a).

2. Find the following "roots":

(**a**) $7^{1/(1+i)}$. (**b**) $(i9)^{1/(i3)}$. (**c**) $(6 + i15)^{1/(1-i\sqrt{2})}$. (**d**) $(e + i\pi)^{1/(e+i\pi)}$.

Answer: We treat these as complex powers rather than roots in the usual sense (which is associated with integers > 1 as explained earlier). We should also note that in the following we use the principal value of the argument of the base (as noted earlier).

(**a**)

$$\begin{aligned} 7^{1/(1+i)} &= 7^{(1-i)/2} = |7|^{1/2}\, 7^{-i/2} = |7|^{1/2}\left(e^{\log_e 7}\right)^{-i/2} = |7|^{1/2}\, e^{-i(\log_e 7)/2} \\ &= |7|^{1/2}\left(\cos\frac{\log_e 7}{2} - i\sin\frac{\log_e 7}{2}\right) \simeq 1.4892 - i2.1869 \end{aligned}$$

(**b**)

$$\begin{aligned} (i9)^{1/(i3)} &= \left(9e^{i\pi/2}\right)^{-i/3} = 9^{-i/3}e^{\pi/6} = \left(e^{\log_e 9}\right)^{-i/3} e^{\pi/6} = e^{\pi/6}e^{-i(\log_e 9)/3} \\ &= e^{\pi/6}\left(\cos\frac{\log_e 9}{3} - i\sin\frac{\log_e 9}{3}\right) \simeq 1.2552 - i1.1288 \end{aligned}$$

(**c**)

$$\begin{aligned} (6 + i15)^{1/(1-i\sqrt{2})} &\simeq \left(\sqrt{261}e^{i1.1903}\right)^{(1+i\sqrt{2})/3} \\ &= \left(\sqrt{261}e^{i1.1903}\right)^{1/3}\left(\sqrt{261}e^{i1.1903}\right)^{i(\sqrt{2}/3)} \\ &= \left(261^{1/6}e^{i0.3968}\right)\left(261^{i(\sqrt{2}/6)}e^{-1.1903(\sqrt{2}/3)}\right) \\ &= \left(261^{1/6}e^{-1.1903(\sqrt{2}/3)}\right)\left(e^{i0.3968}261^{i(\sqrt{2}/6)}\right) \\ &= \left(261^{1/6}e^{-1.1903(\sqrt{2}/3)}\right)\left(e^{i0.3968}e^{i\left[(\sqrt{2}/6)\log_e 261\right]}\right) \\ &= \left(261^{1/6}e^{-1.1903(\sqrt{2}/3)}\right)\left(e^{i\left[0.3968+(\sqrt{2}/6)\log_e 261\right]}\right) \\ &\simeq 1.4424\,(\cos 1.7083 + i\sin 1.7083) \simeq -0.1978 + i1.4288 \end{aligned}$$

(**d**)

$$\begin{aligned}
(e+i\pi)^{1/(e+i\pi)} &\simeq \left(4.1544e^{i0.8575}\right)^{(e-i\pi)/17.2587} \simeq \left(4.1544e^{i0.8575}\right)^{0.1575-i0.1820}\\
&\simeq \left(4.1544e^{i0.8575}\right)^{0.1575}\left(4.1544e^{i0.8575}\right)^{-i0.1820}\\
&\simeq \left(4.1544^{0.1575}e^{i0.1351}\right)\left(4.1544^{-i0.1820}e^{0.1561}\right)\\
&\simeq \left(4.1544^{0.1575}e^{0.1561}\right)\left(e^{i0.1351}e^{-i0.1820\log_e 4.1544}\right)\\
&\simeq \left(4.1544^{0.1575}e^{0.1561}\right)\left(e^{i0.1351}e^{-i0.2592}\right) \simeq 1.4629e^{-i0.1242}\\
&\simeq 1.4629\,(\cos 0.1242 - i\sin 0.1242) \simeq 1.4516 - i0.1812
\end{aligned}$$

3. Analyze Eq. 70.
Answer: We can conclude from this equation the following points:
• The condition $m = 0, 1, \cdots, n-1$ means that any complex number ($\neq 0$) has exactly n n^{th} roots. Since real numbers are a subset of complex numbers, this also applies to real numbers.
• Geometrically, the n^{th} roots of a complex number z can be seen as vectors of length $|z|^{1/n}$ that are evenly distributed around the origin of the complex plane. In other words, they are "radii" of an origin-centered circle of radius $|z|^{1/n}$ where each one of these radii is obtained from its neighbor by a rotation of $2\pi/n$ (or $-2\pi/n$) around the origin. They may also be considered as the vertices of an origin-centered regular n-polygon.
• The root corresponding to $m = 0$ (which we may call the principal root) has an argument (or phase angle) of θ/n (and hence it is uniquely determined in magnitude and argument). Therefore, the other roots are also determined uniquely by the aforementioned rotation (as explained in the previous point).

4. Investigate the effects of the following operations on the given curves and lines:
(**a**) The effect of taking the square root on the upper half of the origin-centered circles.
(**b**) The effect of taking the cubic root on the semi-line $\operatorname{Im} z = -\operatorname{Re} z$ $(\operatorname{Re} z < 0)$.
(**c**) The effect of "taking the i^{th} root" (i.e. exponentiation by $1/i$) on the origin-centered unit circle.
Answer: As noted earlier, we restrict the question and answer to the "principal parameterization".
(**a**) The upper half of an origin-centered circle of radius r is given by $z = re^{i\theta}$ ($r > 0$ and $0 \le \theta \le \pi$). On taking the square root we get (see Eq. 70):

$$z^{1/2} = r^{1/2}e^{i(\theta+2m\pi)/2} = r^{1/2}e^{i(\theta/2+m\pi)} \qquad (0 \le \theta \le \pi \text{ and } m = 0, 1)$$

So, we have two mappings: one corresponds to $m = 0$ and one corresponds to $m = 1$. The first mapping is given by:[80]

$$z_0^{1/2} = r^{1/2}e^{i\theta/2} \qquad (0 \le \theta \le \pi)$$

which is an origin-centered quarter-circle of radius $r^{1/2}$ in the first quadrant. The second mapping is given by:

$$z_1^{1/2} = r^{1/2}e^{i(\theta/2+\pi)} \qquad (0 \le \theta \le \pi)$$

which is an origin-centered quarter-circle of radius $r^{1/2}$ in the third quadrant. As we see, the radius of the quarter-circles is $r^{1/2}$ which means that the origin-centered semi-circles with $0 < r < 1$ expand in radius (but remain inside the origin-centered unit circle) and the origin-centered semi-circles with $r > 1$ shrink in radius (but remain outside the origin-centered unit circle) while the origin-centered unit circle keeps its radius.
(**b**) This semi-line is given by $re^{i3\pi/4}$ ($r > 0$). On taking the cubic root we get:

$$z^{1/3} = r^{1/3}e^{i(3\pi/4+2m\pi)/3} = r^{1/3}e^{i(\pi/4+2m\pi/3)} \qquad (r^{1/3} > 0 \text{ and } m = 0, 1, 2)$$

So, we have three mappings corresponding to $m = 0, 1, 2$. The first mapping is given by:

$$z_0^{1/3} = r^{1/3}e^{i\pi/4} \qquad (r^{1/3} > 0)$$

[80] Symbols like $z_0^{1/2}$ should be interpreted as $\left(z^{1/2}\right)_0$ rather than $(z_0)^{1/2}$.

which is the semi-line $\operatorname{Im} z = \operatorname{Re} z$ $(\operatorname{Re} z > 0)$, i.e. the part of the line $y = x$ in the first quadrant (or similarly the line $\operatorname{Arg} z = \pi/4$).[81] The second mapping is given by:

$$z_1^{1/3} = r^{1/3} e^{i(\pi/4+2\pi/3)} = r^{1/3} e^{i11\pi/12} \qquad (r^{1/3} > 0)$$

which is the semi-line $\operatorname{Im} z \simeq -0.2679 \operatorname{Re} z$ $(\operatorname{Re} z < 0)$, i.e. the part of the line $y \simeq -0.2679\, x$ in the second quadrant (or similarly the line $\operatorname{Arg} z = 11\pi/12$). The third mapping is given by:

$$z_2^{1/3} = r^{1/3} e^{i(\pi/4+4\pi/3)} = r^{1/3} e^{i19\pi/12} = r^{1/3} e^{-i5\pi/12} \qquad (r^{1/3} > 0)$$

which is the semi-line $\operatorname{Im} z \simeq -3.7321 \operatorname{Re} z$ $(\operatorname{Re} z > 0)$, i.e. the part of the line $y \simeq -3.7321\, x$ in the fourth quadrant (or similarly the line $\operatorname{Arg} z = -5\pi/12$).
Noting that a point with modulus r is mapped (in all three mappings) onto a point with modulus $r^{1/3}$, we can see that the points with $0 < r < 1$ will expand in modulus (but remain < 1) and the points with $r > 1$ will shrink in modulus (but remain > 1) while the point with $r = 1$ will keep its modulus.
(**c**) This circle is given by $e^{i\theta}$ $(-\pi < \theta \le \pi)$ and hence on exponentiating this by $1/i$ we get $\left(e^{i\theta}\right)^{1/i} = e^{\theta}$ $(-\pi < \theta \le \pi)$. So, the effect of exponentiation by $1/i$ is to map this circle on the real axis. Accordingly, the origin-centered unit circle will be mapped onto the line segment on the positive real axis that is defined by the real interval $e^{-\pi} < x \le e^{\pi}$.

5. Show that $z^{1/n}$ ($z \ne 0$ with n being integer > 1) has exactly n distinct (complex) roots using the fact (which will be established in Problem 1 of § 7.1) that an n^{th} degree polynomial function has exactly n zeros (or roots).
Answer: If $w = z^{1/n}$ then $w^n = z$. Hence, for a fixed z the equation $w^n - z = 0$ is an n^{th} degree polynomial equation in w and therefore (by the given fact) it should have exactly n zeros (or roots), i.e. $w = z^{1/n}$ has exactly n values (no more and no less). So, what we need to do to complete the proof is to show that these n values (or roots) are distinct from each other (i.e. they are not repetitive). Now, from Eq. 70 (noting that the condition $m = 0, 1, \cdots, n-1$ is an acceptable realization of "exactly n zeros") the n zeros should be distinct from each other (i.e. they are not repetitive). Also, see Problem 4 of § 2.2.
Note: "distinct" may be interpreted to mean simultaneously: any presumed extra roots (i.e. beyond the n roots) are not different from the n roots, and the n roots are different from each other.
6. What is the relation between exponentiation (i.e. raising to power) and rooting (i.e. taking root)?
Answer: Raising to power n and taking the n^{th} root (where n is an integer greater than 1) can be seen as inverse operations (or functions) because:

$$\left(z^{1/n}\right)^n = (z^n)^{1/n} = z^{n/n} = z^1 = z$$

In fact, this should be obvious from the definition of the n^{th} root of a complex number (as given in the first sentence of the present subsection).
7. Identify the branch cut and the branch point of $f(z) = \sqrt{5 - z}$.
Answer: Let $Z = 5 - z$. The branch cut of $\sqrt{Z}$ is obviously $Z \in \mathbb{R}$ and $Z \le 0$. So, the branch cut of f is $(5 - z) \in \mathbb{R}$ and $(5 - z) \le 0$, i.e. $z \in \mathbb{R}$ and $z \ge 5$. In more simple terms, it is the semi-line $x \ge 5$ on the real axis.
From the definition(s) of branch point (see § 1.5 and Problem 21 of § 1.5 in particular), the branch point of $f(z) = \sqrt{5 - z}$ is $z = 5$.
8. Show that the n n^{th} roots of any (non-zero) complex number can be obtained from the n n^{th} roots of 1 by multiplying the roots of 1 by a modulus factor $r^{1/n}$ and an argument factor $e^{i\theta/n}$.
Answer: The n n^{th} roots of 1 are:

$$1^{1/n} = e^{i2m\pi/n} \qquad (m = 0, 1, \cdots, n-1)$$

[81] In Problems like this where we have mapping (which is usually from the z plane to the w plane) it is more appropriate to use the w plane and its variables (i.e. u and v) for representing the images of the mapping (and hence we say for example $\operatorname{Im} w = \operatorname{Re} w$ and $v = u$). However, for simplicity and to avoid distraction (especially at this stage in the book where detailed investigation of complex functions as mappings is still waiting) we use the z plane and its variables (i.e. x and y) instead.

while the n n^{th} roots of a (non-zero) complex number are:

$$z^{1/n} = r^{1/n}e^{i(\theta+2m\pi)/n} = r^{1/n}e^{i\theta/n}\left[e^{i2m\pi/n}\right] = r^{1/n}e^{i\theta/n}\left[1^{1/n}\right] \qquad (m = 0, 1, \cdots, n-1)$$

which establishes the claim.

1.8.12 Complex Numbers as a Group

The set of complex numbers forms an infinite Abelian group under addition since it meets the conditions of closure, associativity, identity (which is 0) and inverse (which is the negative) as well as commutativity. The set of complex numbers (excluding 0) also forms an infinite Abelian group under multiplication (where the identity is 1 and the inverse is the reciprocal).[82]

Problems

1. Show that the set of complex numbers is closed under addition and multiplication.
   Answer: This can be easily seen from Eq. 20 for addition and Eq. 21 (or Eq. 22) for multiplication.
2. Give some examples of subsets of complex numbers that form groups under certain operations.
   Answer: For example:
   • The set of real numbers under addition (infinite Abelian group with 0 identity and negative inverse).
   • The set of real numbers (excluding 0) under multiplication (infinite Abelian group with 1 identity and reciprocal inverse).
   • The set of imaginary numbers under addition (infinite Abelian group with 0 identity and negative inverse).
   • The set $\{1, -1, i, -i\}$ under multiplication (finite Abelian group with 1 identity, -1 as its own inverse and $i, -i$ as inverses of each other).
3. Show that the reciprocal z^{-1} of a complex number $z \neq 0$ is its multiplicative inverse, i.e. $zz^{-1} = z^{-1}z = 1$.
   Answer:[83] We use the polar form of z, that is:

$$zz^{-1} = re^{i\theta}\left(re^{i\theta}\right)^{-1} = re^{i\theta}r^{-1}e^{-i\theta} = r^{1-1}e^{i\theta-i\theta} = r^0e^0 = 1 \times 1 = 1$$

   where we used the established rules of exponents (as well as the established rules of real numbers such as $1 \times 1 = 1$). We can similarly obtain $z^{-1}z = 1$.
   Note: we may also use the generic form of z (which includes both the Cartesian and polar forms), that is:

$$zz^{-1} = z \times \frac{1}{z} = z \times \frac{z^*}{zz^*} = \frac{zz^*}{zz^*} = \frac{|z|^2}{|z|^2} = 1$$

   where in the last step we used the fact that $|z|^2$ is real. However, this demonstration seems more problematic because the multiplication of the numerator and denominator by z^* may depend on the claim that the reciprocal is the multiplicative inverse (as well as other potential circularities).

1.9 Limits of Complex Variables

In this section we present a general investigation of the concept and techniques of limits in complex analysis. We also investigate certain aspects and issues related to limits such as continuity, analyticity and boundedness of complex functions which are strongly linked to the paradigm of limit. However, we

[82] It is straightforward to show the closure of complex numbers under addition and multiplication (see Problem 1). The other conditions of Abelian group have already been demonstrated (see Eqs. 25-34 and Problem 3 of § 1.8.5). We should also note that the set of complex numbers (with the above operations) also forms a field (in the technical sense of "field" as an algebraic structure according to its formal definition in abstract algebra).

[83] We note that this is a demonstration (or "show" according to the question) rather than a proof (and hence the purpose is to demonstrate consistency) because some of the steps in this demonstration can be circular.

should remark that due to restrictions on the space, scope and objectives of this book we do not discuss the issue of limits and the related issues extensively although we provide a sufficient background about these subjects that includes basic definitions, rules and theorems as well as solved Problems and examples.[84]

In brief, we can say that the paradigm and techniques of limits are at the heart of complex analysis as they are central to real analysis. We can also say that the rules of limits in the complex domain are generally similar to the rules in the real domain. For example, if $L_1 = \lim_{z\to z_0} f_1(z)$ and $L_2 = \lim_{z\to z_0} f_2(z)$ (where z_0 is a given point in the z plane, L_1 and L_2 are given points in the w plane, and f_1 and f_2 are complex functions) then:

$$\lim_{z\to z_0} a = a \qquad (a \text{ is complex constant}) \tag{71}$$

$$\lim_{z\to z_0} z = z_0 \tag{72}$$

$$\lim_{z\to z_0} \big[bf_1(z)\big] = b \lim_{z\to z_0} f_1(z) = bL_1 \qquad (b \text{ is complex constant}) \tag{73}$$

$$\lim_{z\to z_0} \big[f_1(z) \pm f_2(z)\big] = \lim_{z\to z_0} f_1(z) \pm \lim_{z\to z_0} f_2(z) = L_1 \pm L_2 \tag{74}$$

$$\lim_{z\to z_0} \big[f_1(z) \times f_2(z)\big] = \lim_{z\to z_0} f_1(z) \times \lim_{z\to z_0} f_2(z) = L_1 \times L_2 \tag{75}$$

$$\lim_{z\to z_0} \left[\frac{f_1(z)}{f_2(z)}\right] = \frac{\lim_{z\to z_0} f_1(z)}{\lim_{z\to z_0} f_2(z)} = \frac{L_1}{L_2} \qquad (L_2 \neq 0) \tag{76}$$

$$\lim_{z\to z_0} |f_1(z)| = \left|\lim_{z\to z_0} f_1(z)\right| = |L_1| \tag{77}$$

$$\lim_{z\to z_0} \big[f_1(f_2(z))\big] = f_1\left(\lim_{z\to z_0} f_2(z)\right) = f_1(L_2) \qquad (f_1 \text{ has a limit at } L_2) \tag{78}$$

We should now draw the attention to the following important remarks:

• If a complex function has a limit at a given point in the complex plane then it should converge to the same value regardless of the direction from which this point is approached (i.e. the limit should be unique).[85] For example, if a function converges to the value 1 when it approaches the point $z = 0$ horizontally while it converges to the value -1 when it approaches this point vertically then this function has no limit at $z = 0$. Some examples of functions with no limits at certain points or regions (due to the dependence of the value of the limit on the direction of approach) will be given in the Problems of the present section and later in the book.[86]

• The value of the limit (if it exists) at a given point z_0 can be different from the value of the function at z_0 (assuming it is defined at z_0). In fact, the limit may exist even if the function is not defined at z_0. Yes, when the two values exist and they are the same then the function is continuous at that point (see Problem 8 of § 1.5).

• The limit of a multi-valued function at a given point may depend on the particular branch and hence the value of the limit on two branches can be different (noting that multi-valued functions are not strictly "functions"). Yes, the value on each individual branch should be unique (as indicated above) if the limit should *exist* (i.e. the "unique existence") for that branch.

Problems

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[84] In addition to what is provided in this section about the aforementioned background, these subjects are also partly investigated earlier in the book (see for example § 1.5) as well as in the coming parts of the book. In fact, these subjects are at the center of complex analysis and hence they are present and discussed (explicitly or implicitly) in many parts of this book.

[85] So, when we say "a function $f(z)$ has a limit at a given point z_0" or "the limit of $f(z)$ at z_0 does exist" it means the limit of f exists and it has one (and only one) value at z_0, i.e. this is a statement of existence and uniqueness of limit although it is usually expressed as an existence statement (with an implicit understanding of the uniqueness condition).

[86] In fact, there are other reasons for the failure of having limit (and hence examples will be given for several types of functions without limits at certain points or regions which could be the entire complex plane). In brief, failure of having limit could be due to failure of existence or to failure of uniqueness.

1. Compare the concept of limit in real analysis and in complex analysis.
Answer: We may say:
• In real analysis the point at which the limit is taken can be approached only from two directions (since the domain in real analysis is the 1D real line), while in complex analysis the point at which the limit is taken can be approached from infinitely-many directions (since the domain in complex analysis is the 2D complex plane).
• In real analysis we evaluate the limit of a single function, while in complex analysis we actually evaluate the limit of two (real) functions (i.e. the real and imaginary parts of the complex function) and hence we may write:
$$\lim_{z\to z_0} w = \lim_{z\to z_0} \mathrm{Re}(w) + i \lim_{z\to z_0} \mathrm{Im}(w)$$
where $w = f(z)$ is a complex function of z.
2. Describe the concept of limit using the concept of neighborhood.
Answer: If L is the limit of a complex function $f(z)$ at a given point z_0, i.e. $\lim_{z\to z_0} f(z) = L$, then we can say: for every neighborhood of L in the w plane (notated as N_L) there is a deleted neighborhood of z_0 in the z plane (notated as N_{z_0}) such that f is in N_L whenever z is in N_{z_0}.
3. Give a sensible interpretation to the following limits that involve infinity:
$$\lim_{z\to\infty} f(z) = L \qquad \lim_{z\to z_0} f(z) = \infty \qquad \lim_{z\to\infty} f(z) = \infty$$
Answer: These limits can be interpreted respectively as (or seen as equivalent to):
$$\lim_{z\to 0} f\left(\frac{1}{z}\right) = L \qquad \lim_{z\to z_0} \frac{1}{f(z)} = 0 \qquad \lim_{z\to 0} \frac{1}{f(1/z)} = 0$$
4. Verify the following rules of limits (where a is a complex constant and P is a complex polynomial):
(**a**) $\lim_{z\to z_0} z^n = z_0^n$. (**b**) $\lim_{z\to z_0} (az^n) = az_0^n$. (**c**) $\lim_{z\to z_0} P(z) = P(z_0)$.
(**d**) $\lim_{z\to z_0} e^z = e^{z_0}$. (**e**) $\lim_{z\to z_0}(\cos z) = \cos z_0$. (**f**) $\lim_{z\to z_0}(\sin z) = \sin z_0$.
Answer: We use the rules of limits which we stated in Eqs. 71-78.
(**a**) This is just an extension of the product rule of limits (i.e. Eq. 75) with the use of the rule of Eq. 72, that is:
$$\lim_{z\to z_0} z^n = \lim_{z\to z_0} \left(z \times z^{n-1}\right) = \lim_{z\to z_0} z \times \lim_{z\to z_0} z^{n-1} = z_0 \times \lim_{z\to z_0} z^{n-1}$$
The same will apply to $\lim_{z\to z_0} z^{n-1}$ (as well as to the lower powers until reaching $\lim_{z\to z_0} z$) and hence we get $\lim_{z\to z_0} z^n = z_0 \times z_0 \times \cdots \times z_0 = z_0^n$.
We may also use the composition rule of limits (i.e. Eq. 78) with the use of the rule of Eq. 72, that is:
$$\lim_{z\to z_0} z^n = \left(\lim_{z\to z_0} z\right)^n = z_0^n$$
(**b**) This is a combination of the product rule of limits (i.e. Eq. 75) with the use of Eq. 71 and the result of part (a), that is:
$$\lim_{z\to z_0} (az^n) = \lim_{z\to z_0} a \times \lim_{z\to z_0} z^n = a \times z_0^n = az_0^n$$
We may also consider this as a combination of the scaling rule of limits (i.e. Eq. 73) and the result of part (a) of this Problem (noting that Eq. 73 is actually an application of Eq. 75 with the use of Eq. 71).
(**c**) This is just a combination of the sum rule of limits (i.e. Eq. 74 where it can be easily extended, if required, by iteration to more than two terms) and the result of part (b) noting that a polynomial is an algebraic sum of terms of the form az^n.

(**d**) From Eq. 6 and the sum rule of limits (as represented by Eq. 74)[87] plus the result of part (b) we have:

$$\lim_{z\to z_0} e^z = \lim_{z\to z_0}\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right) = \sum_{n=0}^{\infty}\lim_{z\to z_0}\left(\frac{z^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{z_0^n}{n!} = e^{z_0}$$

We may also use the composition rule of limits (i.e. Eq. 78) in association with Eq. 72, that is:[88]

$$\lim_{z\to z_0} e^z = e^{\lim_{z\to z_0} z} = e^{z_0}$$

(**e**) From the power series of $\cos z$ (see § 1.4) and the sum rule of limits plus the result of part (b) we have:

$$\lim_{z\to z_0}\cos z = \lim_{z\to z_0}\left(\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}\right) = \sum_{n=0}^{\infty}\lim_{z\to z_0}\left(\frac{(-1)^n z^{2n}}{(2n)!}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n z_0^{2n}}{(2n)!} = \cos z_0$$

We may also use the composition rule of limits (i.e. Eq. 78) in association with Eq. 72, that is:

$$\lim_{z\to z_0}(\cos z) = \cos\left(\lim_{z\to z_0} z\right) = \cos z_0$$

(**f**) From the power series of $\sin z$ (see § 1.4) and the sum rule of limits plus the result of part (b) we have:

$$\lim_{z\to z_0}\sin z = \lim_{z\to z_0}\left(\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty}\lim_{z\to z_0}\left(\frac{(-1)^n z^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n z_0^{2n+1}}{(2n+1)!} = \sin z_0$$

We may also use the composition rule of limits (i.e. Eq. 78) in association with Eq. 72, that is:

$$\lim_{z\to z_0}(\sin z) = \sin\left(\lim_{z\to z_0} z\right) = \sin z_0$$

Note: we should remark that n in part (a) and in part (b) of this Problem (and even in part c although implicitly) can be a non-negative integer (despite the seeming suggestion that n is an integer greater than 1) although we should introduce slight modifications to the above arguments in the case of $n = 0, 1$ (with the exclusion of the origin in some cases). Accordingly, the rules of Eqs. 71 and 72 can be instances of the rules of parts (b) and (a) of this Problem (and in fact even Eq. 72 can be an instance of the rule of part b, noting that part a is an instance of part b corresponding to $a = 1$).

5. Find the following limits:

(**a**) $\lim_{z\to i}\left(3z^3 - z + 6\right)$. (**b**) $\lim_{z\to(1-i)}\left(\frac{z^2-i3}{z+2}\right)$. (**c**) $\lim_{z\to -i}\left(\frac{z^3-i}{z^2+1}\right)$.

(**d**) $\lim_{z\to(2+i)}\left(\frac{\mathrm{Re}\, z}{\mathrm{Im}\, z^*}\right)$. (**e**) $\lim_{z\to\infty}\left(\frac{5z^3+z-1}{-z^3+i8}\right)$. (**f**) $\lim_{z\to\infty}\left(\frac{z^2-i\pi}{7z^3}\right)$.

(**g**) $\lim_{z\to 0}\left(\frac{iz^5}{z^2+9}\right)$. (**h**) $\lim_{z\to(1+i\pi/4)}\left(e^z\right)$. (**i**) $\lim_{z\to(3-i\pi)}\left(e^{z+i}\right)$.

(**j**) $\lim_{z\to(5-i\pi/3)}(\cos z)$. (**k**) $\lim_{z\to i3}\left(\cos z^2\right)$. (**l**) $\lim_{z\to i\sqrt{2}}\left(\sin e^{z^4}\right)$.

Answer:[89] We use the rules of limits which were given earlier in this section (see Eqs. 71-78 and Problem 4) as well as mathematical results and identities that have already been established (e.g. in

[87] In fact, we need to extend Eq. 74 from two aspects: first to more than two terms (which can be trivially achieved by iteration) and second to infinite sums (which may require a somewhat intricate argument).

[88] Although the use of the composition rule looks more tidy and elegant, it may require more effort to establish that e^z has a limit at z_0 (although this should be fairly obvious).

[89] We should remark that infinity (i.e. ∞) in the complex domain is not as simple as in the real domain (noting that it is not simple even in the real domain) and hence we may need to consider more than one type of infinity depending on the circumstances and contexts. So, to avoid many complications and distractions (as well as lengthy explanations and

§ 1.4).
(a)

$$\begin{aligned}\lim_{z\to i}\left(3z^3 - z + 6\right) &= \lim_{z\to i}\left(3z^3\right) + \lim_{z\to i}(-z) + \lim_{z\to i}(6) = 3\lim_{z\to i}\left(z^3\right) - \lim_{z\to i}(z) + \lim_{z\to i}(6)\\ &= 3\,(i)^3 - (i) + 6 = -i3 - i + 6 = 6 - i4\end{aligned}$$

This result can also be obtained more directly from the result of part (c) of Problem 4.
(b)

$$\begin{aligned}\lim_{z\to(1-i)}\left(\frac{z^2-i3}{z+2}\right) &= \frac{\lim_{z\to(1-i)}\left(z^2-i3\right)}{\lim_{z\to(1-i)}(z+2)} = \frac{\lim_{z\to(1-i)}\left(z^2\right)+\lim_{z\to(1-i)}(-i3)}{\lim_{z\to(1-i)}(z)+\lim_{z\to(1-i)}(2)}\\ &= \frac{(1-i)^2-i3}{(1-i)+2} = \frac{-i2-i3}{3-i} = \frac{-i5\,(3+i)}{10} = \frac{1-i3}{2}\end{aligned}$$

(c)

$$\begin{aligned}\lim_{z\to -i}\left(\frac{z^3-i}{z^2+1}\right) &= \lim_{z\to -i}\left(\frac{(z+i)\left(z^2-iz+i^2\right)}{(z+i)(z-i)}\right) = \lim_{z\to -i}\left(\frac{z^2-iz-1}{z-i}\right) = \frac{\lim_{z\to -i}\left(z^2-iz-1\right)}{\lim_{z\to -i}(z-i)}\\ &= \frac{(-i)^2-i(-i)-1}{-i-i} = \frac{-1-1-1}{-i2} = \frac{3}{i2} = -i\frac{3}{2}\end{aligned}$$

(d)

$$\lim_{z\to(2+i)}\left(\frac{\operatorname{Re} z}{\operatorname{Im} z^*}\right) = \frac{\lim_{z\to(2+i)}\operatorname{Re} z}{\lim_{z\to(2+i)}\operatorname{Im} z^*} = \frac{\operatorname{Re}\left(\lim_{z\to(2+i)} z\right)}{\operatorname{Im}\left(\lim_{z\to(2+i)} z^*\right)} = \frac{\operatorname{Re}(2+i)}{\operatorname{Im}(2+i)^*} = \frac{\operatorname{Re}(2+i)}{\operatorname{Im}(2-i)} = \frac{2}{-1} = -2$$

(e)

$$\lim_{z\to\infty}\left(\frac{5z^3+z-1}{-z^3+i8}\right) = \lim_{z\to\infty}\left(\frac{5+\frac{1}{z^2}-\frac{1}{z^3}}{-1+i\frac{8}{z^3}}\right) = \frac{\lim_{z\to\infty}\left(5+\frac{1}{z^2}-\frac{1}{z^3}\right)}{\lim_{z\to\infty}\left(-1+i\frac{8}{z^3}\right)} = \frac{5+0-0}{-1+i0} = -5$$

(f)

$$\lim_{z\to\infty}\left(\frac{z^2-i\pi}{7z^3}\right) = \lim_{z\to\infty}\left(\frac{z^2}{7z^3}-i\frac{\pi}{7z^3}\right) = \lim_{z\to\infty}\left(\frac{1}{7z}\right) - i\lim_{z\to\infty}\left(\frac{\pi}{7z^3}\right) = 0 - i0 = 0$$

(g)

$$\lim_{z\to 0}\left(\frac{iz^5}{z^2+9}\right) = \frac{\lim_{z\to 0}\left(iz^5\right)}{\lim_{z\to 0}\left(z^2+9\right)} = \frac{i0^5}{0^2+9} = \frac{0}{9} = 0$$

(h)

$$\lim_{z\to(1+i\pi/4)}\left(e^z\right) = e^{\lim_{z\to(1+i\pi/4)} z} = e^{1+i\pi/4} = e\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = \frac{e}{\sqrt{2}} + i\frac{e}{\sqrt{2}} \simeq 1.9221 + i1.9221$$

This result can also be obtained more directly from the result of part (d) of Problem 4.
(i)

$$\begin{aligned}\lim_{z\to(3-i\pi)}\left(e^{z+i}\right) &= e^{\lim_{z\to(3-i\pi)}(z+i)} = e^{(3-i\pi)+i} = e^{3+i(1-\pi)} = e^{3+i}e^{-i\pi}\\ &= -e^{3+i} = -e^3(\cos 1 + i\sin 1) \simeq -10.8523 - i16.9014\end{aligned}$$

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discussions) we simply restrict our attention (in the questions of the present Problem and in similar questions) to the first quadrant (i.e. when z goes to infinity from within the first quadrant) although this restriction may not be needed in most cases. It is noteworthy that infinity may be defined (seemingly unambiguously and uniquely) by $\lim_{z\to 0}\frac{1}{z}$ although this does not seem very relevant to our purpose and in our context. In fact, representing the infinity by a single point (as done for instance in stereographic projection and as indicated in § 1.5) is useful but it cannot solve every problem. Anyway, representing the infinity by a single point is commonly used in complex analysis and this could make the above comment redundant (or at least simplistic).

This result can also be obtained more directly from the result of part (d) of Problem 4 (in association with Eq. 73 noting that $e^{z+i} = e^i e^z$).

(**j**)

$$\begin{aligned}\lim_{z\to(5-i\pi/3)}(\cos z) &= \cos\left(\lim_{z\to(5-i\pi/3)} z\right) = \cos(5-i\pi/3) = \frac{e^{i(5-i\pi/3)}+e^{-i(5-i\pi/3)}}{2}\\ &= \frac{e^{\pi/3+i5}+e^{-\pi/3-i5}}{2} \simeq 0.4539 - i1.1980\end{aligned}$$

where we used Eq. 11 to evaluate $\cos(5-i\pi/3)$. This result can also be obtained more directly from the result of part (e) of Problem 4.

(**k**)

$$\lim_{z\to i3}\left(\cos z^2\right) = \cos\left(\lim_{z\to i3} z^2\right) = \cos(i^2 3^2) = \cos(-9) = \cos(9) \simeq -0.9111$$

(**l**)

$$\lim_{z\to i\sqrt{2}}\left(\sin e^{z^4}\right) = \sin\left(\lim_{z\to i\sqrt{2}} e^{z^4}\right) = \sin\left(e^{\lim_{z\to i\sqrt{2}}(z^4)}\right) = \sin\left(e^4\right) \simeq -0.9288$$

6. Determine if the following limits do exist or not:

(**a**) $\lim_{z\to0}\left(\frac{z^*}{z}\right)$. (**b**) $\lim_{z\to0}\left(\frac{z}{\operatorname{Re} z}\right)$. (**c**) $\lim_{z\to0}\left(\frac{\operatorname{Im} z^*}{z}\right)$.

(**d**) $\lim_{z\to0}\left(\frac{z^4+4}{z^3}\right)$. (**e**) $\lim_{z\to(2+i)}\left(\frac{z}{\operatorname{Re} z}\right)$. (**f**) $\lim_{z\to\infty}\left(\frac{z}{e^z}\right)$.

Answer:

(**a**)

$$\lim_{z\to0}\left(\frac{z^*}{z}\right) = \lim_{x,y\to0}\left(\frac{x-iy}{x+iy}\right)$$

Now, if we approach $z=0$ horizontally along the real axis (and hence $y=0$) then:

$$\lim_{z\to0}\left(\frac{z^*}{z}\right) = \lim_{x\to0}\left(\frac{x}{x}\right) = \lim_{x\to0}(1) = 1$$

On the other hand, if we approach $z=0$ vertically along the imaginary axis (and hence $x=0$) then:

$$\lim_{z\to0}\left(\frac{z^*}{z}\right) = \lim_{y\to0}\left(\frac{-iy}{iy}\right) = \lim_{y\to0}(-1) = -1$$

Because a limit does exist only if the function converges to the same value regardless of the direction of approach, we conclude that this limit does not exist.

(**b**)

$$\lim_{z\to0}\left(\frac{z}{\operatorname{Re} z}\right) = \lim_{x,y\to0}\left(\frac{x+iy}{x}\right) = \lim_{x,y\to0}\left(1+i\frac{y}{x}\right)$$

Now, if we approach $z=0$ along the line $y=x$ then:

$$\lim_{z\to0}\left(\frac{z}{\operatorname{Re} z}\right) = \lim_{x\to0}\left(1+i\frac{x}{x}\right) = \lim_{x\to0}(1+i) = 1+i$$

On the other hand, if we approach $z=0$ along the line $y=-x$ then:

$$\lim_{z\to0}\left(\frac{z}{\operatorname{Re} z}\right) = \lim_{x\to0}\left(1+i\frac{-x}{x}\right) = \lim_{x\to0}(1-i) = 1-i$$

Because the value of the limit depends on the direction of approach, this limit does not exist.

(**c**)

$$\lim_{z\to0}\left(\frac{\operatorname{Im} z^*}{z}\right) = \lim_{x,y\to0}\left(\frac{-y}{x+iy}\right)$$

Now, if we approach $z = 0$ horizontally along the real axis (and hence $y = 0$) then:

$$\lim_{z\to 0}\left(\frac{\operatorname{Im} z^*}{z}\right) = \lim_{x\to 0}\left(\frac{0}{x}\right) = \lim_{x\to 0}(0) = 0$$

On the other hand, if we approach $z = 0$ vertically along the imaginary axis (and hence $x = 0$) then:

$$\lim_{z\to 0}\left(\frac{\operatorname{Im} z^*}{z}\right) = \lim_{y\to 0}\left(\frac{-y}{iy}\right) = \lim_{y\to 0}(i) = i$$

Because the value of the limit depends on the direction of approach, this limit does not exist.
(**d**)

$$\lim_{z\to 0}\left(\frac{z^4+4}{z^3}\right) = \lim_{z\to 0}\left(\frac{z^4}{z^3}+\frac{4}{z^3}\right) = \lim_{z\to 0}(z) + \lim_{z\to 0}\left(\frac{4}{z^3}\right) = 0 + \lim_{z\to 0}\left(\frac{4}{z^3}\right) = \lim_{z\to 0}\left(\frac{4}{z^3}\right)$$

Now, $\lim_{z\to 0}\left(4/z^3\right)$ does not exist (because the denominator vanishes at $z = 0$) and hence there is no limit. In fact, we may also write:

$$\lim_{z\to 0}\left(\frac{z^4+4}{z^3}\right) = \lim_{z\to 0}\left(\frac{4}{z^3}\right) = \infty$$

However, we should note that ∞ is not a limit in the technical sense.
(**e**)

$$\lim_{z\to(2+i)}\left(\frac{z}{\operatorname{Re} z}\right) = \lim_{x\to 2,y\to 1}\left(\frac{x+iy}{x}\right) = \lim_{x\to 2,y\to 1}\left(1+i\frac{y}{x}\right)$$

Now, if we approach $z = 2 + i$ from any direction within the z plane we get the same value, i.e. $\lim_{x\to 2,y\to 1}\left(1+i\frac{y}{x}\right) = 1+i\frac{1}{2}$. Because this value is independent of the direction of approach this limit does exist. We note that this result may also be obtained more simply by using the quotient rule of limits (i.e. Eq. 76), that is:

$$\lim_{z\to(2+i)}\left(\frac{z}{\operatorname{Re} z}\right) = \frac{\lim_{z\to(2+i)} z}{\lim_{z\to(2+i)} \operatorname{Re} z} = \frac{2+i}{2} = 1+i\frac{1}{2}$$

(**f**)

$$\begin{aligned}
\lim_{z\to\infty}\left(\frac{z}{e^z}\right) &= \lim_{z\to\infty}\left(\frac{z}{1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\cdots}\right) = \lim_{z\to\infty}\left(\frac{1}{\frac{1}{z}+1+\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\cdots}\right)\\
&= \frac{1}{\lim_{z\to\infty}\left(\frac{1}{z}+1+\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\cdots\right)}\\
&= \frac{1}{\lim_{z\to\infty}\left(\frac{1}{z}\right)+\lim_{z\to\infty}\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\cdots\right)} = \frac{1}{0+\infty} = \frac{1}{\infty} = 0
\end{aligned}$$

So, this limit does exist.[90]

7. Verify the following statements about complex functions $f(z)$:[91]
(**a**) If f is analytic then it is continuous.
(**b**) If f is continuous then it is bounded.
(**c**) If f is analytic then it is bounded.
(**d**) If f is not bounded then it is not continuous.

[90] In fact, this result can be obtained more simply by using L'Hospital's rule (which is not within the scope of this book although it should be known from calculus).

[91] We are assuming proper domains over which these statements apply. We also note that the purpose of the present Problem (and its alike) is to provide a general factual statements about limit-related attributes without the distraction of going through the intricate technical details of these statements. We also refer the reader to § 1.5 for the technical meaning of the terms involved in these statements.

(**e**) If f is not bounded then it is not analytic.
Answer:
(**a**) If f is analytic at a given point z_0 then $f(z_0)$ exists (see Problem 9 of § 1.5). So, all we need to do to establish that f is continuous is to show that $\lim_{z\to z_0} f(z) - f(z_0) = 0$ (see Eq. 15), that is:

$$\begin{aligned}
\lim_{z\to z_0} f(z) - f(z_0) &= \lim_{z\to z_0} f(z) - \lim_{z\to z_0} f(z_0) && \big[f(z_0) \text{ is fixed; see Eq. 71}\big] \\
&= \lim_{z\to z_0} [f(z) - f(z_0)] && \text{(see Eq. 74)} \\
&= \lim_{z\to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} \times (z - z_0)\right] && (z - z_0 \neq 0 \text{ according to definition of limit}) \\
&= \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \times \lim_{z\to z_0} (z - z_0) && \text{(see Eq. 75)} \\
&= f'(z_0) \times \lim_{z\to z_0} (z - z_0) && (f \text{ is analytic; also see Eq. 17}) \\
&= f'(z_0) \times 0 && \text{(see Eqs. 74, 72 and 71)} \\
&= 0
\end{aligned}$$

We would like to remark that $z - z_0 \neq 0$ (in line 3 above) and $\lim_{z\to z_0} (z - z_0) = 0$ (according to line 6 above) are consistent because the latter is about the value of the limit (see Problem 2).
(**b**) If f is continuous over a region R then for every point z_0 in R the limit $\lim_{z\to z_0} f(z)$ does exist (which means it is finite) and we have (see Eq. 15 and the surrounding text):

$$f(z_0) = \lim_{z\to z_0} f(z)$$

which means $f(z_0)$ is finite.[92] Accordingly, $|f(z_0)|$ is finite and hence according to the definition of "bounded" (noting that z_0 represents the entire R) f is bounded over R.
(**c**) If f is analytic then it is continuous (according to part a) and hence it is bounded (according to part b).
(**d**) This is the contrapositive of the statement of part (b) and hence its truthfulness follows from the truthfulness of the statement of part (b) which we already established.
(**e**) This is the contrapositive of the statement of part (c) and hence its truthfulness follows from the truthfulness of the statement of part (c) which we already established.

8. Give an example of a complex function that is analytic nowhere in the complex plane.
Answer: The obvious example of such a function is $f(z) = z^*$. This is because for any point z_0 in the complex plane we have (see Eq. 16):

$$\begin{aligned}
\frac{df}{dz} &= \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z\to 0} \frac{(z_0 + \Delta z)^* - z_0^*}{\Delta z} = \lim_{\Delta z\to 0} \frac{z_0^* + \Delta z^* - z_0^*}{\Delta z} \\
&= \lim_{\Delta z\to 0} \frac{\Delta z^*}{\Delta z} = \lim_{\Delta x,\Delta y\to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}
\end{aligned}$$

Now, if we approach z_0 horizontally (where $\Delta y = 0$) then:

$$\frac{df}{dz} = \lim_{\Delta x\to 0} \frac{\Delta x}{\Delta x} = \lim_{\Delta x\to 0} 1 = 1$$

while if we approach z_0 vertically (where $\Delta x = 0$) then:

$$\frac{df}{dz} = \lim_{\Delta y\to 0} \frac{-i\Delta y}{+i\Delta y} = \lim_{\Delta y\to 0} -1 = -1$$

Because this limit fails to exist (since its value depends on the direction of approach), $\frac{df}{dz}$ does not exist at any point in the complex plane and hence the function $f(z) = z^*$ is analytic nowhere. The reader is also referred to part (d) of Problem 7 of § 3.1 about the non-analyticity of z^* (also see Problem 19 of § 3.1 for a more general result about z^*).

[92] Being finite means both its real and imaginary parts are finite.

1.10 The Calculus of Complex Variables

We mean by this title the operations of differentiation and integration of complex functions of complex variables. In general, the calculus of complex variables follows similar rules to those of the calculus of real variables (with some exceptions that will be clarified later). We will discuss these issues in more details in the Problems of this section as well as in the upcoming parts of the book. However, before that we should remark that the calculus in the complex domain (like the calculus in the real domain) is based on the concept and techniques of limits (which we investigated rather briefly but sufficiently in § 1.9). We should also remark that there are two main approaches (or forms) in the formulations of the calculus of complex variables.[93] One approach is based on dealing with the complex functions as functions of a single variable (i.e. the complex variable z) while the other approach is based on dealing with the complex functions as functions of multiple variables (i.e. the real and imaginary variables x, y in the Cartesian form, and the modulus and argument variables r, θ in the polar form).[94] The first approach formulates the given problem using one complex independent variable (i.e. z) and one complex dependent variable (i.e. w) where each one of these variables combines the real and imaginary parts[95] in a single package and hence we essentially deal with problems of the type $w = f(z)$. The second approach formulates the given problem using two scalar[96] independent variables (i.e. the real variable x and the imaginary variable y) and two scalar dependent variables (i.e. the real variable u and the imaginary variable v) and hence we essentially deal with problems of the type $u + iv = f(x + iy)$ which can be split into two scalar functions representing two parts: real part $u = f_1(x, y)$ and imaginary part $v = f_2(x, y)$. For example, in the differential calculus of complex variables we may deal with the problem of differentiating the function $w = f(z) = z^2$ as a complex problem and hence we formulate it as $\frac{dw}{dz} = 2z$. We may also deal with it as a scalar problem where $z^2 = u + iv = (x^2 - y^2) + i2xy$ and hence we formulate it as (see § Problem 1 of § 3.1):[97]

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i2y \qquad \text{and} \qquad \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = 2x + i2y \tag{79}$$

Similarly, in the integral calculus of complex variables we may deal with the problem of integrating the function z^2 as a complex problem and hence we formulate it as $\int w\,dz = \frac{z^3}{3} + C$ (with C being a complex constant). We may also deal with it as a scalar problem where $w = u + iv = (x^2 - y^2) + i2xy$ and hence we formulate it as:

$$\int w\,dz = \int (u + iv)\,(dx + i\,dy) = \int (u\,dx - v\,dy) + i \int (u\,dy + v\,dx) \tag{80}$$

which may be separated into two scalar integrals: a real part integral

$$\int (u\,dx - v\,dy) = \int (x^2 - y^2)\,dx - \int 2xy\,dy = \frac{x^3}{3} - xy^2 + C_1$$

and an imaginary part integral

$$\int (u\,dy + v\,dx) = \int (x^2 - y^2)\,dy + \int 2xy\,dx = -\frac{y^3}{3} + x^2y + C_2$$

[93] In fact, these two approaches are not restricted to the calculus of complex variables but they permeate almost all aspects and topics of complex analysis, as we saw earlier and will see more later on (refer for example to § 1.11).

[94] The reader should note that in this book we deal only with single complex variable problems (see § 1.1) and hence the above remark should be understood within this context.

[95] In the rest of this remark we restrict our attention to the Cartesian form and its variables (to avoid complicating the text) noting that we also have polar form (as indicated above) with different types of variables (i.e. r and θ) although even in the polar form we still have real and imaginary parts (which may not be separated explicitly as can be seen for example in $re^{i\theta} = r\cos\theta + ir\sin\theta$ where they are separated explicitly on the right but not on the left).

[96] Refer to 1.1 for the meaning of scalar.

[97] We may also formulate it as four separate equations, that is:

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 2y \qquad \frac{\partial v}{\partial y} = 2x$$

where C_1 and C_2 are real constants (with $C = C_1 + iC_2$).

We should now draw the attention to the following useful remarks:

• Higher order derivatives of a complex function are calculated by repeated application of the differentiation operation (subject to the stated rules of differentiation) since the higher order derivatives are just derivatives of the lower order derivatives (which are just complex functions).

• An important difference between differentiable real functions and differentiable (or analytic) complex functions is that the latter are infinitely differentiable (i.e. they have derivatives of all orders) while the former are not necessarily so (i.e. we have some differentiable real functions which have derivatives only to a certain finite order; see Problem 6 of § 1.5). Also, see Problem 6 of § 4.3.

• As in real analysis, we have complex definite and indefinite integrals as well as complex line (or curve or path or contour) integrals where all these integrals are defined and formulated similarly to their real counterparts. Also, the relation between differentiation and integration (as reverse operations) is reflected in complex analysis as in real analysis (subject, of course, to certain conditions and restrictions).

• An important remark (reflecting the correspondence and similarity between the calculus in the real domain and in the complex domain) is the issue of line (or path or contour) integral which is intimately linked to the concept of definite integral. In brief, if we assume the function $w(z)$ to be single-valued and continuous in a given region R in the complex plane then we can define a line integral over a given curve C inside R, i.e. $\int_C w\,dz$.[98] This integral can be path-dependent or path-independent and this should lead us to the concept of definite integral, i.e. $\int_a^b w\,dz$ where a and b are given complex numbers representing the start and end points of the path C. As we will see (refer for instance to § 4.1 and Problem 4 of § 3.2), the existence of such an integral requires w to have an analytic antiderivative and this requires w to be analytic (i.e. over a simply-connected region in which the integral is supposed to be path-independent).

• The vital importance and usefulness of complex integration (and contour integration in particular) is not restricted to the evaluation and calculation of specific integrals, but it is an effective tool for obtaining numerous properties of complex functions and deriving many theoretical results, and hence it is met throughout the entire subject of complex analysis. As we will see (refer to § 4.2), Cauchy's theorem (which is the main pillar of complex analysis and enters directly and indirectly in the derivation of many theorems and theoretical results) is an *integral* theorem. This also applies to many other theorems of complex analysis.

Problems

1. It is stated in the text that "the calculus of complex variables follows similar rules to those of the calculus of real variables". Provide more clarification about this issue with regard to the differentiation of functions of complex variables.

   Answer: We can say briefly: if $f(z)$ is a function of a complex variable z then its derivative has similar definition and formulation to those of the derivative of its real counterpart $f(x)$ (where x is a real variable), that is:
   $$\frac{df(z)}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{81}$$
   assuming this limit does exist.[99] Accordingly, the rules of differentiating complex polynomial or exponential functions for instance are essentially identical to the rules of differentiating their real counterparts, e.g. $\frac{dz^2}{dz} = 2z$ and $\frac{de^z}{dz} = e^z$.

2. Show that the following rules of differentiation apply to complex variables as to real variables (assuming that the basic definition of derivative as a limit and the usual rules of limits apply; see for example Eq. 16 as well as Eqs. 71-78 and Problem 4 of § 1.9):

   (**a**) Product rule. (**b**) Quotient rule. (**c**) Chain rule. (**d**) Sum rule.
   (**e**) Constant rule. (**f**) Multiple constant rule. (**g**) Power rule.

[98] It should be understood that C in all line integrals (like this one) should be smooth (at least piecewise). Moreover, the integrand (i.e. w) is assumed to be continuous over C.

[99] As hinted earlier and will be elaborated further later on, this assumption is more restrictive in the case of functions of complex variables than functions of real variables, and this in fact is the origin of many differences between the calculus of real functions and the calculus of complex functions.

Answer:[100] Let $f(z), f_1(z), f_2(z), \cdots, f_n(z)$ be complex functions of the complex variable $z = x + iy$ and let the prime mean total derivative with respect to z, i.e. d/dz.
(**a**) If $f = f_1 f_2$ then from the definition of derivative (see Eq. 16) we have:

$$\begin{aligned}
\frac{df}{dz} &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{f_1(z+\Delta z)f_2(z+\Delta z) - f_1(z)f_2(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{f_1(z+\Delta z)f_2(z+\Delta z) - f_1(z+\Delta z)f_2(z) + f_1(z+\Delta z)f_2(z) - f_1(z)f_2(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \left[f_1(z+\Delta z)\frac{f_2(z+\Delta z) - f_2(z)}{\Delta z} + \frac{f_1(z+\Delta z) - f_1(z)}{\Delta z} f_2(z) \right] \\
&= f_1 \frac{df_2}{dz} + \frac{df_1}{dz} f_2 = f_1 f_2' + f_1' f_2
\end{aligned}$$

which is the familiar product rule of differentiation. This can be easily generalized (by iteration and induction) to products of more than two functions (i.e. $f = f_1 \times f_2 \times \cdots \times f_n$).
(**b**) If $f = f_1/f_2$ ($f_2 \neq 0$) then from the definition of derivative we have:

$$\begin{aligned}
\frac{df}{dz} &= \lim_{\Delta z \to 0} \frac{\frac{f_1(z+\Delta z)}{f_2(z+\Delta z)} - \frac{f_1(z)}{f_2(z)}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\frac{f_1(z+\Delta z)f_2(z)}{f_2(z+\Delta z)f_2(z)} - \frac{f_1(z)f_2(z+\Delta z)}{f_2(z)f_2(z+\Delta z)}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\frac{f_1(z+\Delta z)f_2(z)}{f_2(z+\Delta z)f_2(z)} - \frac{f_1(z)f_2(z)}{f_2(z+\Delta z)f_2(z)} + \frac{f_1(z)f_2(z)}{f_2(z+\Delta z)f_2(z)} - \frac{f_1(z)f_2(z+\Delta z)}{f_2(z)f_2(z+\Delta z)}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \left[\frac{\frac{f_1(z+\Delta z)f_2(z)}{f_2(z+\Delta z)f_2(z)} - \frac{f_1(z)f_2(z)}{f_2(z+\Delta z)f_2(z)}}{\Delta z} - \frac{\frac{f_1(z)f_2(z+\Delta z)}{f_2(z)f_2(z+\Delta z)} - \frac{f_1(z)f_2(z)}{f_2(z+\Delta z)f_2(z)}}{\Delta z} \right] \\
&= \lim_{\Delta z \to 0} \left[\frac{\frac{f_1(z+\Delta z)-f_1(z)}{\Delta z} f_2(z)}{f_2(z+\Delta z)f_2(z)} - \frac{f_1(z)\frac{f_2(z+\Delta z)-f_2(z)}{\Delta z}}{f_2(z)f_2(z+\Delta z)} \right] \\
&= \frac{\frac{df_1}{dz} f_2}{f_2^2} - \frac{f_1 \frac{df_2}{dz}}{f_2^2} = \frac{f_1' f_2 - f_1 f_2'}{f_2^2}
\end{aligned}$$

which is the familiar quotient rule of differentiation.
(**c**) If f is an analytic function of g which is an analytic function of z, i.e. $f = f(g(z))$, then from the definition of derivative we have:[101]

$$\begin{aligned}
\frac{df}{dz} &= \lim_{\Delta z \to 0} \frac{f(g(z+\Delta z)) - f(g(z))}{\Delta z} \\
&= \lim_{\Delta z \to 0} \left[\frac{f(g+\Delta g) - f(g)}{g(z+\Delta z) - g(z)} \times \frac{g(z+\Delta z) - g(z)}{\Delta z} \right] \\
&= \lim_{\Delta z \to 0} \frac{f(g+\Delta g) - f(g)}{\Delta g} \times \lim_{\Delta z \to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z}
\end{aligned}$$

[100] We should note that although the following proofs are identical to the proofs in real analysis (with the replacement of x by z), we are using the rules of manipulating complex variables (as demonstrated earlier) and hence these proofs are showing the applicability of the same proofs (and hence the rules of differentiation) to complex functions as to real functions.
[101] We should note that in this proof we avoid going through some technicalities and hence the proof is just an outline.

Now, if we note that g is a continuous[102] function of z (and hence $\Delta g \to 0$ as $\Delta z \to 0$) then we have:

$$\frac{df}{dz} = \lim_{\Delta g \to 0} \frac{f(g+\Delta g) - f(g)}{\Delta g} \times \lim_{\Delta z \to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z} = \frac{df}{dg} \times \frac{dg}{dz}$$

which is the familiar chain rule of differentiation.
(**d**) If $f = f_1 \pm f_2$ then from the definition of derivative we have:

$$\begin{aligned}
\frac{df}{dz} &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\left[f_1(z+\Delta z) \pm f_2(z+\Delta z)\right] - \left[f_1(z) \pm f_2(z)\right]}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\left[f_1(z+\Delta z) - f_1(z)\right] \pm \left[f_2(z+\Delta z) - f_2(z)\right]}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{f_1(z+\Delta z) - f_1(z)}{\Delta z} \pm \lim_{\Delta z \to 0} \frac{f_2(z+\Delta z) - f_2(z)}{\Delta z} \\
&= \frac{df_1}{dz} \pm \frac{df_2}{dz} = f_1' \pm f_2'
\end{aligned}$$

which is the familiar sum rule of differentiation. This can be easily generalized (by iteration and induction) to sums of more than two functions (i.e. $f = f_1 \pm f_2 \pm \cdots \pm f_n$).
(**e**) If $f = C$ (with C being a complex constant) then from the definition of derivative we have:

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{C - C}{\Delta z} = \lim_{\Delta z \to 0} \frac{0}{\Delta z} = 0$$

which is the familiar constant rule of differentiation.
(**f**) If $f = Cf_1$ then by the product and constant rules (which we proved in parts a and e) the derivative of f is:

$$\frac{df}{dz} = f_1 \frac{dC}{dz} + C\frac{df_1}{dz} = 0 + C\frac{df_1}{dz} = C\frac{df_1}{dz} = Cf_1'$$

which is the familiar multiple constant rule of differentiation.
(**g**) If $f = z$ then from the definition of derivative we have:

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z+\Delta z) - z}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = 1$$

Similarly, if $f = z^2$ then by the product rule (plus the result of $f = z$) the derivative of f is:

$$\frac{df}{dz} = \frac{d}{dz}(zz) = \frac{dz}{dz}z + z\frac{dz}{dz} = 1 \times z + z \times 1 = z + z = 2z$$

Thus, for some n (i.e. $n = 1, 2$) we have $dz^n/dz = nz^{n-1}$. Now, we show that if this rule applies to n then it should also apply to $n+1$ and hence by induction the power rule is valid. It is obvious that if $f = z^{n+1}$ then by the product rule (plus $dz/dz = 1$ and $dz^n/dz = nz^{n-1}$) we have:

$$\frac{df}{dz} = \frac{dz^{n+1}}{dz} = \frac{d}{dz}(zz^n) = \frac{dz}{dz}z^n + z\frac{dz^n}{dz} = z^n + znz^{n-1} = z^n + nz^n = (n+1)z^n$$

As we see, if $f = z^{n+1}$ then $df/dz = (n+1)z^n$ (assuming the validity of $dz^n/dz = nz^{n-1}$ for some integers) and hence by induction the power rule is valid.

[102] Since g is analytic it is continuous (see part a of Problem 7 of § 1.9).

3. Find the derivatives (with respect to z) of the following complex functions (assuming that the known rules of differentiation, as established in real analysis, apply to complex functions as to real functions):[103]

(a) $f = (-z^3 + 3 - i\pi)^4$. **(b)** $f = \cos^2(3z^2 - i2z + e)$. **(c)** $f = z^5 e^{3z^2 - z - i2}$.

(d) $f = \sinh(-z)\cosh(iz^5)$. **(e)** $f = \sinh^3(e^{\cos(i3z)})$. **(f)** $f = ze^{\sin z} - z^2 e^{-iz}$.

Answer:[104]

$$\textbf{(a)}\quad \frac{df}{dz} = 4(-z^3 + 3 - i\pi)^3(-3z^2) = -12z^2(-z^3 + 3 - i\pi)^3 = 12z^2(z^3 - 3 + i\pi)^3$$

$$\textbf{(b)}\quad \frac{df}{dz} = 2\cos(3z^2 - i2z + e)\left[-\sin(3z^2 - i2z + e)\right](6z - i2)$$
$$= (i4 - 12z)\cos(3z^2 - i2z + e)\sin(3z^2 - i2z + e)$$

$$\textbf{(c)}\quad \frac{df}{dz} = 5z^4 e^{3z^2 - z - i2} + z^5 e^{3z^2 - z - i2}(6z - 1) = \left(6z^6 - z^5 + 5z^4\right) e^{3z^2 - z - i2}$$

$$\textbf{(d)}\quad \frac{df}{dz} = -\cosh(-z)\cosh(iz^5) + i5z^4\sinh(-z)\sinh(iz^5)$$

$$\textbf{(e)}\quad \frac{df}{dz} = 3\sinh^2(e^{\cos(i3z)})\cosh(e^{\cos(i3z)})\left(e^{\cos(i3z)}\{-i3\sin(i3z)\}\right)$$
$$= -i9\,e^{\cos(i3z)}\sinh^2(e^{\cos(i3z)})\cosh(e^{\cos(i3z)})\sin(i3z)$$

$$\textbf{(f)}\quad \frac{df}{dz} = e^{\sin z} + ze^{\sin z}\cos z - 2ze^{-iz} + iz^2 e^{-iz} = (1 + z\cos z)\,e^{\sin z} + \left(iz^2 - 2z\right)e^{-iz}$$

4. Evaluate the following complex definite integrals (assuming that the known rules of integration, as established in the real analysis, apply to complex integrals):[105]

(a) $\int_0^3 (2 - i\pi)\,dz$. **(b)** $\int_{-1}^{+1}(z - i)^3\,dz$. **(c)** $\int_0^{\pi/2} e^{i6z}\,dz$. **(d)** $\int_i^{i\pi}\sin(i2z)\,dz$.

(e) $\int_{i2}^{i9} z^{1/3}\,dz$. **(f)** $\int_{-2-i5}^{-6+i} z^2\,dz$. **(g)** $\int_1^i (z/e^{z^2})\,dz$. **(h)** $\int_{i3}^{i9}\cosh(iz)\,dz$.

Answer:

$$\textbf{(a)}\quad \int_0^3 (2 - i\pi)\,dz = \big[(2 - i\pi)\,z\big]_0^3 = 3\,(2 - i\pi) - 0 = 6 - i3\pi$$

$$\textbf{(b)}\quad \int_{-1}^{+1}(z - i)^3\,dz = \left[\frac{(z - i)^4}{4}\right]_{-1}^{+1} = \frac{(1 - i)^4}{4} - \frac{(-1 - i)^4}{4} = \frac{-4}{4} - \frac{-4}{4} = 0$$

$$\textbf{(c)}\quad \int_0^{\pi/2} e^{i6z}\,dz = \left[\frac{e^{i6z}}{i6}\right]_0^{\pi/2} = \frac{e^{i3\pi}}{i6} - \frac{e^{i0}}{i6} = \frac{\cos 3\pi + i\sin 3\pi}{i6} - \frac{e^0}{i6} = \frac{-1}{i6} - \frac{1}{i6} = -\frac{1}{i3} = \frac{i}{3}$$

$$\textbf{(d)}\quad \int_i^{i\pi}\sin(i2z)\,dz = \left[\frac{-\cos(i2z)}{i2}\right]_i^{i\pi} = \frac{-\cos(-2\pi)}{i2} - \frac{-\cos(-2)}{i2} = \frac{-1 + \cos(2)}{i2}$$

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[103] In fact, we have two types of rules of differentiation: general rules (i.e. not function-specific) such as the product and sum rules, and specific (or function-specific) rules such as the rules of differentiating exponential and cosine functions. The general rules (or at least what we need of these rules) have already been investigated in Problem 2 (noting that some of the rules in Problem 2 are not completely general) where we found that these rules apply to complex functions as to real functions. Regarding the specific rules, we will show later in the book (see for example chapter 2) that these rules apply to complex functions as to real functions (at least for the functions that we use here). So, the present Problem is just a warming up exercise. In fact, some of these rules can be gathered from what have been established already (e.g. the differentiation rule of polynomials which is based on the sum, multiple constant and power rules which we established in Problem 2).

[104] The results in some parts of this answer are kept in almost their basic form because although more simplification is possible it requires pending investigations. Moreover, the purpose of this Problem is to introduce the subject of differentiation of complex functions and its real-like style and hence no benefit is gained from further simplification.

[105] Again, this is a warming up exercise.

$$= i\frac{1-\cos(2)}{2} \simeq i0.7081$$

(e) $$\int_{i2}^{i9} z^{1/3}\,dz = \left[\frac{3}{4}z^{4/3}\right]_{i2}^{i9} = \frac{3}{4}(i9)^{4/3} - \frac{3}{4}(i2)^{4/3} = \frac{3}{4}\left(9^{4/3} - 2^{4/3}\right) i^{4/3} \simeq 12.1507\, i^{4/3}$$
$$= 12.1507\left(e^{i\pi/2}\right)^{4/3} = 12.1507\, e^{i2\pi/3} = 12.1507\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$
$$= 12.1507\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \simeq -6.07534 + i10.5228$$

(f) $$\int_{-2-i5}^{-6+i} z^2\,dz = \left[\frac{z^3}{3}\right]_{-2-i5}^{-6+i} = \frac{(-6+i)^3}{3} - \frac{(-2-i5)^3}{3} = \frac{-198+i107}{3} - \frac{142+i65}{3}$$
$$= -\frac{340}{3} + i14$$

(g) $$\int_1^i \frac{z}{e^{z^2}}\,dz = \int_1^i z e^{-z^2}\,dz = \left[\frac{-e^{-z^2}}{2}\right]_1^i = \frac{-e^1}{2} - \frac{-e^{-1}}{2} = \frac{e^{-1}-e^1}{2} = \sinh(-1) \simeq -1.1752$$

(h) $$\int_{i3}^{i9} \cosh(iz)\,dz = \Big[-i\sinh(iz)\Big]_{i3}^{i9} = -i\sinh(-9) + i\sinh(-3) = i(\sinh 9 - \sinh 3) \simeq i4041.5240$$

5. Give some general properties of complex integrals and compare these properties to the corresponding properties of real integrals.
Answer: Some of the general properties of complex integrals are:
• Linearity which means that the integral of a linear combination of functions $f_n(z)$ is the linear combination of their integrals, i.e.

$$\int\left(\sum_n a_n f_n\right)dz = \sum_n a_n\left(\int f_n dz\right) \qquad (a_n \text{ are constants})$$

• Reversibility which means that the path integral of a function $f(z)$ over a given curve C in a given direction $\downarrow$ is the negative of that path integral over that curve in the opposite direction $\uparrow$, i.e.

$$\int_{C\downarrow} f\,dz = -\int_{C\uparrow} f\,dz$$

• Additivity which means that the path integral of a function $f(z)$ over a given curve C that is made of a union of n curves (i.e. $C = C_1 \cup C_2 \cup \cdots \cup C_n$) is the sum of the integrals of that function over these individual n curves, i.e.

$$\int_C f\,dz = \int_{C_1} f\,dz + \int_{C_2} f\,dz + \cdots + \int_{C_n} f\,dz$$

• Boundedness which means that if the modulus (or magnitude) of a given function $f(z)$ over a given curve C of length l does not exceed a given number M (i.e. $|f| \le M$) then the modulus of the path integral of f over C is bounded and is restricted by the relation:

$$\left|\int_C f\,dz\right| \le Ml$$

All these properties of complex integrals are similar to these properties in real integrals.

1.11 Complex Functions as Mappings

As indicated earlier (see § 1.5), complex function can be defined generically by the following relation:

$$w(z) = f(z) = u(x,y) + iv(x,y) \tag{82}$$

where w $(= u + iv)$ and z $(= x + iy)$ are complex variables while u, v, x, y are real variables (with each of u and v being a function of x and y in general as indicated in the above equation). Accordingly, complex function can be seen as a mapping (or transformation) from the z plane to the w plane, as depicted graphically in Figure 11. In chapter 2 we will investigate some of the complex functions that are commonly encountered in complex analysis. However, we should note that almost all commonly-known real functions have complex counterparts and hence what is in chapter 2 is just a sample. Moreover, the investigation of complex functions (either explicitly or implicitly) is everywhere in complex analysis (as well as in this book) and hence we do not need to go through detailed investigations here although we will give some illuminating examples and instances in the Problems of this section. Also, the types of transformation achieved by complex functions and relations will be investigated more thoroughly in the future (see for instance chapter 6).

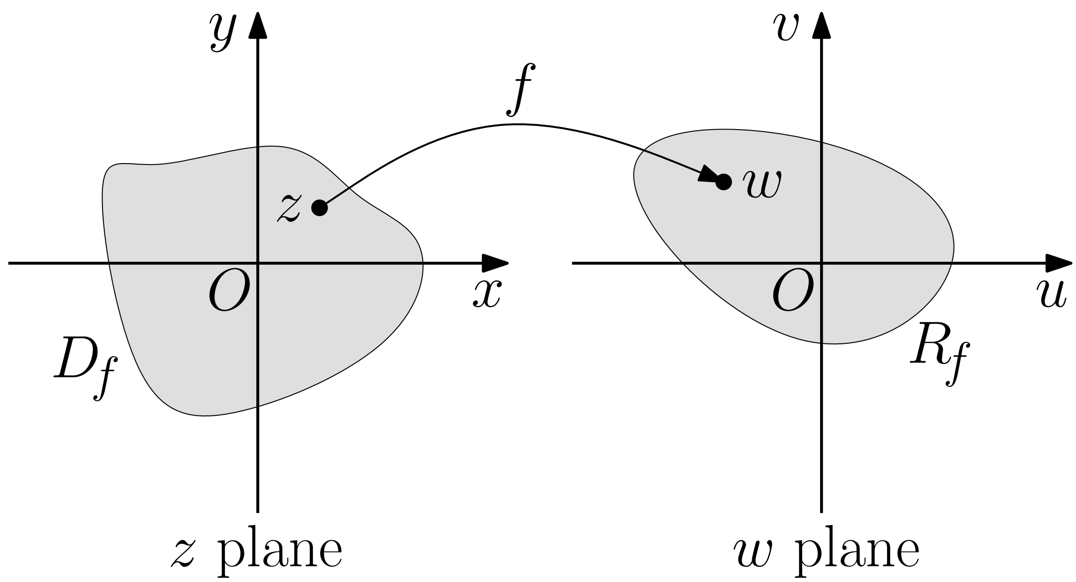

Figure 11: A complex function f depicted as a mapping (or transformation) from its domain D_f in the z plane to its range R_f in the w plane. See § 1.11.

We should now draw the attention to the following points about complex functions:

• It is important to remind the reader here of our earlier remark (see § 1.10) about the two main approaches in the formulations of complex analysis where in one approach a single complex dependent variable w is considered as an image (or map) of a single function of another single independent complex variable z and hence we write $w = f(z)$ which expresses a "lump" relation between the complex variable w and the complex variable z represented by a single complex function f, while in another approach a complex dependent variable is considered as a composite made of two real variables u, v that depend on other two real independent variables x, y and hence we write $u(x, y) + iv(x, y) = f(x + iy)$ where each of u and v is a function of x and y and hence we may write $u = f_1(x, y)$ and $v = f_2(x, y)$ to express "individual" relations between the real u, v and the real x, y by two real functions f_1, f_2. In fact, we already met many examples of both of these two approaches in the past and will meet many more in the future. The distinction between these two approaches may be specially obvious with regard to the calculus of complex variables as we saw earlier and will see more later (refer for example to § 1.10, § 3.1 and § 3.2).

• Although graphic representation of complex functions (i.e. functions that map complex variables from the z plane onto complex variables in the w plane) requires two 2D plots (as demonstrated in Figure 11) because it is impossible to have more than three variables on an ordinary graph, it is still possible to use 3D plots to graphically represent and illustrate complex functions where two (independent) variables (i.e. x and y) are used to represent the z plane while a third (dependent) variable is used to represent a complex variable w or an attribute of a complex variable. For example, if w is real (i.e. it has no

imaginary part) or imaginary (i.e. it has no real part) then we can use a 3D plot to graphically represent a complex function. Similarly, if w is strictly complex (i.e. it has both real and imaginary parts) then we can use the third variable to represent an attribute of w such as its modulus or argument or real part or imaginary part. Some of these 3D graphic plots and illustrations will be given in the future (see for example Figures 16, 17 and 31). In fact, the use of 3D plots to represent complex variables provides a handy and effective method for visualizing and appreciating complex functions (which the above method of using two 2D plots, as demonstrated in Figure 11, fails to do) although we generally need two 3D plots to fully represent the function (i.e. one plot for the real part or modulus and one plot for the imaginary part or argument).

• In complex analysis we have two types of functions: single-valued functions and multi-valued functions.[106] For example, polynomials are single-valued functions because for each given value of the independent variable z we have a single value of the function (e.g. if $z = 1 + i$ then the linear polynomial $2z - i5$ evaluates to the single value $2 - i3$), while the n^{th} root functions have n distinct values (see § 1.8.11). In fact, the multiple values can be infinite (especially when considering non-distinct values) and hence we may classify multi-valued complex functions as finitely multi-valued functions and infinitely multi-valued functions (where "finitely" and "infinitely" refer to the number of values) beside the classification of complex functions in general as single-valued and multi-valued. As we saw earlier (refer to § 1.8.7), $\arg(z)$ is an example of a function of z with infinite number of values. However, it is important to notice that "finitely multi-valued" and "infinitely multi-valued" are used in the literature of complex analysis in two meanings: distinct values and non-distinct values. For example, both the n^{th} root function and the $\arg(z)$ function are infinitely multi-valued when we look to their values in general (i.e. regardless of being distinct or not) but the n^{th} root function is finitely multi-valued and the $\arg(z)$ function is infinitely multi-valued when we look to their distinct values since the former has only n distinct values (because the other values are repetitive) while the latter has infinitely-many values.

• Based on the previous point, each value (or cycle or member) of a multi-valued function is commonly labeled as a branch of the function.[107] Moreover, one of these values (or branches) is chosen conventionally as the principal or main value (or branch) where it is commonly used to represent the function.[108] The principal branch (or value) is commonly distinguished by an uppercase initial, e.g. $\text{Arg}(z)$ and $\text{Ln}(z)$ are the principal branches of $\arg(z)$ and $\ln(z)$.[109] The purpose of representing the function by a (single) principal branch is to achieve continuity and uniqueness (i.e. being single-valued) as well as maintaining consistency and avoiding vagueness.[110]

• A complex function $w = f(z)$ has an inverse function $z = f^{-1}(w)$ if f is one-to-one (where the domain and range are exchanged by this inversion). The mathematical form of the inverse function f^{-1} is usually obtained by algebraic manipulation (if possible) of the expression of the original function f through solving for z to obtain an expression for z in terms of w (as done in real analysis with real functions). For example, if $w = f(z) = \frac{1}{z-1}$ then $z = f^{-1}(w) = \frac{1}{w} + 1$. There are some standard functions with standard inverses like the exponential and natural logarithm functions (see § 2.2) or the trigonometric and inverse trigonometric functions (see § 2.3 and 2.4).

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[106] We remind the reader of our previous remark (see footnote [37] on page 18) about the eligibility of multi-valued relations to be titled "functions".

[107] In fact, this statement is rather loose because more restrictions are required for the technical definition of branch (as we saw and will see). However, this is not the main objective of our discussion here and hence we tolerate this laxity.

[108] More clearly and precisely, a branch of a complex function is the function as represented continuously by a single cycle over its domain, i.e. each branch (or cycle) of the function represents the entire function uniquely and continuously as a single-valued function over its domain. Accordingly, the principal branch of a complex function is the function as represented by the cycle of its principal value over its (modified) domain. Also, see § 1.5 about the terminology of branches of multi-valued complex functions.

[109] It should be noted that uppercase initialization (as in $\text{Arg}\, z$ and $\text{Ln}\, z$) may be used rather loosely to indicate the principal value without considering the strict technical definition of branch. It should also be noted that there are other conventions about the use of uppercase initialization.

[110] We should note that the intended "continuity" in the above discussion may have two different meanings (or rather instantiations) since it partly belongs to the relation between the different branches and partly belongs to the continuity of the principal branch itself (and indeed any branch) which is achieved by modifying the domain of the function, i.e. by excluding the branch cut (see § 1.5). For example, to make a continuous principal branch of $\sqrt{z}$ the negative real axis (which is the branch cut of $\sqrt{z}$) should be removed from the domain of $\sqrt{z}$ (see Problem 10).

• Noting that a complex function $f(z)$ can be seen as a mapping (or transformation), an inverse mapping (or transformation) can be defined by the inverse of f (i.e. f^{-1}) when f is invertible (as explained in the previous point).
• The behavior of a complex function $f(z)$ at infinity (i.e. when $z \to \infty$) can be determined from the behavior of the function $f(1/z)$ at 0 (i.e. when $z \to 0$). For example, the limit of $f(z) = (2/z) + 1$ as $z \to \infty$ can be obtained from taking the limit of $f(1/z) = 2z + 1$ at $z = 0$, i.e. $\lim_{z\to\infty} \left[(2/z) + 1\right] = \lim_{z\to 0} \left[2z + 1\right] = 1$ (see for example Problem 3 of § 1.9).
• As indicated earlier (see Problem 1 of § 1.2), a correct general complex formulation should yield its real version when the complex variable is replaced by a real variable and hence if a general complex formulation failed to produce its real counterpart when the complex variable is replaced by a real variable then this formulation should be rejected without further ado. For example, if our mathematical derivation led us to the equation $\cos^2 z + \sin^2 z = 2$ then this result should be discarded immediately because $\cos^2 x + \sin^2 x = 1 \neq 2$. Accordingly, all the complex functions which are defined on a domain that includes real variables should produce their real counterparts correctly (with all their characteristic behavior and consequences). Any complex function that fails to meet this criterion and pass this test should be rejected. In fact, this correspondence between the real and complex formulations may also be used (with certain conditions and restrictions) to obtain one of these formulations if the other is known.

Problems

1. What is the difference in the graphic representation of real functions[111] and complex functions?
   Answer: While the representation of real functions requires a single 2D plot (e.g. y versus x in the Cartesian form), the representation of complex functions requires two 2D plots (i.e. one represents the z plane and the other represents the w plane). In fact, this similarly applies to the 3D graphic representation where we generally need two 3D plots to fully represent the function, i.e. one plot for the real part or modulus and one plot for the imaginary part or argument.
   Note 1: although in principle graphic representation of complex functions requires two 2D or 3D plots, in practice a single 2D or 3D plot can be sufficient by using multiple labeling for the axes (e.g. x, u label for the independent-variable axis and y, v label for the dependent-variable axis in the case of 2D, and x label for one axis and y label for a second axis as well as $\operatorname{Re} z, \operatorname{Im} z$ for the third axis in the case of 3D). However, this technique of multiple labeling does not work in all cases due to practical reasons such as large difference in the range of the variables and obstruction of the view (due to overlapping) when the independent and dependent variables represent regions and surfaces rather than curves (although different coloring and transparency or different perspective point can be used in some cases to overcome some of these difficulties). In Figure 12 the technique of multiple labeling is demonstrated for the case of 2D plots.
   Note 2: the advantage of the technique of multiple labeling (when it is viable) is that it is compact (i.e. it demands less space) since it requires only one plot. Moreover, it illustrates the relationship between the pre-image (or inverse image or source) and the image more clearly and precisely since they are both on the same plot and hence their relative size and position are visualized and realized more realistically and proportionately than in the case of using two plots.
2. Mention one limitation of complex functions.
   Answer: They are basically restricted to 2D correlations and mappings (where both the independent and dependent variables are made of real and imaginary parts) although generalizations and extensions may be made.
3. Find u and v of the following complex functions (of $z = x + iy$).
   (**a**) $w = 3z - i\pi + 2$. (**b**) $w = z^2 - 1$. (**c**) $w = (z^*)^2$. (**d**) $w = \left(z^2\right)^*$.
   (**e**) $w = 2z^3 + 5z^2 - z - 7$. (**f**) $w = (z - 3 + i)^{-1}$. (**g**) $w = e^{z^2}$. (**h**) $w = dz^3/dz$.
   (**i**) $w = \cos\left(\operatorname{Re} z^2\right)$. (**j**) $w = |z|\, z$. (**k**) $w = e^{-i\operatorname{Arg} z}$.
   Answer:
   (**a**) $w = 3(x + iy) - i\pi + 2 = (3x + 2) + i(3y - \pi)$. Hence, $u = 3x + 2$ and $v = 3y - \pi$.

[111] Real functions here (and in similar contexts) means single-variable scalar functions, i.e. with one independent variable and one dependent variable.

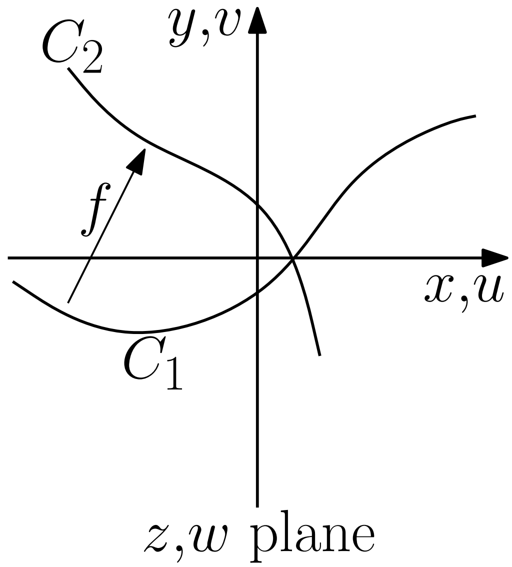

Figure 12: Graphic illustration of the technique of multiple labeling of axes where the curve C_1 in the z plane is mapped by the function f onto the curve C_2 in the w plane, and hence the x, y labeling of the axes belongs to C_1 while the u, v labeling of the axes belongs to C_2. See Problem 1 of § 1.11.

(**b**) $w = (x+iy)^2 - 1 = x^2 + i2xy - y^2 - 1 = (x^2 - y^2 - 1) + i(2xy)$. Hence, $u = x^2 - y^2 - 1$ and $v = 2xy$.
(**c**) $w = (x-iy)^2 = (x^2 - y^2) - i2xy$. Hence, $u = x^2 - y^2$ and $v = -2xy$.
(**d**) $w = \left[(x+iy)^2\right]^* = \left[(x^2-y^2) + i2xy\right]^* = (x^2 - y^2) - i2xy$. Hence, $u = x^2 - y^2$ and $v = -2xy$.
(**e**)

$$\begin{aligned} w &= 2(x+iy)^3 + 5(x+iy)^2 - (x+iy) - 7 \\ &= 2(x^3 + i3x^2y - 3xy^2 - iy^3) + 5(x^2 + i2xy - y^2) - (x+iy) - 7 \\ &= 2x^3 + i6x^2y - 6xy^2 - i2y^3 + 5x^2 + i10xy - 5y^2 - x - iy - 7 \\ &= \left(2x^3 - 6xy^2 + 5x^2 - 5y^2 - x - 7\right) + i\left(6x^2y - 2y^3 + 10xy - y\right) \end{aligned}$$

Hence, $u = 2x^3 - 6xy^2 + 5x^2 - 5y^2 - x - 7$ and $v = 6x^2y - 2y^3 + 10xy - y$.
(**f**)

$$\begin{aligned} w &= \frac{1}{x+iy-3+i} = \frac{1}{(x-3)+i(y+1)} = \frac{(x-3)-i(y+1)}{(x-3)^2+(y+1)^2} \\ &= \frac{x-3}{(x-3)^2+(y+1)^2} - i\frac{y+1}{(x-3)^2+(y+1)^2} \end{aligned}$$

Hence, $u = \frac{x-3}{(x-3)^2+(y+1)^2}$ and $v = \frac{-(y+1)}{(x-3)^2+(y+1)^2}$.
(**g**)

$$w = e^{(x+iy)^2} = e^{(x^2-y^2)+i2xy} = e^{(x^2-y^2)}\left[\cos(2xy) + i\sin(2xy)\right] = e^{(x^2-y^2)}\cos(2xy) + ie^{(x^2-y^2)}\sin(2xy)$$

Hence, $u = e^{(x^2-y^2)}\cos(2xy)$ and $v = e^{(x^2-y^2)}\sin(2xy)$.
(**h**) $w = dz^3/dz = 3z^2 = 3\,(x+iy)^2 = 3\left(x^2 + i2xy - y^2\right) = 3x^2 + i6xy - 3y^2 = (3x^2 - 3y^2) + i6xy$. Hence, $u = 3x^2 - 3y^2$ and $v = 6xy$.
(**i**) We have $z^2 = (x^2-y^2) + i2xy$ and hence $\operatorname{Re} z^2 = x^2 - y^2$. Therefore, $w = \cos\left(\operatorname{Re} z^2\right) = \cos(x^2-y^2) = \cos(x^2-y^2) + i0$. Hence, $u = \cos(x^2-y^2)$ and $v = 0$.

(**j**) $w = |z|\,z = \sqrt{x^2+y^2}(x+iy) = (x\sqrt{x^2+y^2}) + i(y\sqrt{x^2+y^2})$. Hence, $u = x\sqrt{x^2+y^2}$ and $v = y\sqrt{x^2+y^2}$.
(**k**) Noting that $\operatorname{Arg} z$ is real (since $-\pi < \operatorname{Arg} z \le \pi$), we have $w = e^{-i\operatorname{Arg} z} = \cos(\operatorname{Arg} z) - i\sin(\operatorname{Arg} z)$. Hence, $u = \cos(\operatorname{Arg} z)$ and $v = -\sin(\operatorname{Arg} z)$.

4. What are the images (in the w plane) of the following complex numbers (in the z plane) under the given complex functions (or mappings):
(**a**) $z = 1 - i$ under the function $f(z) = 5z^2 - 2z + i7$.
(**b**) $z = 2 + i5$ under the function $f(z) = e^{2z}$.
(**c**) $z = \pi + i2$ under the function $f(z) = 1/z^2$.
Answer:
(**a**) $w = f(1-i) = 5\,(1-i)^2 - 2\,(1-i) + i7 = 5\,(1 - i2 - 1) - 2\,(1-i) + i7 = -2 - i$.
(**b**) $w = f(2+i5) = e^{2(2+i5)} = e^{4+i10} = e^4\,(\cos 10 + i\sin 10) \simeq -45.8118 - i29.7025$.
(**c**) $w = f(\pi + i2) = \frac{1}{(\pi+i2)^2} = \frac{1}{\pi^2-4+i4\pi} = \frac{\pi^2-4-i4\pi}{(\pi^2-4)^2+(4\pi)^2} \simeq 0.03051 - i0.06533$.

5. What are the images (in the w plane) of the following sets of complex numbers (in the z plane) under the given complex functions (or mappings):
(**a**) The straight line that connects the points $z_1 = -2 - i$ and $z_2 = 2 + i3$ under the function $f(z) = -z + 7 + i3$.
(**b**) The straight line $\operatorname{Re}(z) = 2$ under the function $f(z) = z^2$.
(**c**) The square with vertices $z_1 = -1 - i$, $z_2 = 1 - i$, $z_3 = 1 + i$ and $z_4 = -1 + i$ under the function $f(z) = i2z$.
Also, demonstrate these mappings (or transformations) graphically.
Answer:
(**a**) The equation of a straight line in the z plane is $y = ax + b$ (with a and b being real constants) and hence the straight line in the z plane can be represented by the complex equation $z = x + i(ax+b)$. On applying this equation to z_1 and z_2 (since they are on the straight line) respectively we get:

$$\begin{aligned} -1 = y &= ax + b = a(-2) + b = -2a + b \\ 3 = y &= ax + b = a(+2) + b = +2a + b \end{aligned}$$

On solving these equations simultaneously we get $a = b = 1$. So, the straight line in the z plane that connects z_1 and z_2 is represented by the complex equation $z = x + i\,(x+1)$.
Now, to obtain the shape of the image we map a general point on the line in the z plane onto the w plane, that is:

$$w = f(z) = f(x + i\,[x+1]) = -(x + i\,[x+1]) + 7 + i3 = (7 - x) + i(2 - x) = u + iv$$

Accordingly, $u = 7 - x$ (and hence $x = 7 - u$) and $v = 2 - x$ (and hence $x = 2 - v$). Since, $x = x$ we should have $7 - u = 2 - v$ and hence $v = u - 5$ (which is an equation of a straight line in the w plane). Therefore, the straight line in the z plane that connects z_1 and z_2 is mapped under this function onto the straight line $w = u + i(u-5)$ in the w plane. This mapping is demonstrated graphically in Figure 13.
(**b**) We have:

$$w = f(z) = z^2 = (x+iy)^2 = x^2 + i2xy + (iy)^2 = \left(x^2 - y^2\right) + i2xy$$

and hence $u = x^2 - y^2$ and $v = 2xy$. Now, $x = \operatorname{Re}(z) = 2$ (as given in the question) leads (by substitution) to $u = 4 - y^2$ and $v = 4y$. Accordingly, $y = v/4$ which we substitute into $u = 4 - y^2$ to get:

$$u = 4 - \left(\frac{v}{4}\right)^2 = 4 - \frac{v^2}{16}$$

This is the required image in the w plane (since it is a relation between u and v). As we see, the image is a parabola whose axis of symmetry is along the u axis of the w plane with vertex at $(u, v) = (4, 0)$

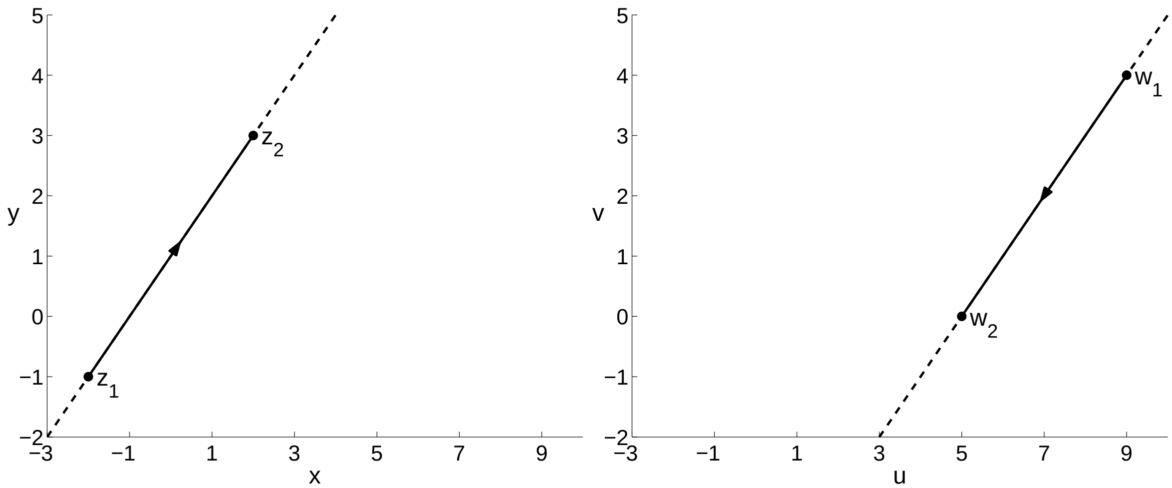

Figure 13: Graphic representation of the mapping of part (a) of Problem 5 of § 1.11 where the straight line that connects the points $z_1 = -2 - i$ and $z_2 = 2 + i3$ in the z plane (left frame) is mapped under the function $f(z) = -z + 7 + i3$ onto the straight line that connects the points $w_1 = f(z_1) = 9 + i4$ and $w_2 = f(z_2) = 5 + i0$ in the w plane (right frame). The direction of progression is indicated by the arrow heads, i.e. as z moves from z_1 towards z_2 along the line $\overline{z_1 z_2}$ in the z plane, w moves from w_1 towards w_2 along the line $\overline{w_1 w_2}$ in the w plane.

and it opens to the left. This mapping is demonstrated graphically in Figure 14.[112]
(**c**) If we follow a similar approach to that of part (a) we will find that this type of mapping (i.e. linear) transforms straight lines to straight lines. Therefore, all we need to do is to find the images of the vertices of the square and connect them by straight line segments (corresponding to the sides of the transformed square). Now:

$$\begin{aligned} w_1 &= f(z_1) = i2(-1 - i) = +2 - i2 \\ w_2 &= f(z_2) = i2(+1 - i) = +2 + i2 \\ w_3 &= f(z_3) = i2(+1 + i) = -2 + i2 \\ w_4 &= f(z_4) = i2(-1 + i) = -2 - i2 \end{aligned}$$

Accordingly, the square with vertices z_1, z_2, z_3, z_4 in the z plane is mapped onto the square with vertices w_1, w_2, w_3, w_4 in the w plane. This mapping is demonstrated graphically in Figure 15.

6. Make 3D plots of the following (and comment on the plots):
(**a**) The modulus of e^z over the region $-2 \le x \le 2$ and $-2 \le y \le 2$.
(**b**) The real part of e^{-iz} over the region $-2\pi \le x \le 2\pi$ and $-1 \le y \le 2$.
Answer: To avoid any potential confusion we should clarify a (rather obvious) point that is: when we talk in this Problem (and its alike) about real and imaginary or modulus and argument (or any similar attribute) it should be obvious that they belong to the image of the function in the w plane and hence the real and imaginary are u and v respectively, and the modulus and argument are $|w|$ and $\arctan(v/u)$.
(**a**) The modulus $|e^z|$ of the function e^z is given by:

$$|e^z| = \left|e^{x+iy}\right| = \left|e^x e^{iy}\right| = |e^x| \left|e^{iy}\right| = |e^x| \left|\cos y + i \sin y\right| = |e^x| \sqrt{\cos^2 y + \sin^2 y} = |e^x| = e^x$$

[112] In Figure 14 we selected a line segment connecting three specific points (i.e. z_1, z_2, z_3) to demonstrate some of the features of this mapping (noting that the mapping in the question is not restricted to this line segment). This also applies to Figure 13 of part (a) where the segment connecting z_1 and z_2 is highlighted for the same purpose (although the subject of mapping in the question is the entire line).

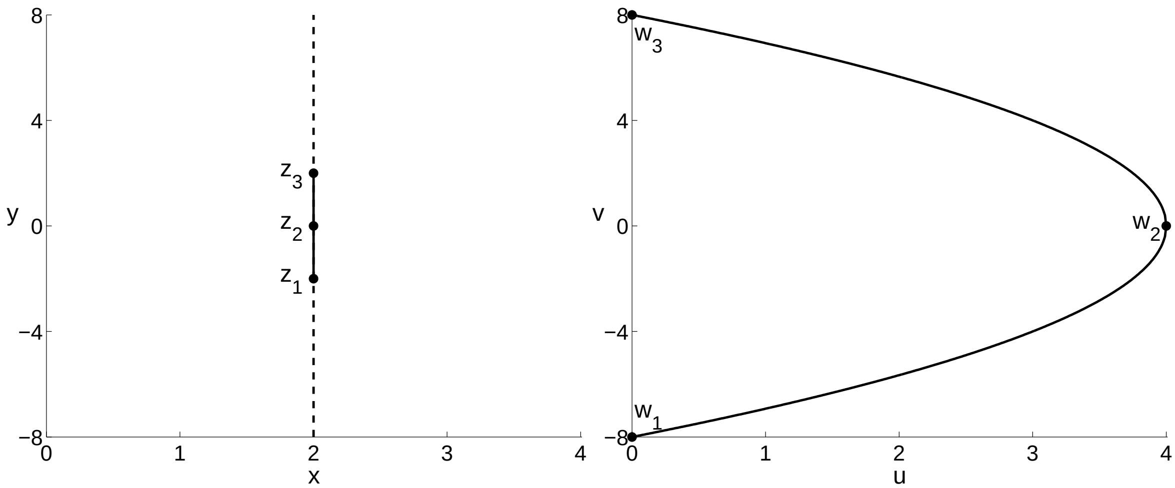

Figure 14: Graphic representation of the mapping of part (b) of Problem 5 of § 1.11 where the straight line $\text{Re}(z) = 2$ in the z plane (left frame) is mapped under the function $f(z) = z^2$ onto the parabola $u = 4 - \frac{v^2}{16}$ in the w plane (right frame). As we see, the points $z_1 = 2 - i2$, $z_2 = 2 + i0$ and $z_3 = 2 + i2$ are mapped (respectively) onto the points $w_1 = 0 - i8$, $w_2 = 4 + i0$ and $w_3 = 0 + i8$. As z progresses in the direction $z_1 \to z_2 \to z_3$ along the line $\overline{z_1 z_2 z_3}$ in the z plane, w progresses in the sense $w_1 \to w_2 \to w_3$ (i.e. anticlockwise) along the parabola in the w plane.

This is plotted over the region $-2 \le x \le 2$ and $-2 \le y \le 2$ in Figure 16.
Comment: $|e^z| = e^x$ is an exponential function of x alone with no dependency on y and hence what we have in Figure 16 is an exponential rise in the positive x direction which forms an "exponential cylinder". Accordingly, as we move along lines of constant x in the y direction we see straight lines parallel to the y axis and the xy plane, while as we move along lines of constant y in the positive x direction we see ascending exponential curves where all these exponential curves are identical.
(**b**) The function e^{-iz} is given by:

$$e^{-iz} = e^{-i(x+iy)} = e^{y-ix} = e^y(\cos x - i\sin x) = e^y \cos x - ie^y \sin x$$

Hence, its real part is $\text{Re}(e^{-iz}) = e^y \cos x$. This is plotted over the region $-2\pi \le x \le 2\pi$ and $-1 \le y \le 2$ in Figure 17.
Comment: $\text{Re}(e^{-iz}) = e^y \cos x$ is a superposition of a "wavy" cosine function in the x direction and an exponential function in the y direction and hence the cosine waves in the x direction are moderated by an exponential function in the y direction where the positive peaks and the negative troughs of the waves determine whether the exponential ascends or descends (i.e. rises up or drops down in the positive y direction). Accordingly, as we move along lines of constant x in the positive y direction we see (ascending or descending) exponential curves but they become straight lines when $\cos x = 0$, i.e. when $x = \left(n + \frac{1}{2}\right)\pi$, while as we move along lines of constant y in the x direction we see ordinary cosine waves whose magnitudes are scaled by the constant e^y.

7. Given that $f(z) = u(x, y) + iv(x, y)$ is a complex function (with u and v being real functions of x and y), show that f has a limit L at a given point z_0 *iff* both u and v have limits L_u and L_v at z_0, that is:

$$\lim_{z\to z_0} f(z) = L \qquad \textit{iff} \qquad \lim_{z\to z_0} u = L_u \quad \text{and} \quad \lim_{z\to z_0} v = L_v$$

where $L = L_u + iL_v$.
Answer: If $\lim_{z\to z_0} f(z) = L$ then we have:

$$L_u + iL_v = L = \lim_{z\to z_0} f(z) = \lim_{z\to z_0} \left[u + iv\right] = \lim_{z\to z_0} u + i \lim_{z\to z_0} v$$

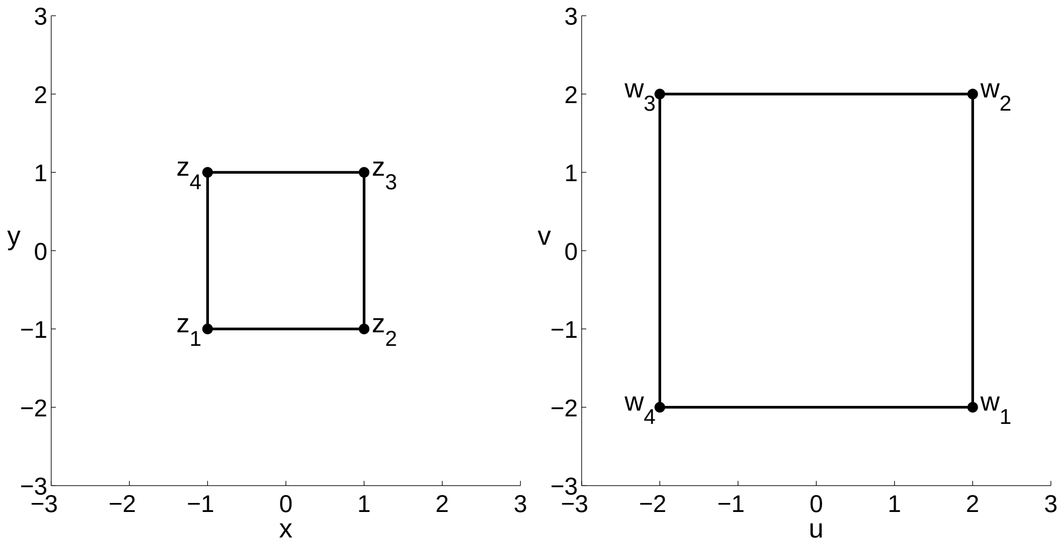

Figure 15: Graphic representation of the mapping of part (c) of Problem 5 of § 1.11 where the square with vertices $z_1 = -1 - i$, $z_2 = 1 - i$, $z_3 = 1 + i$ and $z_4 = -1 + i$ in the z plane (left frame) is mapped under the function $f(z) = i2z$ onto the square with vertices $w_1 = 2 - i2$, $w_2 = 2 + i2$, $w_3 = -2 + i2$ and $w_4 = -2 - i2$ in the w plane (right frame). As z progresses in the sense $z_1 \to z_2 \to z_3 \to z_4 \to z_1$ (i.e. anticlockwise) along the square in the z plane, w progresses in the sense $w_1 \to w_2 \to w_3 \to w_4 \to w_1$ (i.e. anticlockwise) along the square in the w plane.

where the sum and multiple constant rules of limits (see Eqs. 74 and 73) are used in the last equality. On comparing the left side and the right side we get $\lim_{z\to z_0} u = L_u$ and $\lim_{z\to z_0} v = L_v$.
On the other hand, if $\lim_{z\to z_0} u = L_u$ and $\lim_{z\to z_0} v = L_v$ then we have:

$$L_u + iL_v = \lim_{z\to z_0} u + i \lim_{z\to z_0} v = \lim_{z\to z_0} \big[u + iv\big] = \lim_{z\to z_0} f(z) = L$$

8. Show that the statement of Problem 7 also applies to continuity, that is:

$f(z)$ is continuous at z_0 *iff* u is continuous at z_0 and v is continuous at z_0

Answer: This is because if $f(z)$ is continuous at z_0 then $\lim_{z\to z_0} f(z) = f(z_0)$ and hence (noting that $z_0 = x_0 + iy_0$):

$$u(x_0, y_0) + iv(x_0, y_0) = f(z_0) = \lim_{z\to z_0} f(z) = \lim_{z\to z_0} \big[u + iv\big] = \lim_{z\to z_0} u + i \lim_{z\to z_0} v = \lim_{\substack{x\to x_0\\ y\to y_0}} u + i \lim_{\substack{x\to x_0\\ y\to y_0}} v$$

On comparing the left side and the right side we get:

$$\lim_{\substack{x\to x_0\\ y\to y_0}} u = u(x_0, y_0) \qquad \text{and} \qquad \lim_{\substack{x\to x_0\\ y\to y_0}} v = v(x_0, y_0)$$

i.e. both u and v are continuous at $z_0 = x_0 + iy_0$.
Similarly, if both u and v are continuous at z_0 then $\lim_{x\to x_0, y\to y_0} u = u(x_0, y_0)$ and $\lim_{x\to x_0, y\to y_0} v = v(x_0, y_0)$ and hence:

$$f(z_0) = u(x_0, y_0) + iv(x_0, y_0) = \lim_{\substack{x\to x_0\\ y\to y_0}} u + i \lim_{\substack{x\to x_0\\ y\to y_0}} v = \lim_{z\to z_0} u + i \lim_{z\to z_0} v = \lim_{z\to z_0} \big[u + iv\big] = \lim_{z\to z_0} f(z)$$

where the equality $f(z_0) = \lim_{z\to z_0} f(z)$ means that $f(z)$ is continuous at z_0.

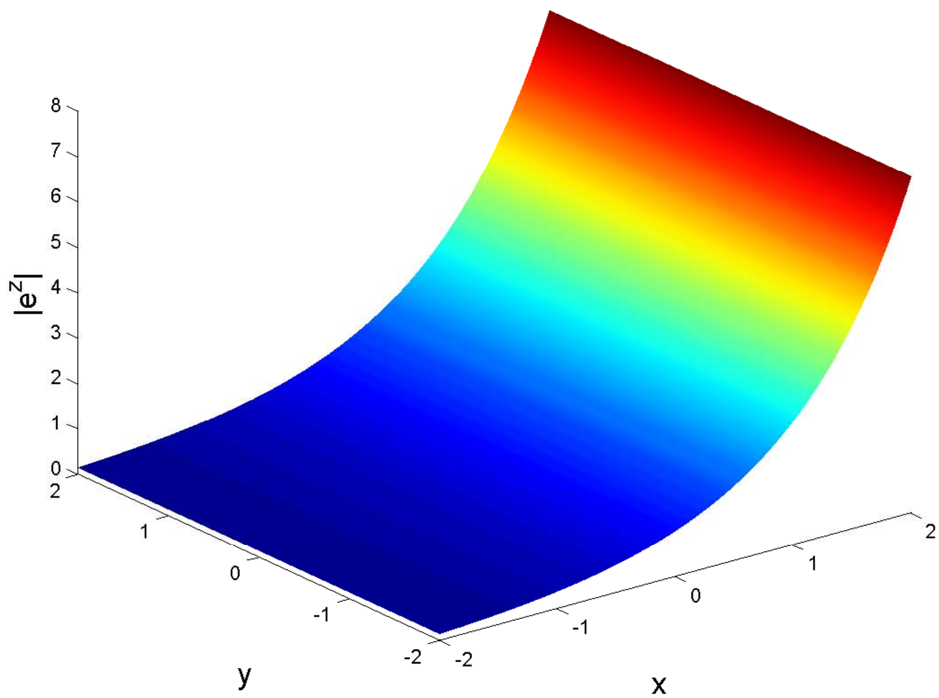

Figure 16: Graphic illustration of the modulus $|e^z| = e^x$ of the complex exponential function e^z over the square region in the z plane defined by $-2 \le x \le 2$ and $-2 \le y \le 2$. See part (a) of Problem 6 of § 1.11.

9. Given that $f(z) = u(x, y) + iv(x, y)$ is a complex function (with u and v being real functions of x and y) that is continuous on a bounded and closed region R, show that f is bounded on R, that is:

$$|f(z)| \le M \qquad \text{(for all } z \text{ in } R)$$

where M is a positive real constant.

Answer: Because f is continuous on R then u and v are continuous on R (see Problem 8) and hence $|f| = \sqrt{u^2 + v^2}$ is continuous on R (since $\sqrt{u^2 + v^2}$ is a composition of a continuous function with the sum of squares of continuous functions; also see Problem 13 of § 1.5). Accordingly, $|f|$ is bounded on R (i.e. $|f| \le M$) and hence by definition f is bounded on R (see § 1.5).

Note: the statement in this Problem is essentially the same as the statement in part (b) of Problem 7 of § 1.9. However, the two statements differ in certain attributes as well as in the method of proving (which follows the difference in the style of phrasing and formulation where in Problem 7 of § 1.9 we follow the "lump" or z approach while here we follow the "split" or u-v approach). We also note that an important difference between the essence of the two proofs (which is based on the difference in formulation) is that in the present proof we employ a fact from real analysis (i.e. if $|f|$ is continuous then $|f|$ is bounded) as the fundamental argument in the proof rather than using the complex argument (in association with the boundedness of $|f|$ and the definition of "bounded") as in Problem 7 of § 1.9. Hence, the proof in the present Problem is essentially based on real analysis (which we take for granted).

10. Determine the points of discontinuity of the complex square root function, i.e. $\sqrt{z}$. Hence, determine the principal branch and the branch cut of this function.

Answer: The discontinuity of $\sqrt{z}$ is on the negative real axis (noting that we restrict our attention here to the discontinuity of the principal branch of $\sqrt{z}$). So, let r be a positive real number and hence the negative real axis is represented by $-r$. Now, let r represent the radius of an origin-centered circle C and let approach the point $-r$ (which is on the negative real axis) along C once from above in the second quadrant and once from below in the third quadrant. Accordingly, if we use the polar form to represent $\sqrt{z}$ (i.e. $\sqrt{re^{i\theta}}$ with θ representing θ_p) and find the limit as the point $-r$ is approached from

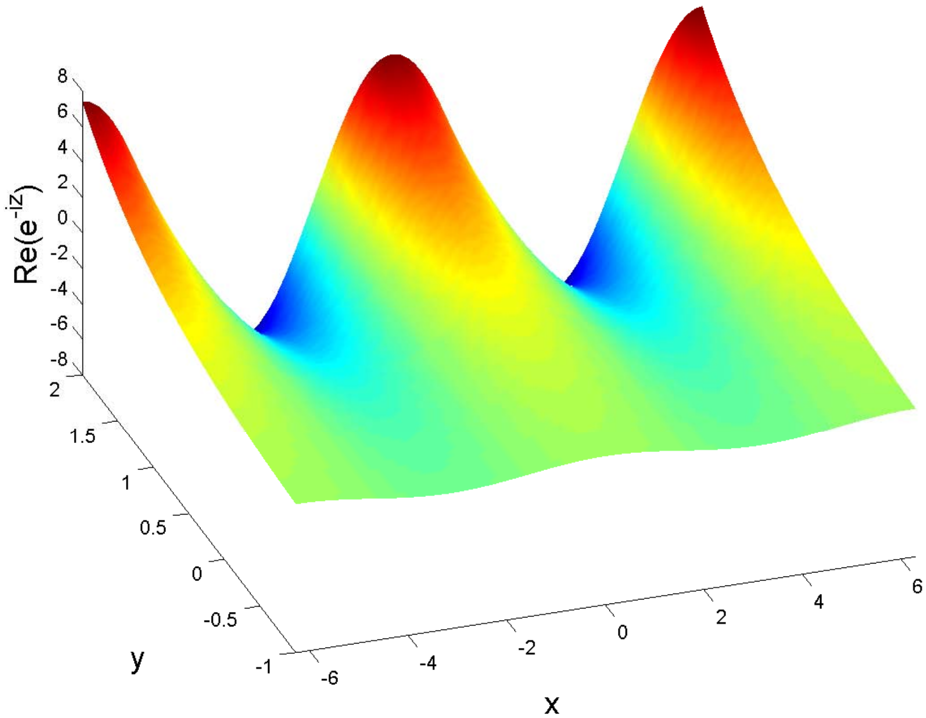

Figure 17: Graphic illustration of the real part $\mathrm{Re}(e^{-iz}) = e^y \cos x$ of the complex exponential function e^{-iz} over the rectangular region in the z plane defined by $-2\pi \le x \le 2\pi$ and $-1 \le y \le 2$. See part (b) of Problem 6 of § 1.11.

above and from below then we get:

$$\begin{aligned}
\lim_{z \to -r\downarrow} \sqrt{z} &= \lim_{\theta \to +\pi} \sqrt{re^{i\theta}} = \lim_{\theta \to +\pi} \sqrt{r} e^{i\theta/2} = \sqrt{r} e^{+i\pi/2} = +i\sqrt{r} \\
\lim_{z \to -r\uparrow} \sqrt{z} &= \lim_{\theta \to -\pi} \sqrt{re^{i\theta}} = \lim_{\theta \to -\pi} \sqrt{r} e^{i\theta/2} = \sqrt{r} e^{-i\pi/2} = -\,i\sqrt{r}
\end{aligned}$$

As we see, we have two different limits and hence the point $-r$ (i.e. any point on the negative real axis which is represented by $-r$) is a point of discontinuity for $\sqrt{z}$. Hence, $\sqrt{z}$ is discontinuous on the entire negative real line. The reader is also referred to the lower frame of Figure 21 which gives an idea about this type of discontinuity (although the Figure belongs to a different function). We should finally note that this sort of discontinuity is not caused by our convention $-\pi < \theta_p \le \pi$ about the range of θ_p (with θ_p being the principal argument) because even if we adopt the other convention (i.e. $0 \le \theta_p < 2\pi$) we get this sort of discontinuity but this time on the positive real axis, that is:

$$\begin{aligned}
\lim_{z \to r\downarrow} \sqrt{z} &= \lim_{\theta \to 0} \sqrt{re^{i\theta}} = \lim_{\theta \to 0} \sqrt{r} e^{i\theta/2} = \sqrt{r} e^{i0} = +\sqrt{r} \\
\lim_{z \to r\uparrow} \sqrt{z} &= \lim_{\theta \to 2\pi} \sqrt{re^{i\theta}} = \lim_{\theta \to 2\pi} \sqrt{r} e^{i\theta/2} = \sqrt{r} e^{i\pi} = -\,\sqrt{r}
\end{aligned}$$

where the arrow $\downarrow$ belongs to the first quadrant while the arrow $\uparrow$ belongs to the fourth quadrant. We should also note that from the above discussion and results we conclude that the range of the argument $\arg(z)$ of the principal branch of $\sqrt{z}$ is $-\pi < \arg(z) < \pi$ inline with the removal of the branch cut which is the negative real axis (or rather the non-positive real axis to include the branch point at the origin which is equivalent to imposing the additional condition $|z| \ne 0$).

Note: although it is generally recognized in the literature that the branch cut can be chosen freely as

long as it achieves its objective of making the function single-valued (by providing a cut that prevents the encircling of the branch point by a complete circle), in the above explanation we assumed that the branch cut is a specific curve (or at least it is more natural than others) to make the concept more comprehensible and to be inline with our convention about the range of the principal argument (which identifies the principal branch and hence the other branches).

11. Referring to the discussion and results of Problem 10, identify two other branches of $\sqrt{z}$. Also, investigate the relation between the different branches of a multi-valued function (as exemplified by the function $\sqrt{z}$).
Answer: For example, one branch corresponds to $-3\pi < \theta < -\pi$ and a second branch corresponds to $\pi < \theta < 3\pi$ (noting that $|z| > 0$ and $\theta = \arg z$). These branches have the same branch cut as the principal branch (which is investigated in Problem 10).
Regarding the relation between the different branches of a multi-valued function, we should note that they are not repetitive in general, i.e. the values of two branches corresponding to a given point in the domain are not necessarily the same (as indicated earlier in Problem 4 of § 1.8.2). For example, the value of the principal branch $\big[$i.e. the branch with $-\pi < \theta < \pi\big]$ of $\sqrt{z}$ at point $z_0 = i$ is:

$$\sqrt{i} = \sqrt{|i|\, e^{i\pi/2}} = \sqrt{|i|}e^{i\pi/4} = e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

while the value of the next branch $\big[$i.e. the branch with $\pi < \theta < 3\pi\big]$ of $\sqrt{z}$ at point $z_0 = i$ is:

$$\sqrt{i} = \sqrt{|i|\, e^{i5\pi/2}} = \sqrt{|i|}e^{i5\pi/4} = e^{i5\pi/4} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

We should also note that with regard to $\sqrt{z}$ (and its alike) the range of the argument of each branch is simply obtained from the range of the argument of the previous branch by adding 2π. For example, the range of the argument of the principal branch is $-\pi < \theta < \pi$ and hence the range of the argument of the next branch is $(-\pi + 2\pi) < \theta < (\pi + 2\pi)$, i.e. $\pi < \theta < 3\pi$ (as seen above for the second branch). In fact, this should indicate (with regard to $\sqrt{z}$) that each branch is the negative of its neighboring branch (on either sides) because since the difference in the range of the argument (which represents z) between two neighboring branches is 2π, then the difference between the values of the two branches (which represent $\sqrt{z}$) is a multiplicative factor of $e^{i(2\pi)/2} = e^{i\pi} = -1$ where the division by 2 comes from the action of the square root function on the argument of z (as seen above). We should therefore have a repetitive cycle of $\cdots + - + - \cdots$ over the consecutive branches. This $+-$ cycle over the consecutive branches may be seen as representing (in the complex analysis) the phenomenon of "positive" and "negative" square roots which we are familiar with in real analysis (where the domain of the square root function is restricted to non-negative or positive real numbers).
We should finally note that since a non-zero complex number has n distinct n^{th} roots (see § 1.8.11), then the n^{th} root function should have n distinct branches where these n branches are repeated in a cycle of order n. We should also note that in accord with this, branch points may be assigned orders corresponding to the number of branches in each cycle, i.e. a branch point is of order n if it requires exactly n (i.e. no less than n) circuits around the point to return to the original value[113] of the multi-valued function. For example, the branch point of $z^{1/2}$ is of order 2, the branch point of $z^{1/3}$ is of order 3, and so on. Similarly, the branch points of $\arg z$ and $\ln z$ are of infinite order because these functions never return to their previous values.

12. Discuss the concept of "Riemann surface" to make a multi-valued function single-valued (and continuous as well).
Answer: If we represent each value (or branch) of a multi-valued function by a sheet corresponding to the entire complex plane (where these sheets are on the top of each other) and then we made a cut along a line or a curve of all these sheets and connected one side of the cut of each sheet to the opposite side of the cut of the next sheet then we will get a continuous surface (like spiral). This construction is supposed to make a multi-valued function single-valued because each one of the multiple values will

[113] "Original value" here means the value of the first branch in the n-order cycle.

belong to only one sheet (i.e. the function is single-valued on each sheet). This construction will also maintain continuity since (through connecting the sheets along the cut) no jump will be required from one branch to the next or from one side of the cut to the next.

Note 1: if the multi-valued function has only finitely many (distinct) values (say n) then only n consecutive sheets (representing the n distinct values) are required to construct the Riemann surface of the function. In this case the unconnected edges of the bottom and top sheets (in the above description) are also connected to each other (in a rather mysterious way) to make a totally connected multi-layer surface. In fact, this should apply even to the infinitely multi-valued functions where their "bottom" and "top" sheets (at infinity) are also connected (noting that in this case we have infinitely-many sheets).

Note 2: the construction of Riemann surface (as described above) is supposed to be an alternative approach to the branch cut approach for making a multi-valued function single-valued although there is an obvious difference between the two, i.e. the branch cut approach makes the function single-valued on the individual values or branches while the Riemann surface approach makes the function single-valued on the entire (distinct) values or "branches". The supposed advantage of the Riemann surface approach (over the branch cut approach) is that it recovers all the (distinct) values of the multi-valued function in one go and in a continuous manner (with no removal of any part).

13. How many sheets the Riemann surface has for the functions $f_1(z) = z^{1/2}$ and $f_2(z) = z^{1/3}$ and why? What about $f_3(z) = z^{1/2} + z^{1/3}$?
Answer: The Riemann surface of f_1 has two sheets and the Riemann surface of f_2 has three sheets. This is because f_1 has two distinct branches while f_2 has three distinct branches (see § 1.8.11). Regarding f_3, its Riemann surface has six sheets because $z^{1/2}$ has a cycle of two distinct branches and $z^{1/3}$ has a cycle of three distinct branches and hence when they are added their sum will have a cycle of six distinct branches (i.e. all the other branches are repetitive of these six distinct branches).[114]

1.12 Useful Identities and Formulae

To finalize this preliminary chapter, we provide in the following list some basic identities and formulae about complex numbers and variables. Many of these identities and formulae are either self evident (from the basic definitions and rules of mathematics in general and complex analysis in particular) or can be proved easily by using the rules and formalism of manipulating complex numbers that we discussed and developed earlier. However, we will provide in the Problems proofs or verifications to these identities and formulae (mainly for the purpose of practice and demonstration).

We should remark that some of these identities and formulae are repetitive (i.e. they are essentially the same but in different forms) and hence the justification for listing them is that they occur frequently (in their different forms) and hence it is useful to be familiar with these different forms. We should also remark that in the case of dealing (directly or indirectly) with the polar form, the principal argument is generally employed (when we have the choice to do so considering the intended objective) to avoid distractive complications.

$$i^0 = 1 \tag{83}$$
$$i^1 = i = \sqrt{-1} \tag{84}$$
$$i^2 = i \times i = \sqrt{-1} \times \sqrt{-1} = -1 \tag{85}$$
$$i^3 = i^2 \times i = -i \tag{86}$$
$$i^4 = i^2 \times i^2 = (-1) \times (-1) = 1 \tag{87}$$
$$i^{4n} = 1 \tag{88}$$
$$i^{4n+1} = i \tag{89}$$
$$i^{4n+2} = -1 \tag{90}$$

[114] The number "six" is obtained by taking the least common multiple of 2 and 3 noting that they are relatively prime.

$$
\begin{aligned}
i^{4n+3} &= -i && && (91)\\
i^{n} &= i^{(n \bmod 4)} && (n \geq 0) && (92)\\
i &= e^{i\pi/2} && && (93)\\
i &= \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) && && (94)\\
i^{n} &= \cos\left(\frac{n\pi}{2}\right) + i\sin\left(\frac{n\pi}{2}\right) && && (95)\\
\frac{i}{i} &= i^{1-1} = i^{0} = 1 && && (96)\\
i^{-1} &= \frac{1}{i} = \frac{i}{ii} = \frac{i}{-1} = -i && && (97)\\
i^{-2} &= i^{-1} \times i^{-1} = (-i) \times (-i) = i^{2} = -1 && && (98)\\
i^{-3} &= i^{-1} \times i^{-2} = (-i) \times (-1) = i && && (99)\\
i^{-4} &= i^{-2} \times i^{-2} = (-1) \times (-1) = 1 && && (100)\\
i^{-n} &= i^{-(n \bmod 4)} && (n \geq 0) && (101)\\
\sqrt{i} &= i^{1/2} = \pm\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) && && (102)\\
e^{iz} &= \cos z + i\sin z && && (103)\\
e^{-iz} &= \cos z - i\sin z && && (104)\\
1^{i} &= 1 && && (105)\\
i^{i} &= e^{-\pi/2} && && (106)\\
z^{in} &= \cos(n\ln z) + i\sin(n\ln z) && && (107)\\
\sqrt[in]{z} &= \cos\left(\frac{\ln z}{n}\right) - i\sin\left(\frac{\ln z}{n}\right) && && (108)\\
e^{i\pi} + 1 &= 0 && && (109)\\
\left|e^{i\theta}\right| &= 1 && (\theta \text{ is real}) && (110)\\
|e^{z}| &= e^{x} && (z = x + iy) && (111)\\
e^{z+i2n\pi} &= e^{z} && && (112)\\
e^{i(\theta+2n\pi)} &= e^{i\theta} && && (113)\\
(\cos\theta + i\sin\theta)^{n} &= \cos n\theta + i\sin n\theta && && (114)\\
\cos z &= \frac{e^{iz} + e^{-iz}}{2} && && (115)\\
\sin z &= \frac{e^{iz} - e^{-iz}}{i2} && && (116)\\
\mathrm{Re}(iz) &= -\mathrm{Im}(z) && && (117)\\
\mathrm{Im}(iz) &= \mathrm{Re}(z) && && (118)
\end{aligned}
$$

Problems

1. Discuss (and verify if necessary) the identities and formulae of the list that we provided in this section.
   Answer: We discuss this list in the following points:[115]

[115] We note that the following proofs and verifications generally indicate the main and specific justifications and hence general justifications, like the use of the rules of indices, are generally taken for granted (and hence they are ignored). We also note that for multi-valued functions we may use the principal value only (to avoid complications that will be investigated later on).

• Regarding Eqs. 83-87, these formulae (of integer powers of i) are obvious (or justified on the spot) and they are mainly based on the basic definition of i (i.e. $i = \sqrt{-1}$).
• Regarding Eqs. 88-91, we have (where we use Eqs. 84-87):

$$i^{4n} = \left(i^4\right)^n = 1^n = 1, \qquad i^{4n+1} = i^{4n}\, i^1 = i, \qquad i^{4n+2} = i^{4n}\, i^2 = -1, \qquad i^{4n+3} = i^{4n}\, i^3 = -i$$

• Regarding Eq. 92, we have (noting that n and m are non-negative integers and we use Eq. 87):

$$i^n = i^{4m+(n \bmod 4)} = i^{4m}\, i^{(n \bmod 4)} = (i^4)^m\, i^{(n \bmod 4)} = 1^m\, i^{(n \bmod 4)} = i^{(n \bmod 4)}$$

• Regarding Eq. 93, we use Eq. 7 (with $y = \pi/2$), that is:

$$e^{i\pi/2} = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) = 0 + i\,1 = i$$

• Regarding Eq. 94, it is shown in the previous point.
• Regarding Eq. 95, we use Eqs. 93 and 7, that is:

$$i^n = \left(e^{i\pi/2}\right)^n = e^{i(n\pi/2)} = \cos\left(\frac{n\pi}{2}\right) + i \sin\left(\frac{n\pi}{2}\right)$$

• Regarding Eqs. 96-100, they are obvious (or rather justified on the spot).
• Regarding Eq. 101, we use Eq. 92, that is:

$$i^{-n} = (i^n)^{-1} = \left(i^{(n \bmod 4)}\right)^{-1} = i^{-(n \bmod 4)}$$

• Regarding Eq. 102, we use Eqs. 93 and 70, that is:

$$\sqrt{i} = i^{1/2} = (1\, e^{i(\pi/2)})^{1/2} = |1|^{1/2}\, e^{i(\pi/2+2m\pi)/2} = e^{i(\pi/4+m\pi)} \qquad (m = 0, 1)$$

So, for $m = 0$ we have $\sqrt{i} = e^{i(\pi/4)} = \cos(\pi/4) + i \sin(\pi/4) = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ and for $m = 1$ we have $\sqrt{i} = e^{i(\pi/4+\pi)} = e^{i(5\pi/4)} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$.
• Regarding Eqs. 103 and 104, they have been verified earlier (see Eq. 8 and part d of Problem 1 of § 1.4).
• Regarding Eq. 105, we have:

$$1^i = (e^{i0})^i = e^0 = 1$$

• Regarding Eq. 106, we have (see Eq. 93):

$$i^i = (i)^i = (e^{i\pi/2})^i = e^{-\pi/2}$$

• Regarding Eq. 107, it is based on Eq. 103, that is:[116]

$$z^{in} = \left(e^{\ln z}\right)^{in} = e^{i(n \ln z)} = \cos(n \ln z) + i \sin(n \ln z)$$

• Regarding Eq. 108, it is based on Eq. 104, that is:

$$\sqrt[in]{z} = z^{1/(in)} = z^{-i/n} = \left(e^{\ln z}\right)^{-i/n} = e^{-i(\frac{\ln z}{n})} = \cos\left(\frac{\ln z}{n}\right) - i \sin\left(\frac{\ln z}{n}\right)$$

• Regarding Eq. 109 (which may be more familiar in the form $e^{i\pi} = -1$), it is justified by Eq. 103, that is:

$$e^{i\pi} + 1 = (\cos \pi + i \sin \pi) + 1 = -1 + i0 + 1 = 0$$

[116] The relation $z = e^{\ln z}$ will be fully investigated in § 2.2. However, it should be recognized from a general background in analysis (knowing that the exponential and logarithm functions are inverses). Also, see Problem 6 of § 1.8.10.

• Regarding Eq. 110, it is based on Eq. 103 (as well as an obvious trigonometric identity), that is:

$$\left|e^{i\theta}\right| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$$

• Regarding Eq. 111, we have:

$$|e^z| = \left|e^{x+iy}\right| = |e^x|\left|e^{iy}\right| = |e^x|\sqrt{\cos^2 y + \sin^2 y} = |e^x| = e^x$$

It is worth noting that $|e^z| = e^x$ means that the modulus of the complex exponential function is the same as the value of its real counterpart.
• Regarding Eq. 112, we have:

$$e^{z+i2n\pi} = e^z e^{i2n\pi} = e^z(\cos 2n\pi + i\sin 2n\pi) = e^z(1 + i0) = e^z$$

where we used Eq. 103 and the periodicity of the (real) cosine and sine functions.
• Regarding Eq. 113, it is similar to Eq. 112, that is:

$$e^{i(\theta+2n\pi)} = e^{i\theta}e^{i2n\pi} = e^{i\theta}(\cos 2n\pi + i\sin 2n\pi) = e^{i\theta}(1 + i0) = e^{i\theta}$$

• Regarding Eq. 114, it is De Moivre's formula which we verified in Problem 3 of § 1.8.10.
• Regarding Eq. 115, it was verified in part (g) of Problem 1 of § 1.4.
• Regarding Eq. 116, it was verified in part (h) of Problem 1 of § 1.4.
• Regarding Eq. 117, we have $z = x + iy$ and hence:

$$\mathrm{Re}(iz) = \mathrm{Re}(i[x+iy]) = \mathrm{Re}(ix - y) = \mathrm{Re}(-y + ix) = -y = -\mathrm{Im}(z)$$

• Regarding Eq. 118, we have $z = x + iy$ and hence:

$$\mathrm{Im}(iz) = \mathrm{Im}(i[x+iy]) = \mathrm{Im}(ix - y) = \mathrm{Im}(-y + ix) = x = \mathrm{Re}(z)$$

2. Check that the square roots of the imaginary unit i are as given by Eq. 102 and obtained in Problem 1.
Answer: As seen in Problem 1, i has two square roots which are $\pm\frac{(1+i)}{\sqrt{2}}$. This result can be easily checked by squaring these roots, that is:

$$\begin{aligned}\left[+\frac{(1+i)}{\sqrt{2}}\right]^2 &= (+1)^2\,\frac{1+2i+i^2}{2} = \frac{1+2i-1}{2} = \frac{2i}{2} = i\\ \left[-\frac{(1+i)}{\sqrt{2}}\right]^2 &= (-1)^2\,\frac{1+2i+i^2}{2} = \frac{1+2i-1}{2} = \frac{2i}{2} = i\end{aligned}$$

Note: the method of check of the present Problem is general, and is not restricted to the square roots, i.e. we can check that the obtained n^{th} roots are correct by raising these roots to the power n to obtain the radicand (see for example Problem 3).
3. Give and verify the cube roots of the imaginary unit i.
Answer: i has three distinct cube roots which are $-i$, $\frac{\sqrt{3}+i}{2}$ and $\frac{-\sqrt{3}+i}{2}$. This can be easily checked by cubing these roots, that is:

$$\begin{aligned}(-i)^3 &= (-1)^3(i)^3 = -i^3 = -(-i) = i\\ \left(\frac{\sqrt{3}+i}{2}\right)^3 &= \frac{\left(\sqrt{3}\right)^3 + 3\left(\sqrt{3}\right)^2 i + 3\sqrt{3}i^2 + i^3}{8} = \frac{3\sqrt{3} + 9i - 3\sqrt{3} - i}{8} = \frac{8i}{8} = i\\ \left(\frac{-\sqrt{3}+i}{2}\right)^3 &= \frac{\left(-\sqrt{3}\right)^3 + 3\left(-\sqrt{3}\right)^2 i + 3\left(-\sqrt{3}\right)i^2 + i^3}{8} = \frac{-3\sqrt{3} + 9i + 3\sqrt{3} - i}{8} = \frac{8i}{8} = i\end{aligned}$$

4. What is wrong in the following reasoning and results:

$$\begin{aligned} i \times i &= \sqrt{-1} \times \sqrt{-1} = \sqrt{(-1) \times (-1)} = \sqrt{1} = 1 \\ \frac{1}{i} &= \frac{\sqrt{1}}{\sqrt{-1}} = \sqrt{\frac{1}{-1}} = \sqrt{-1} = i \end{aligned}$$

Answer: The imaginary unit i stands for $\sqrt{-1}$ as it is and hence it does not follow the rules of square root, i.e. its argument -1 cannot be treated as an independent number. In other words, the product of the square roots in the first equation is not the same as the square root of the product (as it is the case in the ordinary square root function). Similarly, the quotient of the square roots in the second equation is not the same as the square root of the quotient.

Chapter 2
Common Functions

We dedicate this chapter to the investigation and discussion of certain types of functions that are commonly used in complex analysis. To be more specific, the focus of investigation in this chapter is the so-called "elementary functions" (and hence we are investigating commonly used elementary functions). In fact, being commonly used is an indication to their special importance and widespread occurrence in the theory and application of this branch of mathematics. Generally, these functions are no more than extensions and generalizations of similar well known real-valued functions that are widely used in real analysis.

2.1 Polynomial Functions

A complex function $f(z)$ is a polynomial function $P_n(z)$ of degree n if it has the following form:

$$f(z) \equiv P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 = \sum_{k=0}^{n} a_k z^k \qquad (119)$$

where the subscripted a's are complex constants (with $a_n \neq 0$) and n is a non-negative integer.[117] Polynomial functions are analytic over the entire complex plane and hence they are entire functions. Due to their favorable characteristics (especially entirety) and ease of mathematical manipulation (e.g. by algebraic operations and differentiation and integration), they are widely used in complex analysis (as elsewhere).

Problems

1. Show that the differentiation rules of polynomial functions apply to complex polynomials as to real polynomials, that is:

$$\frac{df}{dz} \equiv \frac{dP_n}{dz} = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \cdots + 2 a_2 z + a_1 = \sum_{k=1}^{n} k\, a_k z^{k-1}$$

Answer: Referring to Problem 2 of § 1.10, the rules of differentiating polynomial functions are no more than a combination of the sum, multiple constant and power rules of differentiation (and hence no additional proof is required).

2. Find the roots (i.e. the zeros or solutions) of the following polynomial functions:

(**a**) $3z^2 + i5z - 6 + i$. (**b**) $z^3 - 3 - i4$. (**c**) $z^4 + 16e^{i\pi/2}$. (**d**) $z^5 - 4\sqrt{2}$.

Answer: The roots of a polynomial function $f(z)$ are the solutions of the equation $f(z) = 0$, i.e. those values of z that make f zero. Hence, all we need to do is to solve the equation $f(z) = 0$. It should be obvious that the obtained roots can be easily checked by substitution in the equation $f(z) = 0$ to produce an identity, i.e. $0 = 0$.

(**a**) We use the quadratic formula with $a = 3$, $b = i5$ and $c = -6 + i$, that is:

$$z = \frac{-i5 \pm \sqrt{-25 + (72 - i12)}}{6} = \frac{-i5 \pm \sqrt{47 - i12}}{6}$$

Now, from Eq. 70 (using the polar form of $47 - i12$) we have:

$$\sqrt{47 - i12} \simeq \sqrt{48.5077} e^{i(-0.2500 + 2m\pi)/2} = \sqrt{48.5077} e^{i(-0.1250 + m\pi)} \qquad (m = 0, 1)$$

[117] This is to include constant (non-zero) functions such as $f(z) = 3$ as polynomials; otherwise n should be a positive integer.

$$= \quad \sqrt{48.5077}e^{-i0.1250}e^{im\pi} = \pm\sqrt{48.5077}e^{-i0.1250} \simeq \pm(6.9104 - i0.8683)$$

Therefore, we get (considering all the four sign combinations which reduce to only two distinct combinations):[118]

$$z \simeq \frac{-i5 \pm \left[\pm(6.9104 - i0.8683)\right]}{6} = \frac{-i5 \pm (6.9104 - i0.8683)}{6} = \frac{\pm 6.9104 + i\,(-5 \mp 0.8683)}{6}$$

i.e. we have two distinct roots (corresponding to the two opposite sign combinations):

$$z_1 \simeq \frac{6.9104 - i5.8683}{6} \simeq 1.1517 - i0.9780 \qquad \text{and} \qquad z_2 \simeq \frac{-6.9104 - i4.1317}{6} \simeq -1.1517 - i0.6886$$

(**b**) We have $z^3 = 3 + i4 \simeq 5e^{i0.9273}$ (where we use the principal polar form of $3 + i4$). Hence, from Eq. 70 (with z^3 here corresponding to z there) we get:

$$z \simeq \sqrt[3]{5}e^{i(0.9273+2m\pi)/3} = \sqrt[3]{5}e^{i(0.3091+2m\pi/3)} \qquad (m = 0, 1, 2)$$

i.e. we have three distinct roots (corresponding to $m = 0, 1, 2$): $z_1 \simeq 1.6289 + i0.5202$, $z_2 \simeq -1.2650 + i1.1506$ and $z_3 \simeq -0.3640 - i1.6708$.
(**c**) We have $z^4 = -16e^{i\pi/2} = 16e^{i\pi/2}e^{-i\pi} = 16e^{-i\pi/2} = 2^4 e^{-i\pi/2}$ (where we use the principal polar form of $-16e^{i\pi/2}$). Hence, from Eq. 70 (with z^4 here corresponding to z there) we get:

$$z = 2e^{i(-\pi/2+2m\pi)/4} = 2e^{i(-\pi/8+m\pi/2)} \qquad (m = 0, 1, 2, 3)$$

i.e. we have four distinct roots (corresponding to $m = 0, 1, 2, 3$): $z_1 \simeq 1.8478 - i0.7654$, $z_2 \simeq 0.7654 + i1.8478$, $z_3 \simeq -1.8478 + i0.7654$ and $z_4 \simeq -0.7654 - i1.8478$.
(**d**) We have $z^5 = 4\sqrt{2} = 2^{5/2}e^{i0}$ (where we use the principal polar form of $4\sqrt{2}$). Hence, from Eq. 70 (with z^5 here corresponding to z there) we get:

$$z = 2^{1/2}e^{i(0+2m\pi)/5} = \sqrt{2}e^{i(2m\pi/5)} \qquad (m = 0, 1, 2, 3, 4)$$

i.e. we have five roots (corresponding to $m = 0, 1, 2, 3, 4$): $z_1 = \sqrt{2} \simeq 1.4142$, $z_2 \simeq 0.4370 + i1.3450$, $z_3 \simeq -1.1441 + i0.8313$, $z_4 \simeq -1.1441 - i0.8313$ and $z_5 \simeq 0.4370 - i1.3450$.

3. Verify the following:
(**a**) $(z^n)^* = (z^*)^n$ where n is an integer.
(**b**) $(az^n)^* = a\,(z^*)^n$ where a is real and n is an integer.
(**c**) $\left[P_n(z)\right]^* = P_n(z^*)$ where P_n is an n^{th} order polynomial with real coefficients.
(**d**) $P_n(z^*) = 0$ *iff* $P_n(z) = 0$ where P_n is as in part (c).[119]
(**e**) The complex conjugate root theorem which states that if P_n is a polynomial with real coefficients and z_0 is a given root of P_n then its conjugate z_0^* is also a root of P_n.[120]
Answer: We are considering in this answer the case where n is a positive integer because that is what we need for the objective of this Problem.
(**a**) We have:

$$(z^n)^* = (z \times z \times \cdots \times z)^* = z^* \times z^* \times \cdots \times z^* = (z^*)^n$$

[118] Considering both roots of $\sqrt{47 - i12}$ by using Eq. 70 (as done above) is redundant (and hence we did this for the purpose of demonstration). This is because a polynomial of order n has exactly n roots (as will be shown later in Problem 1 of § 7.1) and hence it is sufficient to consider one root of $\sqrt{47 - i12}$ with the sign combination of z.

[119] Considering the anonymity of z and z^*, "if" should be sufficient. However, we used *iff* for clarity and demonstration. We should also note that z here represents the roots of P_n in general (as will be stated explicitly in part e).

[120] This may be expressed more compactly by saying: the roots of polynomials with real coefficients occur in conjugate pairs. It should be noted that in this theorem we implicitly assume z_0 to have a distinct conjugate (i.e. z_0 is complex or imaginary and not real although real can also be included if repetition is not implied).

where we used the fact that the conjugate of a product is the product of conjugates (see § 1.8.8).
(**b**) We have:

$$(az^n)^* = a^*(z^n)^* = a(z^n)^* = a(z^*)^n$$

where in step 1 we used the rule of conjugate of product (see § 1.8.8), in step 2 we used the fact that a is real and hence its conjugate is itself, and in step 3 we used the result of part (a).
(**c**) This is no more than a combination of the rule of conjugate of sum (see § 1.8.8) and the result of part (b), that is:

$$\left[P_n(z)\right]^* = \left[\sum_{k=0}^{n} a_k z^k\right]^* = \sum_{k=0}^{n} \left(a_k z^k\right)^* = \sum_{k=0}^{n} a_k (z^*)^k = P_n(z^*)$$

(**d**) If $P_n(z) = 0$ then on taking the conjugate of both sides we get:

$$\begin{aligned}\left[P_n(z)\right]^* &= 0^* \\ P_n(z^*) &= 0\end{aligned}$$

where in line 2 we used the result of part (c) and the fact that 0 is real. Similarly, if $P_n(z^*) = 0$ then from the result of part (c) we have $\left[P_n(z)\right]^* = 0$ and hence on taking the conjugate of both sides we get $P_n(z) = 0$ (where we use the fact that the conjugate of the conjugate of a number is the number itself as well as the fact that 0 is real).
(**e**) This theorem is essentially the same as the result of part (d), i.e. $P_n(z_0^*) = 0$ *iff* $P_n(z_0) = 0$. This means that z_0^* is a root of P_n whenever z_0 is a root of P_n, as required by the complex conjugate root theorem. We should finally note that the presentation of this theorem in two parts (i.e. d and e) rather than in one part is for the purpose of clarity.

4. Evaluate the following complex polynomial integrals:
(**a**) $\int_{-3}^{11}(i8z + i5)\,dz$. (**b**) $\int_{\pi-i2}^{3+i}(3z^2 - i2z + \pi)\,dz$. (**c**) $\int_{7-i}^{5-i8}\left[(5 - i25)z^4 + (22 - i13)\right]\,dz$.
Answer:
(**a**)

$$\int_{-3}^{11}(i8z + i5)\,dz = \left[i4z^2 + i5z\right]_{-3}^{11} = \left[i484 + i55\right] - \left[i36 - i15\right] = i518$$

(**b**)

$$\begin{aligned}\int_{\pi-i2}^{3+i}\left(3z^2 - i2z + \pi\right)\,dz &= \left[z^3 - iz^2 + \pi z\right]_{\pi-i2}^{3+i} = \left[(3+i)^3 - i(3+i)^2 + \pi(3+i)\right] - \\ &\quad \left[(\pi - i2)^3 - i(\pi - i2)^2 + \pi(\pi - i2)\right] \simeq 42.8144 + i84.5120\end{aligned}$$

(**c**)

$$\begin{aligned}\int_{7-i}^{5-i8}\left[(5 - i25)z^4 + (22 - i13)\right]\,dz &= \left[(1 - i5)z^5 + (22 - i13)z\right]_{7-i}^{5-i8} \\ &= \left[(1 - i5)(5 - i8)^5 + (22 - i13)(5 - i8)\right] - \\ &\quad \left[(1 - i5)(7 - i)^5 + (22 - i13)(7 - i)\right] = 420718 + i21055\end{aligned}$$

5. Make 3D plots of the following:
(**a**) The imaginary part of the linear polynomial $P_1 = (2 - i)z + 1 + i3$ over the region $-2 \le x \le 2$ and $-2 \le y \le 2$.
(**b**) The (principal) argument of the quadratic polynomial $P_2 = z^2 + i2z + 1$ over the region $-0.4 \le x \le 0.4$ and $-0.4 \le y \le 0.4$.

Answer: Before going through the solution of this Problem, we would like to remind the reader of the remark that we made earlier in Problem 6 of § 1.11, that is: attributes like real and imaginary or modulus and argument belong to the image of the function in the w plane.
(**a**) The linear polynomial P_1 is given by:

$$P_1 = (2-i)(x+iy)+1+i3 = 2x+y-ix+i2y+1+i3 = (2x+y+1)+i(3-x+2y)$$

Hence, its imaginary part is $\mathrm{Im}(P_1) = 3-x+2y$. This is plotted over the region $-2 \le x \le 2$ and $-2 \le y \le 2$ in Figure 18. As we see, $\mathrm{Im}(P_1) = 3-x+2y$ is linear in both x and y and hence what we have is a plane surface descending (with slope -1) in the x direction and ascending (with slope 2) in the y direction. Accordingly, as we move along lines of constant y in the positive x direction we see descending straight lines, while as we move along lines of constant x in the positive y direction we see ascending straight lines.

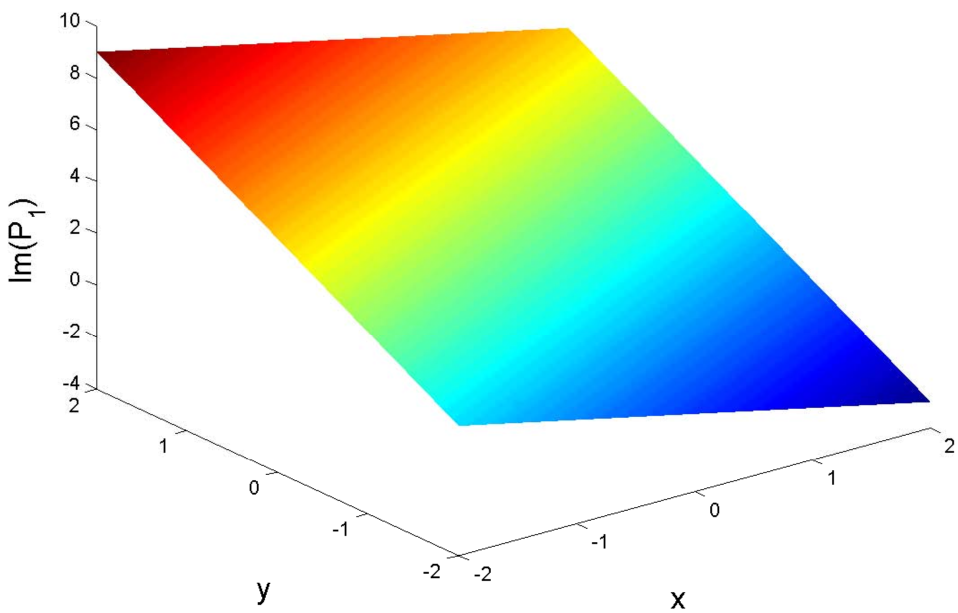

Figure 18: Graphic illustration of the imaginary part $\mathrm{Im}(P_1) = 3-x+2y$ of the complex linear polynomial $P_1 = (2-i)z+1+i3$ over the square region in the z plane defined by $-2 \le x \le 2$ and $-2 \le y \le 2$. See part (a) of Problem 5 of § 2.1.

(**b**) The quadratic polynomial P_2 is given by:

$$P_2 = z^2 + i2z + 1 = x^2 - y^2 + i2xy + i2x - 2y + 1 = (x^2 - y^2 - 2y + 1) + i(2xy + 2x)$$

Hence, its (principal) argument is $\mathrm{Arg}(P_2) = \arctan\left(\frac{2xy+2x}{x^2-y^2-2y+1}\right)$. This is plotted over the region $-0.4 \le x \le 0.4$ and $-0.4 \le y \le 0.4$ in Figure 19.

6. What is the equation whose solutions are represented by the vertices of an origin-centered regular 7-polygon (or heptagon) where one of these vertices is located at $z = i2$?
Answer: Referring to Problem 3 of § 1.8.11, the solutions are the roots of an equation of the form $z^7 + b = 0$. Now, one of these roots is $i2$ and hence it should satisfy this equation, that is:

$$(i2)^7 + b = 0 \qquad \to \qquad -i128 + b = 0 \qquad \to \qquad b = i128$$

Hence, the equation is $z^7 + i128 = 0$.

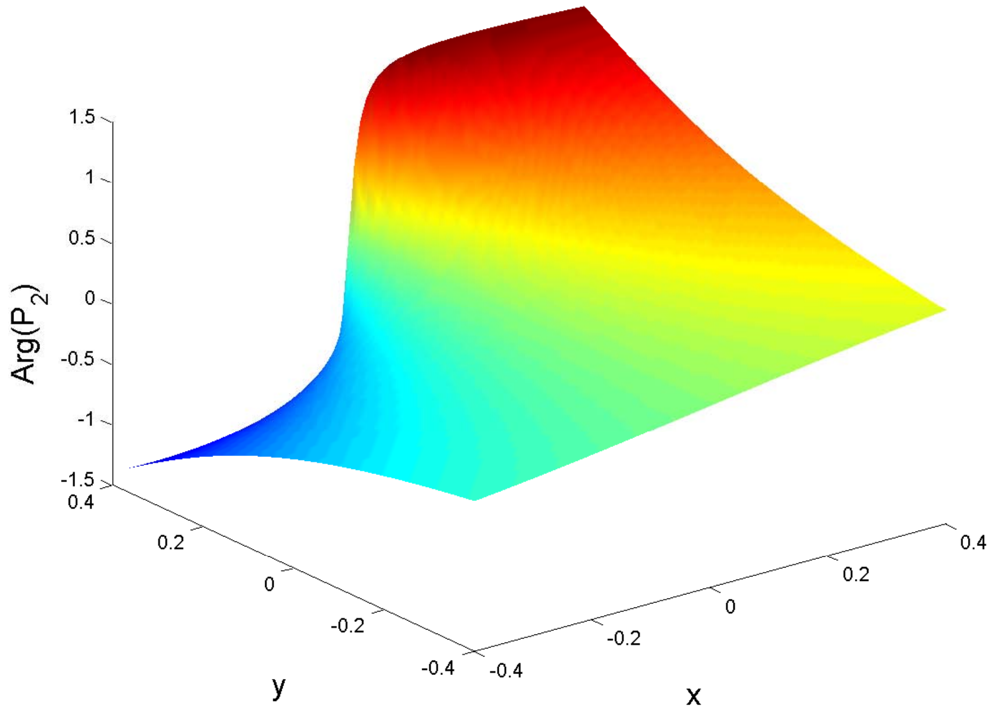

Figure 19: Graphic illustration of the (principal) argument $\text{Arg}(P_2) = \arctan\left(\frac{2xy+2x}{x^2-y^2-2y+1}\right)$ of the complex quadratic polynomial $P_2 = z^2 + i2z + 1$ over the square region in the z plane defined by $-0.4 \le x \le 0.4$ and $-0.4 \le y \le 0.4$. See part (b) of Problem 5 of § 2.1.

7. What "rational function" means?
   Answer: Rational function is a ratio of two polynomials where the denominator is of degree 1 or higher (i.e. the denominator is not a constant polynomial). For example, the following functions are rational:

$$\frac{1}{z+i} \qquad \frac{z^2-3}{z^5-2z^2+i2z} \qquad \frac{3z^4+iz^2-2}{z+1-5i} \qquad \frac{2z}{iz-9}$$

   Rational functions are analytic over the entire complex plane except at the zeros of their denominator where they have isolated singularities.
8. Find a quadratic polynomial whose roots z_1 and z_2 have a product of $-6+i9$ and a sum of $3+i5$. Also, find z_1 and z_2.
   Answer: We have $z_1z_2 = -6+i9$ and $z_1+z_2 = 3+i5$. On substituting from the second into the first we get:

$$\begin{aligned} z_1(3+i5-z_1) &= -6+i9 \\ (3+i5)z_1 - z_1^2 &= -6+i9 \\ z_1^2 - (3+i5)z_1 - 6 + i9 &= 0 \end{aligned}$$

   So, the quadratic polynomial is $z^2 - (3+i5)z - 6 + i9 = 0$. On solving this equation by the quadratic formula we get $z_1 = i3$ and $z_2 = 3+i2$.

2.2 Exponential and Natural Logarithm Functions

The exponential of a complex number $z = x + iy$ is defined as follows (see Eq. 9):[121]

$$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y) \tag{120}$$

The exponential of complex numbers is an analytic function over the entire complex plane and hence it is an entire function. Most of the rules and properties of the real-valued exponential similarly apply to the complex exponential. For example, we have (see parts e and f of Problem 1 of § 1.4):

$$e^{z_1} e^{z_2} = e^{z_1+z_2} \qquad \text{and} \qquad \frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2} \tag{121}$$

As it is the case in real analysis, there is a strong link between the exponential function and the natural logarithm function in complex analysis (as will be seen in the following).[122]

The natural logarithm of a complex number z ($z \neq 0$) is defined as follows:

$$\text{If } z = e^w \text{ then } w \text{ is the natural logarithm of } z, \text{ i.e. } w = \ln z \tag{122}$$

It is obvious that the exponential and natural logarithm functions are inverses of each other and hence we have:

$$e^{\ln z} = \ln e^z = z \tag{123}$$

Using the polar form of complex numbers (i.e. $z = re^{i\theta} = |z| e^{i\theta}$) plus the rules of logarithms and exponents we have:

$$\ln(z) = \ln\left(|z| e^{i\theta}\right) = \ln|z| + \ln e^{i\theta} = \ln|z| + i\theta = \ln|z| + i(\theta_p + 2n\pi) = \ln r + i(\theta_p + 2n\pi) \tag{124}$$

where n is an integer and $-\pi < \theta_p \leq \pi$ is the principal argument which is commonly symbolized as $\operatorname{Arg} z$ (see Problem 2) and where $\ln|z|$ is the *real* logarithm function (as defined in calculus).[123] The principal value of the natural logarithm (which corresponds to the principal argument) is distinguished by using Ln (instead of ln) and hence:[124]

$$\operatorname{Ln}(z) = \ln|z| + i\operatorname{Arg}(z) \tag{125}$$

So, for $z \neq 0$, $\operatorname{Ln}(z)$ is unique (or single-valued) because $\operatorname{Arg}(z)$ is unique. As a matter of terminology, each value of $\ln z$ (corresponding to a specific value of n) is called a branch and thus $\operatorname{Ln}(z)$ is called the principal branch. Accordingly, $\ln(z)$ has an infinite number of branches.[125] It is obvious that the above equations can be easily manipulated to get the commonly used equation:

$$\ln(z) = \operatorname{Ln}(z) + i2n\pi \tag{126}$$

[121] In fact, this definition is largely based on our mathematical foundations which we investigated in § 1.4, and hence it is not a definition in the strict sense. We also note that e^z in this equation and its alike stands for the exponential function $\big[$which may be symbolized as $\exp(z)\big]$ rather than the number e ($\simeq 2.718281828$) raised to the power z although the two are equivalent (noting the extension to irrational and complex powers). We should also note that "the exponential function" is specific to this function $\big[$i.e. the function e^z or $\exp(z)\big]$ and hence any "exponential function" or exponentiation to bases other than e should be referred to with other labels (such as "exponential" without "the").

[122] In fact, this strong link is between any exponential function and logarithm function to a common base since they are inverses of each other (see Problem 6 of § 1.8.10).

[123] As noted earlier, there is another convention about the principal argument $\operatorname{Arg}(z)$, i.e. $0 \leq \operatorname{Arg}(z) < 2\pi$. We should also note that Eq. 124 indicates that $\ln(z)$ is infinitely multi-valued (since n takes infinite number of values and hence $\ln z$ has infinitely-many distinct values). We also note that the use of ln (rather than $\log_e$) in $\ln|z|$ and $\ln r$ will be revised later.

[124] It should be noted that some authors use Ln as a symbol for the complex logarithm function and hence Eq. 124 becomes $\operatorname{Ln}(z) = \ln r + i(\theta_p + 2n\pi)$. Moreover, they use $\ln(z)$ to represent the principal value of $\operatorname{Ln}(z)$.

[125] However, we should note that an extra condition is usually needed to obtain a "branch" in the strict technical sense that is the removal of the branch cut (as well as the branch point which is already excluded) to achieve continuity (since "branch" is distinguished by being single-valued and continuous over its *reduced* domain).

Most of the rules and properties of real-valued logarithms apply to complex logarithms. For example, we have:

$$\ln(z_1 z_2) = \ln(z_1) + \ln(z_2) \qquad \text{and} \qquad \ln\left(\frac{z_1}{z_2}\right) = \ln(z_1) - \ln(z_2) \tag{127}$$

Because $\ln(0)$ is not defined (see Problem 21), $\ln(z)$ is not analytic at $z = 0$ and hence the natural logarithm is not an entire function. Furthermore, $\ln(z)$ is discontinuous at all points of the negative part of the real axis in the complex plane.[126] Therefore, the non-positive part of the real axis is commonly known as a branch cut for the complex logarithm function (see § 1.5). However, the principal branch of $\ln(z)$ is analytic (considering the removal of the branch cut from the domain as explained earlier in § 1.5 and § 1.11).[127] This also applies to the other branches of $\ln(z)$.

It should be noted that the natural logarithm of real positive numbers as a subset of complex numbers is different from their natural logarithm as real numbers by the presence and absence of the imaginary part.[128] For example, if we use the equation $\ln z = \ln r + i(\theta_p + 2n\pi)$ then the natural logarithm of 5 as a complex number (i.e. $5 + 0i$) is $\ln 5 = \ln 5 + i2n\pi$ where the $\ln 5$ on the left side represents the natural logarithm of 5 as a complex number while the $\ln 5$ on the right side represents the natural logarithm of 5 as a real number (i.e. $\ln 5$ on the left is the complex ln while $\ln 5$ on the right is the real ln). In fact, this may cause confusion where some may attempt to cancel $\ln 5$ from both sides to conclude $0 = i2n\pi$ which is wrong in general. In brief, the natural logarithm of 5 as a complex number is $\ln 5 + i2n\pi$ while the natural logarithm of 5 as a real number is $\ln 5$ (where $\ln 5$ in this sentence means its ordinary meaning as used in real analysis, i.e. the real natural logarithm). In other words, the natural logarithm of 5 as a complex number is a complex number (i.e. $\ln 5 + i2n\pi$ which consists of a real part $\ln 5$ and an imaginary part $2n\pi$) while the natural logarithm of 5 as a real number is a purely real number (i.e. $\ln 5$).

Accordingly, a distinction between ln as a complex function and ln as a real function (in the case of natural logarithm of real positive numbers) may be made by reserving ln for the complex and using Ln for the real and hence we may write $\ln 5 = \mathrm{Ln}(5) + i2n\pi$.[129] In fact, this equation yields the trivial identity $\mathrm{Ln}(5) = \mathrm{Ln}(5)$ in the case of the principal branch. A distinction may also be made by writing the real ln as $\log_e$ and hence $\ln 5 = \log_e 5 + i2n\pi$. The latter distinction is more definite, clear, general and reliable and hence in this book we use this form of distinction (i.e. $\log_e$). Therefore, the above equations (e.g. Eq. 124) should be modified accordingly (as we will do in the future; see for example Problem 2).

Finally, it should be obvious that the aforementioned confusion and the need for distinction do not apply to numbers other than the real positive numbers because if a number z is not real positive (whether z is real negative or it is not real) then $\ln z$ and $\ln r$ $(= \ln |z|)$ in the equation $\ln z = \ln r + i(\theta_p + 2n\pi)$ are obviously different although the interpretation of $\ln r$ as real still depends on its context and position. However, for more clarity (as well as coherence) the aforementioned form of distinction (i.e. $\log_e$) will also be used in these (supposedly non-confusing) cases. So in brief, we will use $\log_e$ for the real natural logarithm function and ln for the complex natural logarithm function in all cases.

Problems

1. What are the domain and range of e^z and $\ln z$?
   Answer: The domain of e^z is all $z \in \mathbb{C}$ and its range is all $z \in \mathbb{C}$ excluding zero. The domain of $\ln z$

[126] We are considering here the individual branches and taking into account our convention about the principal argument.

[127] The reader is referred to Problems 22 and 24 of the present section and Problem 5 of § 3.1 for further details and discussions. We also refer to Problem 11 of § 1.8.7 noting that the continuity and uniqueness of $\mathrm{Ln}(z)$ are based on the continuity and uniqueness of $\mathrm{Arg}(z)$.

[128] It is claimed that in this case ln means the real natural logarithm, i.e. it has no imaginary part. However, the literature is not consistent in this regard (as usually in other regards).

[129] The stated condition "in the case of natural logarithm of real positive numbers" is essential in this statement because otherwise this distinction will not be consistent with Eq. 125. Yes, to avoid these complications we may say: "the distinction between Ln and ln (i.e. as principal and non-principal) may be used since Ln of a real positive number is a purely real number and hence we may write $\ln 5 = \mathrm{Ln}(5) + i2n\pi$" although other complications arise from this phrasing (e.g. this does not make a distinction between the complex and real specifically). Anyway, this is a trivial matter (noting the clarity of the distinction and its objective); moreover we have another form of distinction (which will be discussed next) that is free of these complications (and hence we use it in this book).

is all $z \in \mathbb{C}$ excluding zero and its range is all $z \in \mathbb{C}$.[130] This is inline with the fact that e^z and $\ln z$ are inverses.

2. Investigate Eq. 124.
Answer: Referring to Problem 10 of § 1.8.7, we can say: if z is given in polar form then θ is unique (since it is determined explicitly and specifically) and hence:

$$\ln(z) = \ln\left(|z|\, e^{i\theta}\right) = \log_e |z| + \ln e^{i\theta} = \log_e |z| + i\theta = \log_e r + i\theta \tag{128}$$

where we use here $\log_e$ (instead of ln which we used in Eq. 124) for the real natural logarithm function. But if z is given in Cartesian form then θ is not unique and hence:

$$\ln(z) = \ln\left(|z|\, e^{i\theta}\right) = \log_e |z| + \ln e^{i\theta} = \log_e |z| + i\theta = \log_e |z| + i(\theta_p + 2n\pi) = \log_e r + i(\theta_p + 2n\pi) \tag{129}$$

Accordingly, Eq. 124 tries to consider both possibilities (i.e. polar and Cartesian) in one go. However, with regard to the polar form the last two steps of Eq. 124 should be seen as explanatory steps with no added content (since θ is already fixed according to the polar form and hence n is fixed).
Note: from the above investigation we can say that the natural logarithm of a complex number in polar form is unique (i.e. single-valued) while the natural logarithm of a complex number in Cartesian form is not unique (i.e. infinitely multi-valued).

3. Show that:
(**a**) $z^\beta = e^{\beta \ln z}$ ($z, \beta \in \mathbb{C}$ and $z \neq 0$). (**b**) $(e^z)^* = e^{z^*}$.
Answer:
(**a**) We can use the basic rules of manipulating logarithms and indices in real numbers (which are established in algebra) as well as some elementary results that we already obtained about complex numbers. However, it is easier to use the results about exponents and logarithms that we established in § 1.8.10, that is:

$$z^\beta = (z)^\beta = (e^{\ln z})^\beta = e^{\beta \ln z} \tag{130}$$

where in step 2 we use the fact that exponentiation and taking logarithm are inverses (see Problem 6 of § 1.8.10) while in step 3 we use one of the rules of indices (see the note of Problem 5 of § 1.8.10).
(**b**) We use the basic rules of manipulating indices as well as some elementary results that we already obtained about complex numbers (see for instance § 1.4 and § 1.8.8), that is:

$$\begin{aligned} (e^z)^* &= \left(e^{x+iy}\right)^* = \left(e^x e^{iy}\right)^* = (e^x)^* \left(e^{iy}\right)^* = e^x (\cos y + i \sin y)^* = e^x (\cos y - i \sin y) \\ &= e^x (\cos\{-y\} + i \sin\{-y\}) = e^x e^{-iy} = e^{x-iy} = e^{z^*} \end{aligned}$$

4. Show that a non-zero complex number has exactly n distinct n^{th} roots.
Answer: We have (see Eqs. 130 and 129):

$$z^{1/n} = e^{\frac{\ln z}{n}} = e^{\frac{\log_e r + i(\theta_p + 2m\pi)}{n}} = e^{\frac{\log_e r}{n}} \left(\cos\frac{\theta_p + 2m\pi}{n} + i \sin\frac{\theta_p + 2m\pi}{n}\right)$$

Now, for $m = 0, 1, \cdots, (n-1)$ we have n distinct values (i.e. roots) while for all the other values of m we have the same values as those for $m = 0, 1, \cdots, (n-1)$ due to the periodicity of the trigonometric cosine and sine functions[131] which just repeats those m distinct roots. Also, see Problem 5 of § 1.8.11.

5. Write down the results of Problem 7 of § 1.8.10 (which are in terms of general logarithm log) in terms of the natural logarithm ln.
Answer: Noting that ln means $\log_e$ (considering that $\log_e$ here is not for the purpose of specifying the real function) we can easily obtain the following results as instances of the results in Problem 7 of

[130] We note that $\mathbb{C}$ here represents the *finite* complex plane.

[131] What we need here is only the periodicity of the real trigonometric cosine and sine functions (with a period of 2π) although we will show later (see Problem 15 of § 2.3) that this periodicity also applies to the complex trigonometric cosine and sine functions.

§ 1.8.10:

$\ln(\alpha\beta) = \ln\alpha + \ln\beta$ $\quad$ $\ln\left(\frac{\alpha}{\beta}\right) = \ln\alpha - \ln\beta$ $\quad$ $\ln\alpha^\beta = \beta\ln\alpha$

$\ln\left(\sqrt[\beta]{\alpha}\right) = \beta^{-1}\ln\alpha$ $\quad$ $\ln 1 = 0$ $\quad$ $\ln e = 1$

$\log_\beta \alpha = \frac{\ln\alpha}{\ln\beta}$ (and $\ln\alpha = \frac{\log_\gamma \alpha}{\log_\gamma e}$) $\quad$ $\ln\left(\frac{1}{\alpha}\right) = -\ln\alpha$ $\quad$ $\ln\alpha = \frac{1}{\log_\alpha e}$

6. As a consistency check, verify the first two results of Problem 5 $\big[$i.e. $\ln(\alpha\beta) = \ln\alpha + \ln\beta$ and $\ln(\alpha/\beta) = \ln\alpha - \ln\beta\big]$ by using $\alpha = z_1 = r_1 e^{i\theta_1}$ and $\beta = z_2 = r_2 e^{i\theta_2}$ plus Eq. 128 (as well as the basic rules of manipulating complex numbers and known facts from real analysis).
Answer:

$$\begin{aligned}
\ln(z_1 z_2) &= \ln\left(r_1 r_2 e^{i(\theta_1+\theta_2)}\right) && \text{(see Eq. 22)}\\
&= \log_e(r_1 r_2) + i(\theta_1 + \theta_2) && \text{(see Eq. 128)}\\
&= \log_e r_1 + \log_e r_2 + i\theta_1 + i\theta_2 && \text{(from real analysis)}\\
&= (\log_e r_1 + i\theta_1) + (\log_e r_2 + i\theta_2) &&\\
&= \ln(z_1) + \ln(z_2) && \text{(see Eq. 128)}
\end{aligned}$$

$$\begin{aligned}
\ln\left(\frac{z_1}{z_2}\right) &= \ln\left(r_1 r_2^{-1} e^{i(\theta_1-\theta_2)}\right) && \text{(see Eq. 24)}\\
&= \log_e\left(r_1 r_2^{-1}\right) + i(\theta_1 - \theta_2) && \text{(see Eq. 128)}\\
&= \log_e r_1 - \log_e r_2 + i\theta_1 - i\theta_2 && \text{(from real analysis)}\\
&= (\log_e r_1 + i\theta_1) - (\log_e r_2 + i\theta_2) &&\\
&= \ln(z_1) - \ln(z_2) && \text{(see Eq. 128)}
\end{aligned}$$

Note: the above answer should indicate that for the relations $\ln(z_1 z_2) = \ln(z_1) + \ln(z_2)$ and $\ln\left(\frac{z_1}{z_2}\right) = \ln(z_1) - \ln(z_2)$ to hold correctly, unambiguously and unconditionally, z_1 and z_2 (i.e. in $z_1 z_2$ and $\frac{z_1}{z_2}$) should be identified separately in their unique polar form (see Problem 2).

7. Investigate the complex exponentiation function $f(z,\alpha) = z^\alpha$ (where z is complex and α is real) as single- or multi-valued function.
Answer: We have:

$$f(z,\alpha) = z^\alpha = \left(e^{\ln z}\right)^\alpha = \left(e^{\log_e r + i(\theta_p + 2n\pi)}\right)^\alpha = e^{\alpha\log_e r + i(\alpha\theta_p + 2n\pi\alpha)} = e^{\alpha\log_e r + i\alpha\theta_p}\, e^{i2n\pi\alpha}$$

Now, if we note that $e^{\alpha\log_e r + i\alpha\theta_p}$ is single-valued then we have three cases:
• If α is integer then $e^{i2n\pi\alpha} = 1$ and hence f is single-valued.
• If α is rational then $e^{i2n\pi\alpha}$ is finitely multi-valued and hence f is finitely multi-valued.
• If α is irrational then $e^{i2n\pi\alpha}$ is infinitely multi-valued and hence f is infinitely multi-valued.
Note 1: if α is (strictly) complex or imaginary then f is infinitely multi-valued.
Note 2: "single", "finitely multi" and "infinitely multi" in this Problem refers to the *distinct* values (see the bullet points in the text of § 1.11). Also see Problem 1 of § 1.8.10.

8. Find the exponentials of the following numbers:
(**a**) $z = i$. (**b**) $z = -1 - i$. (**c**) $z = 1 + i\sqrt{3}$. (**d**) $z = \pi + ie$.
(**e**) $z = e^\pi$. (**f**) $z = \pi^2 e^{i\pi/4}$. (**g**) $z = e^{3-i\pi/3}$. (**h**) $z = e^{i6+\pi/2}$.
Answer: We use $e^z = e^{x+iy} = e^x(\cos y + i\sin y) = e^x\cos y + ie^x\sin y$ (see Eq. 9).
(**a**) $e^i = e^{0+i1} = e^0\cos 1 + ie^0\sin 1 = \cos 1 + i\sin 1 \simeq 0.5403 + i0.8415$
(**b**) $e^{-1-i} = e^{-1}\cos(-1) + ie^{-1}\sin(-1) = e^{-1}\cos(1) - ie^{-1}\sin(1) \simeq 0.1988 - i0.3096$
(**c**) $e^{1+i\sqrt{3}} = e^1\cos\sqrt{3} + ie^1\sin\sqrt{3} \simeq -0.4364 + i2.6830$

(**d**) $e^{\pi+ie} = e^{\pi}\cos e + ie^{\pi}\sin e \simeq -21.0982 + i9.5058$

(**e**) $e^{e^{\pi}} = e^{e^{\pi}+i0} = e^{e^{\pi}}\cos 0 + ie^{e^{\pi}}\sin 0 = e^{e^{\pi}} \times 1 + i0 \simeq 11216958622.4676$

(**f**) $z = \pi^2 e^{i\pi/4} = \pi^2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{\pi^2}{\sqrt{2}} + i\frac{\pi^2}{\sqrt{2}}$

$e^{z} = e^{\pi^2/\sqrt{2}}e^{i\pi^2/\sqrt{2}} = e^{\pi^2/\sqrt{2}}\left(\cos\frac{\pi^2}{\sqrt{2}} + i\sin\frac{\pi^2}{\sqrt{2}}\right) \simeq 824.1909 + i688.1404$

(**g**) $z = e^{3-i\pi/3} = e^3\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = e^3\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \frac{e^3}{2} - i\frac{e^3\sqrt{3}}{2}$

$e^{z} = e^{e^3/2}e^{-ie^3\sqrt{3}/2} = e^{e^3/2}\left(\cos\frac{e^3\sqrt{3}}{2} - i\sin\frac{e^3\sqrt{3}}{2}\right) \simeq 2656.7587 + i22834.9063$

(**h**) $z = e^{i6+\pi/2} = e^{\pi/2}e^{i6} = e^{\pi/2}(\cos 6 + i\sin 6) = e^{\pi/2}\cos 6 + ie^{\pi/2}\sin 6$

$e^{z} = e^{e^{\pi/2}\cos 6}\left(\cos\left[e^{\pi/2}\sin 6\right] + i\sin\left[e^{\pi/2}\sin 6\right]\right) \simeq 22.7840 - i98.7868$

9. Find the natural logarithms of the following numbers (which are given in Cartesian form):

(**a**) $z = 1.$ (**b**) $z = 2.$ (**c**) $z = -1.$ (**d**) $z = -2.$

(**e**) $z = i.$ (**f**) $z = -1 - i.$ (**g**) $z = 1 + i\sqrt{3}.$ (**h**) $z = \sqrt{\pi} - i7.$

Answer: We use $\ln z = \log_e r + i(\theta_p + 2n\pi) = \log_e\sqrt{x^2+y^2} + i\left(\arctan\left[\frac{y}{x}\right] + 2n\pi\right)$.

(**a**) $z = 1 = 1 + i0 = x + iy$. Hence:

$$\ln(1) = \log_e\sqrt{1^2+0^2} + i\left(\arctan\left[\frac{0}{1}\right] + 2n\pi\right) = \log_e 1 + i\,(0 + 2n\pi) = 0 + i2n\pi = i2n\pi$$

(**b**) $z = 2 = 2 + i0 = x + iy$. Hence:

$$\ln(2) = \log_e\sqrt{2^2+0^2} + i\left(\arctan\left[\frac{0}{2}\right] + 2n\pi\right) = \log_e 2 + i\,(0 + 2n\pi) = \log_e 2 + i2n\pi$$

(**c**) $z = -1 = -1 + i0 = x + iy$. Hence:

$$\begin{aligned}\ln(-1) &= \log_e\sqrt{(-1)^2+0^2} + i\left(\arctan\left[\frac{0}{-1}\right] + 2n\pi\right) = \log_e 1 + i(\pi + 2n\pi)\\ &= 0 + i(\pi + 2n\pi) = i(2n+1)\pi\end{aligned}$$

(**d**) $z = -2 = -2 + i0 = x + iy$. Hence:

$$\ln(-2) = \log_e\sqrt{(-2)^2+0^2} + i\left(\arctan\left[\frac{0}{-2}\right] + 2n\pi\right) = \log_e 2 + i(\pi + 2n\pi) = \log_e 2 + i(2n+1)\pi$$

(**e**) $z = i = 0 + i1 = x + iy$. Hence:

$$\ln(i) = \log_e\sqrt{0^2+1^2} + i\left(\arctan\left[\frac{1}{0}\right] + 2n\pi\right) = \log_e 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = i\left(\frac{4n+1}{2}\right)\pi$$

(**f**) $z = -1 - i = x + iy$. Hence:

$$\begin{aligned}\ln(-1-i) &= \log_e\sqrt{(-1)^2+(-1)^2} + i\left(\arctan\left[\frac{-1}{-1}\right] + 2n\pi\right) = \log_e\sqrt{2} + i\left(-\frac{3\pi}{4} + 2n\pi\right)\\ &= \log_e\sqrt{2} + i\left(\frac{8n-3}{4}\right)\pi\end{aligned}$$

(**g**) $z = 1 + i\sqrt{3} = x + iy$. Hence:

$$\begin{aligned}\ln(1+i\sqrt{3}) &= \log_e \sqrt{1^2 + \left(\sqrt{3}\right)^2} + i\left(\arctan\left[\frac{\sqrt{3}}{1}\right] + 2n\pi\right) = \log_e 2 + i\left(\frac{\pi}{3} + 2n\pi\right) \\ &= \log_e 2 + i\left(\frac{6n+1}{3}\right)\pi\end{aligned}$$

(**h**) $z = \sqrt{\pi} - i7 = x + iy$. Hence:

$$\ln(\sqrt{\pi} - i7) = \log_e \sqrt{\pi + (-7)^2} + i\left(\arctan\left[\frac{-7}{\sqrt{\pi}}\right] + 2n\pi\right) \simeq \log_e \sqrt{\pi + 49} + i\,(-1.3228 + 2n\pi)$$

10. Verify the results of Problem 9.
Answer: We simply take the exponentials of the logarithms that we obtained in Problem 9 to reverse the process of taking natural logarithm (noting that the exponential and natural logarithm functions are inverses of each other) and hence obtain the z's that are given in Problem 9.

(**a**)
$$e^{i2n\pi} = \cos(2n\pi) + i\sin(2n\pi) = 1 + i0 = 1 = z$$

(**b**)
$$e^{\log_e 2 + i2n\pi} = e^{\log_e 2}\left[\cos(2n\pi) + i\sin(2n\pi)\right] = e^{\log_e 2}(1 + i0) = e^{\log_e 2} = 2 = z$$

(**c**)
$$e^{i(\pi + 2n\pi)} = \cos(\pi + 2n\pi) + i\sin(\pi + 2n\pi) = -1 + i0 = -1 = z$$

(**d**)
$$e^{\log_e 2 + i(\pi + 2n\pi)} = e^{\log_e 2}\left[\cos(\pi + 2n\pi) + i\sin(\pi + 2n\pi)\right] = e^{\log_e 2}(-1 + i0) = -e^{\log_e 2} = -2 = z$$

(**e**)
$$e^{i(\pi/2 + 2n\pi)} = \cos\left(\frac{\pi}{2} + 2n\pi\right) + i\sin\left(\frac{\pi}{2} + 2n\pi\right) = 0 + i\,1 = i = z$$

(**f**)
$$\begin{aligned}e^{\log_e \sqrt{2} + i\left(-\frac{3\pi}{4} + 2n\pi\right)} &= e^{\log_e \sqrt{2}}\left[\cos\left(-\frac{3\pi}{4} + 2n\pi\right) + i\sin\left(-\frac{3\pi}{4} + 2n\pi\right)\right] = \sqrt{2}\left[\frac{-1-i}{\sqrt{2}}\right] \\ &= -1 - i = z\end{aligned}$$

(**g**)
$$e^{\log_e 2 + i\left(\frac{\pi}{3} + 2n\pi\right)} = e^{\log_e 2}\left[\cos\left(\frac{\pi}{3} + 2n\pi\right) + i\sin\left(\frac{\pi}{3} + 2n\pi\right)\right] = 2\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] = 1 + i\sqrt{3} = z$$

(**h**)
$$e^{\log_e \sqrt{\pi + 49} + i(-1.3228 + 2n\pi)} = \sqrt{\pi + 49}\big[\cos(-1.3228 + 2n\pi) + i\sin(-1.3228 + 2n\pi)\big] = \sqrt{\pi} - i7 = z$$

11. What are the principal values of the logarithms in Problem 9?
Answer: The principal values correspond to $n = 0$.

(**a**) $\mathrm{Ln}(1) = 0$.
(**b**) $\mathrm{Ln}(2) = \log_e 2$.
(**c**) $\mathrm{Ln}(-1) = i\pi$.
(**d**) $\mathrm{Ln}(-2) = \log_e 2 + i\pi$.
(**e**) $\mathrm{Ln}(i) = i\frac{\pi}{2}$.
(**f**) $\mathrm{Ln}(-1-i) = \log_e \sqrt{2} - i\frac{3\pi}{4}$.
(**g**) $\mathrm{Ln}(1 + i\sqrt{3}) = \log_e 2 + i\frac{\pi}{3}$.
(**h**) $\mathrm{Ln}(\sqrt{\pi} - i7) \simeq \log_e \sqrt{\pi + 49} - i1.3228$.

12. Find the relation between $\log_i x$ and $\log_e x$ where x is a positive real number.
Answer: We use the relation between logarithms to different bases (i.e. $\log_\beta \alpha = \log_\gamma \alpha / \log_\gamma \beta$; see part g of Problem 7 of § 1.8.10) plus the result of part (e) of Problem 9, that is:[132]

$$\log_i x = \frac{\log_e x}{\log_e i} = \frac{\log_e x}{\ln i} = \frac{\log_e x}{i\left(\frac{\pi}{2} + 2n\pi\right)} = -i\frac{\log_e x}{\frac{\pi}{2} + 2n\pi} = -i\frac{2\log_e x}{\pi + 4n\pi}$$

Note: this relation is commonly given as $\log_i x = \frac{2\ln x}{i\pi}$ (considering the principal value which corresponds to $n = 0$ and noting that we use $\log_e x$ for $\ln x$ since it is real according to our consideration which is indicated in the footnote).

13. Given that $z = x + iy$ and $w = u + iv = \ln z$, obtain the following:
(**a**) x and y as functions of u and v. (**b**) u and v as functions of x and y.
Answer:
(**a**) Because $w = \ln z$ we have:

$$z = e^w = e^{u+iv} = e^u e^{iv} = e^u(\cos v + i \sin v) = e^u \cos v + ie^u \sin v = x + iy$$

Hence, $x = e^u \cos v$ and $y = e^u \sin v$.
(**b**) Using the result of part (a) we have $x^2 + y^2 = e^{2u} \cos^2 v + e^{2u} \sin^2 v = e^{2u}(\cos^2 v + \sin^2 v) = e^{2u}$ and hence:

$$2u = \log_e(x^2 + y^2) \qquad \text{that is} \qquad u = \log_e \sqrt{x^2 + y^2}$$

Also, $y/x = \tan v$ and hence $v = \arctan(y/x)$.
Note: to generalize the result of part (b) arctan should be generalized; otherwise the question should be restricted to $\operatorname{Ln} z$.

14. Solve the following equations (for $z \in \mathbb{C}$):
(**a**) $e^z = 3 - i3$. (**b**) $e^z = \pi + i$. (**c**) $\ln z = \sqrt{\pi} - ie$. (**d**) $\ln(3z) = 6 + i\pi$.
Answer:
(**a**) On taking the natural logarithm of both sides we get:

$$z = \ln(3 - i3) = \log_e \sqrt{3^2 + (-3)^2} + i\left(\arctan\left[\frac{-3}{3}\right] + 2n\pi\right) = \log_e \sqrt{18} + i\left(-\frac{\pi}{4} + 2n\pi\right)$$

(**b**) On taking the natural logarithm of both sides we get:

$$z = \ln(\pi + i) = \log_e \sqrt{\pi^2 + 1^2} + i\left(\arctan\left[\frac{1}{\pi}\right] + 2n\pi\right) \simeq \log_e \sqrt{\pi^2 + 1} + i\,(0.3082 + 2n\pi)$$

(**c**) On taking the exponential of both sides we get:

$$z = e^{\sqrt{\pi} - ie} = e^{\sqrt{\pi}} \cos(-e) + ie^{\sqrt{\pi}} \sin(-e) \simeq -5.3658 - i2.4176$$

(**d**) On taking the exponential of both sides we get:

$$3z = e^{6+i\pi} = e^6 \cos \pi + ie^6 \sin \pi = -e^6 \simeq -403.4288$$

Hence, $z = -e^6/3 \simeq -134.4763$.

15. Solve the following systems of simultaneous complex equations (using only the principal value of logarithms if the equations involve logarithm functions):
(**a**) $e^z + iz^2 + 1 + i\pi^2 = 0$ and $3e^z + \pi z + 3 = i\pi^2$.

[132] It should be obvious that $\log_i x$ and $\log_e i$ are not real even though they look similar to $\log_e x$ which we use for real specifically (noting that the real function $\log_e$ is restricted to real positive argument and hence $\log_e i$ cannot be real and therefore there is no ambiguity in our notation). It should also be noted that we consider $\log_e x$ in the numerator above to be the real function (i.e. it is not $\ln x$); otherwise the above result should be modified accordingly.

(**b**) $e^{2z} + e^z - 6 = 0$ and $e^z - i5z - 2 + i\ln(32) = 0$.
(**c**) $\ln z + z - 1 + e - i\pi = 0$ and $7\ln z - z - e - 7 - i7\pi = 0$.

Answer:
(**a**) The first equation can be written as $e^z = -iz^2 - 1 - i\pi^2$. So, on substituting from this into the second equation we get:

$$\begin{aligned} 3\left(-iz^2 - 1 - i\pi^2\right) + \pi z + 3 &= i\pi^2 \\ -i3z^2 - 3 - i3\pi^2 + \pi z + 3 &= i\pi^2 \\ -i3z^2 + \pi z - i4\pi^2 &= 0 \end{aligned}$$

On solving this equation (by the quadratic formula) we get:

$$z = \frac{-\pi \pm \sqrt{\pi^2 - 4 \times (-i3) \times (-i4\pi^2)}}{-i6} = \frac{-\pi \pm \sqrt{49\pi^2}}{-i6} = \frac{-\pi \pm 7\pi}{-i6} = i\frac{-\pi \pm 7\pi}{6}$$

So, $z = i\pi$ or $z = -i\frac{4\pi}{3}$. On substituting $z = i\pi$ and $z = -i\frac{4\pi}{3}$ into the two equations we can easily find that only $z = i\pi$ satisfies these equations and hence we accept this solution and reject the other. So, the solution of these simultaneous equations is $z = i\pi$.
(**b**) The first equation is quadratic in e^z and hence it can be solved by the quadratic formula, that is:

$$e^z = \frac{-1 \pm \sqrt{1 - 4 \times (-6)}}{2} = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2}$$

So, $e^z = 2$ and hence $z = \ln 2$, or $e^z = -3$ and hence $z = \ln(-3)$. We now need to check if these values of z satisfy the second equation or not. On substituting $z = \ln 2$ into the second equation we get:

$$e^{\ln 2} - i5\ln 2 - 2 + i\ln(32) = 2 - i\ln 2^5 - 2 + i\ln 2^5 = 0$$

and hence $z = \ln 2$ is a solution to both equations. However, on substituting $z = \ln(-3)$ into the second equation we get:

$$e^{\ln(-3)} - i5\ln(-3) - 2 + i\ln(32) = -3 - i5\ln(-3) - 2 + i\ln(32) = -5 + i\big[\ln(32) - 5\ln(-3)\big] \neq 0$$

and hence $z = \ln(-3)$ is not a solution to both equations. Therefore, the solution of these simultaneous equations is $z = \ln 2$.
(**c**) On multiplying the first equation by 7 and subtracting the second equation from the product we get:

$$\begin{aligned} 7(\ln z + z - 1 + e - i\pi) - (7\ln z - z - e - 7 - i7\pi) &= 7(0) - 0 \\ 7\ln z + 7z - 7 + 7e - i7\pi - 7\ln z + z + e + 7 + i7\pi &= 0 \\ 8z + 8e &= 0 \\ z + e &= 0 \\ z &= -e \end{aligned}$$

On substituting this value of z into these equations we get:

$$\begin{aligned} \ln(-e) + (-e) - 1 + e - i\pi = \ln(-1) + \ln e - e - 1 + e - i\pi = i\pi + 1 - e - 1 + e - i\pi = 0 \\ 7\ln(-e) - (-e) - e - 7 - i7\pi = 7\ln(-1) + 7\ln e + e - e - 7 - i7\pi = i7\pi + 7 + e - e - 7 - i7\pi = 0 \end{aligned}$$

and hence $z = -e$ is a solution to both equations. Therefore, the solution of these simultaneous equations is $z = -e$.

16. Find z^β for the following pairs of complex numbers:
   (**a**) $z = 2$ and $\beta = 1 + i$. (**b**) $z = -1 - i$ and $\beta = \pi$. (**c**) $z = i$ and $\beta = i4$.
   (**d**) $z = \sqrt{\pi} - i7$ and $\beta = e - i3$. (**e**) $z = i$ and $\beta = 2$.
   Answer: We use $z^\beta = e^{\beta \ln z}$ (see part a of Problem 3) as well as other previously established results (e.g. Eq. 129).
   (**a**) We have:

$$\begin{aligned} z^\beta &= 2^{1+i} = 2^1 2^i = 2e^{i\ln 2} = 2e^{i(\log_e 2 + i2n\pi)} = 2e^{(i\log_e 2 - 2n\pi)} = 2e^{-2n\pi}e^{i\log_e 2} \\ &= 2e^{-2n\pi}\big[\cos(\log_e 2) + i\sin(\log_e 2)\big] \end{aligned}$$

For $n = 0$ we have $z^\beta = 2\big[\cos(\log_e 2) + i\sin(\log_e 2)\big] \simeq 1.5385 + i1.2779$.
(**b**) We have (see part f of Problem 9):

$$\begin{aligned} z^\beta &= (-1-i)^\pi = e^{\pi\ln(-1-i)} = e^{\pi\left[\log_e\sqrt{2} + i\left(-\frac{3\pi}{4} + 2n\pi\right)\right]} = e^{\pi\log_e\sqrt{2} + i\pi\left(-\frac{3\pi}{4}+2n\pi\right)} \\ &= e^{\pi\log_e\sqrt{2}}e^{i\pi\left(-\frac{3\pi}{4}+2n\pi\right)} = 2^{\pi/2}\left[\cos\left(-\frac{3\pi^2}{4} + 2n\pi^2\right) + i\sin\left(-\frac{3\pi^2}{4} + 2n\pi^2\right)\right] \end{aligned}$$

For $n = 0$ we have $z^\beta = 2^{\pi/2}\left[\cos\left(-\frac{3\pi^2}{4}\right) + i\sin\left(-\frac{3\pi^2}{4}\right)\right] \simeq 1.2969 - i2.6726$.
(**c**) We have (see part e of Problem 9):

$$z^\beta = i^{i4} = e^{i4\ln i} = e^{i4[\log_e 1 + i(\pi/2+2n\pi)]} = e^{i4[i(\pi/2+2n\pi)]} = e^{-4(\pi/2+2n\pi)} = e^{-2(1+4n)\pi}$$

For $n = 0$ we have $z^\beta = e^{-2\pi} \simeq 0.001867$.[133]
(**d**) We have (see part h of Problem 9):

$$\begin{aligned} z^\beta &= \left(\sqrt{\pi} - i7\right)^{e-i3} = e^{(e-i3)\ln(\sqrt{\pi}-i7)} \simeq e^{(e-i3)\left[\log_e\sqrt{\pi+49} + i(-1.3228+2n\pi)\right]} \\ &= e^{\left[e\log_e\sqrt{\pi+49} + 3(-1.3228+2n\pi)\right] + i\left[e(-1.3228+2n\pi) - 3\log_e\sqrt{\pi+49}\right]} \end{aligned}$$

For $n = 0$ we have $z^\beta \simeq e^{\left[e\log_e\sqrt{\pi+49}+3(-1.3228)\right]+i\left[e(-1.3228)-3\log_e\sqrt{\pi+49}\right]} \simeq -4.05677 + i0.4149$.
(**e**) We know that by definition $i^2 = -1$. However, to check the consistency of our rules and definitions let apply $z^\beta = e^{\beta\ln z}$, that is (see part e of Problem 9):

$$z^\beta = i^2 = e^{2\ln i} = e^{2i(\pi/2+2n\pi)} = e^{i(\pi+4n\pi)} = \cos(\pi + 4n\pi) + i\sin(\pi + 4n\pi) = -1 + i0 = -1$$

17. Confirm the results of parts (b) and (c) of Problem 16 (for $n = 0$) using the polar form of z.
   Answer:
   (**b**) $(-1-i)^\pi = \left(\sqrt{2}e^{i(-3\pi/4)}\right)^\pi = 2^{\pi/2}e^{-i(3\pi^2/4)} = 2^{\pi/2}\left(\cos\frac{3\pi^2}{4} - i\sin\frac{3\pi^2}{4}\right) \simeq 1.2969 - i2.6726$.
   (**c**) $z^\beta = i^{i4} = \left(e^{i\pi/2}\right)^{i4} = e^{-2\pi} \simeq 0.001867$.
   As we see, these results are the same as those of Problem 16 (for $n = 0$).
18. Verify that e^z is periodic and find its period.
   Answer: If e^z is periodic then for a given complex constant C we should have:

$$e^{z+C} = e^z$$

[133] If we write $z^\beta = i^{i4} = (i^i)^4$ and use Eq. 106 we get $z^\beta = (e^{-\pi/2})^4 = e^{-2\pi}$ (which is the same result). However, if we write $z^\beta = i^{i4} = (i^4)^i = 1^i$ and use Eq. 105 we get $z^\beta = 1$ (which is different). So, the readers should be aware of possible restrictions and conditions on such expressions (as well as potential twists). What we can say is that if we treat $i4$ as a single (imaginary) number then we should get $z^\beta = e^{-2\pi}$. Accordingly, $z^\beta = 1$ is correct only if we mean raising the number i^4 (which is 1) to the power i (rather than raising the number i to the power $i4$). Therefore, $z^\beta = e^{-2\pi}$ if $z^\beta = (i)^{i4} = (i^i)^4$ and $z^\beta = 1$ if $z^\beta = (i^4)^i$. In fact, these considerations are not restricted to this example but they also apply to other similar examples and occurrences throughout the subject of complex analysis. Therefore, extra care is required in interpreting and manipulating such expressions and formulations (especially when dealing with these in physical contexts where each expression and formulation can have a specific physical significance). Also, see part (c) of Problem 17.

$$\begin{aligned} e^C &= 1 && (\div e^z) \\ C &= \ln 1 = \log_e 1 + i2n\pi = i2n\pi \end{aligned}$$

As we see, C does exist and hence e^z is periodic. Now, the period of a function corresponds to the interval between two consecutive cycles and hence the period of e^z is $i2(n+1)\pi - i2n\pi = i2\pi$.
Note 1: referring to Figure 20 (in the upcoming Problem 23), we can see that the real and imaginary parts of e^z are periodic in the y direction with a period of 2π (in agreement with the period $i2\pi$ which we found in the present Problem). Also, see Problem 3 of § 7.1. We note that the periodicity of e^z (with a period of $i2\pi$) means that e^z takes all its possible values within the horizontal strip $-\pi < y \le \pi$ and hence e^z is made of identical copies of this strip (where each strip is displaced from the strip $-\pi < y \le \pi$ by $2n\pi$ in the y direction) and hence the strip $-\pi < y \le \pi$ is commonly described as a fundamental region of e^z. We also note that the periodicity of the complex exponential function e^z is one of the main differences between e^z and e^x (i.e. the real exponential function) because e^x is not periodic.
Note 2: if e^z has a period of $i2\pi$ then e^{-z} should also have a period of $i2\pi$. This can be demonstrated by repeating the above argument, that is:

$$e^{-(z-C)} = e^{-z} \qquad \to \qquad e^C = 1 \qquad \to \qquad C = \ln 1 = i2n\pi$$

Hence, the period of e^{-z} is $i2(n+1)\pi - i2n\pi = i2\pi$.
Also, if e^z has a period of $i2\pi$ then e^{iz} should have a period of 2π. This can be demonstrated by repeating the above argument, that is:

$$e^{i(z+C)} = e^{iz} \qquad \to \qquad e^{iC} = 1 \qquad \to \qquad iC = \ln 1 = i2n\pi \qquad \to \qquad C = 2n\pi$$

Hence, the period of e^{iz} is $2(n+1)\pi - 2n\pi = 2\pi$. Similarly, e^{-iz} should also have a period of 2π as can be demonstrated by repeating the above argument, that is:

$$e^{-i(z-C)} = e^{-iz} \qquad \to \qquad e^{iC} = 1 \qquad \to \qquad iC = \ln 1 = i2n\pi \qquad \to \qquad C = 2n\pi$$

Hence, the period of e^{-iz} is $2(n+1)\pi - 2n\pi = 2\pi$.

19. Verify that the following rules of differentiation apply to complex variables (as to real variables):
(**a**) $\frac{de^z}{dz} = e^z$. (**b**) $\frac{d\ln z}{dz} = \frac{1}{z} \quad (z \ne 0)$. (**c**) $\frac{d}{dz}(z\ln z - z) = \ln z \quad (z \ne 0)$.
Answer:[134]
(**a**) From the definition of e^z (see Eq. 6) plus the sum, multiple constant and power rules of differentiation (see Problem 2 of § 1.10) we have:

$$\frac{de^z}{dz} = \frac{d}{dz}\left[\sum_{n=0}^{\infty}\frac{z^n}{n!}\right] = \sum_{n=0}^{\infty}\frac{d}{dz}\left(\frac{z^n}{n!}\right) = \sum_{n=1}^{\infty}\frac{nz^{n-1}}{n!} = \sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{z^n}{n!} = e^z$$

(**b**) We have:

$$\begin{aligned} e^{\ln z} &= z && \text{(see Eq. 123)} \\ \frac{de^{\ln z}}{dz} &= \frac{dz}{dz} && \text{(taking derivative of both sides)} \\ \frac{de^{\ln z}}{d\ln z} \times \frac{d\ln z}{dz} &= 1 && \text{(chain and power rules; see Problem 2 of § 1.10)} \\ e^{\ln z} \times \frac{d\ln z}{dz} &= 1 && \text{(result of part a)} \\ z\,\frac{d\ln z}{dz} &= 1 && \text{(see Eq. 123)} \end{aligned}$$

[134] The reader should also refer to Problem 14 of § 3.1 for a different proof for parts (a) and (b).

$$\frac{d \ln z}{dz} = \frac{1}{z}$$

We note that this equation (as well as the equation of part c) should apply for individual (continuous) branches of $\ln z$ with the removal of the branch cut (including the origin which is the branch point of $\ln z$). We should also note that this rule of differentiation is an example of the fact that the domain of the derivative (i.e. where it is analytic) can be larger than the domain of the original function (since $1/z$ excludes the origin only while $\ln z$ excludes the branch cut). However, the opposite does not occur.
(**c**) Using the sum, product and power rules of differentiation (see Problem 2 of § 1.10) plus the result of part (b) we have:

$$\frac{d}{dz}(z \ln z - z) = \frac{d}{dz}(z \ln z) - \frac{dz}{dz} = \ln z \frac{dz}{dz} + z \frac{d \ln z}{dz} - \frac{dz}{dz} = \ln z + z\frac{1}{z} - 1 = \ln z + 1 - 1 = \ln z$$

Note: it is obvious that the integration rules that correspond to the above differentiation rules are:
$\int e^z\, dz = e^z + C$ $\quad$ $\int \frac{1}{z}\, dz = \ln z + C$ $\quad$ $\int \ln z\, dz = z \ln z - z + C$

20. Evaluate the following complex exponential and logarithmic integrals:
(**a**) $\int_{-i7}^{11+i4} e^{iz+8}\, dz$. (**b**) $\int_{1+i3}^{2-i2} \mathrm{Ln}(z^2)\, dz$. (**c**) $\int_{\pi/4}^{\pi/3} e^{iz}\, dz$. (**d**) $\int_1^3 \mathrm{Ln}(i\pi z)\, dz$.
Answer:

(**a**)
$$\int_{-i7}^{11+i4} e^{iz+8}\, dz = \left[\frac{e^{iz+8}}{i}\right]_{-i7}^{11+i4} = \frac{e^{i(11+i4)+8}}{i} - \frac{e^{i(-i7)+8}}{i} = \frac{e^{i11+4} - e^{15}}{i}$$
$$= i\left(e^{15} - e^{4+i11}\right) \simeq -54.5976 + i3269017.1308$$

(**b**)
$$\int_{1+i3}^{2-i2} \mathrm{Ln}(z^2)\, dz = \int_{1+i3}^{2-i2} 2\,\mathrm{Ln}(z)\, dz = 2\big[z\,\mathrm{Ln}\, z - z\big]_{1+i3}^{2-i2}$$
$$= 2\Big[(2-i2)\,\mathrm{Ln}(2-i2) - (2-i2)\Big] - 2\Big[(1+i3)\,\mathrm{Ln}(1+i3) - (1+i3)\Big]$$
$$\simeq 2\Big[(2-i2)\,(1.03972 - i0.7854) - (2-i2)\Big]$$
$$- 2\Big[(1+i3)\,(1.1513 + i1.2490) - (1+i3)\Big]$$
$$\simeq 4.2090 - i6.7063$$

(**c**)
$$\int_{\pi/4}^{\pi/3} e^{iz}\, dz = \left[\frac{e^{iz}}{i}\right]_{\pi/4}^{\pi/3} = \frac{e^{i\pi/3}}{i} - \frac{e^{i\pi/4}}{i} = i\left[e^{i\pi/4} - e^{i\pi/3}\right]$$
$$= i\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} - \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = i\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\sqrt{3}-\sqrt{2}}{2} + i\frac{\sqrt{2}-1}{2} \simeq 0.1589 + i0.2071$$

(**d**)
$$\int_1^3 \mathrm{Ln}(i\pi z)\, dz = \big[z\,\mathrm{Ln}(i\pi z) - z\big]_1^3 = \big[3\,\mathrm{Ln}(i\pi 3) - 3\big] - \big[\mathrm{Ln}(i\pi) - 1\big] = 3\,\mathrm{Ln}(i\pi 3) - \mathrm{Ln}(i\pi) - 2$$
$$\simeq 3\left(2.2433 + i\frac{\pi}{2}\right) - \left(1.1447 + i\frac{\pi}{2}\right) - 2 \simeq 3.5853 + i\pi$$

21. Why is the natural logarithm function not defined at the origin?
Answer: If we use the definition of the natural logarithm function then we have:

$$\ln(0 + i0) = \log_e \sqrt{0^2 + 0^2} + i\left[\mathrm{Arg}(0) + 2n\pi\right] = \log_e(0) + i\left[\mathrm{Arg}(0) + 2n\pi\right]$$

As we see, neither $\log_e(0)$ nor $\mathrm{Arg}(0)$ are defined and hence the logarithm function is not defined at the origin.

Note: considering the fact that the exponential and logarithm functions are inverses (as well as the fact that the range of the exponential function is all $z \in \mathbb{C}$ excluding zero; see Problem 1) we can see that the logarithm function should not be defined at the origin.

22. Show that the points of discontinuity of the complex logarithm function are on the negative real axis and hence determine the branch cut of this function.
Answer: If we follow a similar reasoning to that of Problem 10 of § 1.11 then we have (considering the individual branches):

$$\begin{aligned}\lim_{z\to -r\downarrow}\left[\ln z\right] &= \lim_{\theta_p\to +\pi}\left[\ln r + i(\theta_p + 2n\pi)\right] = \ln r + i(+\pi + 2n\pi) = \ln r + i(2n+1)\pi \\ \lim_{z\to -r\uparrow}\left[\ln z\right] &= \lim_{\theta_p\to -\pi}\left[\ln r + i(\theta_p + 2n\pi)\right] = \ln r + i(-\pi + 2n\pi) = \ln r + i(2n-1)\pi\end{aligned}$$

So, the branch cut is the non-positive real axis (where the branch point $z = 0$ is included because $\ln z$ is not defined at the origin as seen in Problem 21).

23. Make 3D plots of the following (and comment on the plots):
(**a**) The real and imaginary parts of the exponential function e^z over the region $-0.5 \le x \le 1$ and $-2\pi \le y \le 2\pi$.
(**b**) The real and imaginary parts of the principal value of the natural logarithm function $\mathrm{Ln}(z)$ over the region $-2 \le x \le 2$ and $-2 \le y \le 2$ (excluding the origin).
Answer:
(**a**) The function e^z is given by (see Eq. 9):

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y) = e^x\cos y + ie^x\sin y$$

Hence, its real part is $\mathrm{Re}(e^z) = e^x\cos y$ and its imaginary part is $\mathrm{Im}(e^z) = e^x\sin y$. These parts are plotted over the region $-0.5 \le x \le 1$ and $-2\pi \le y \le 2\pi$ in Figure 20.
Comment: regarding $\mathrm{Re}(e^z) = e^x\cos y$, it is a superposition of a "wavy" cosine function in the y direction and an exponential function in the x direction and hence the cosine waves in the y direction are moderated by an exponential function in the x direction where the positive peaks and the negative troughs of the waves determine whether the exponential ascends or descends (i.e. rises up or drops down in the positive x direction). Accordingly, as we move along lines of constant x in the y direction we see ordinary cosine waves whose magnitudes are scaled by the constant e^x, while as we move along lines of constant y in the positive x direction we see (ascending or descending) exponential curves but they become straight lines when $\cos y = 0$, i.e. when $y = \left(n + \frac{1}{2}\right)\pi$. Regarding $\mathrm{Im}(e^z) = e^x\sin y$, it is identical to $\mathrm{Re}(e^z)$ but with a phase lag (i.e. along the y direction) of $\pi/2$ due to the relation between the sine and cosine functions, i.e. $\sin\left(y + \frac{\pi}{2}\right) = \cos(y)$.
(**b**) The function $\mathrm{Ln}(z)$ is given by:

$$\mathrm{Ln}(z) = \log_e|z| + i\mathrm{Arg}(z) = \log_e\sqrt{x^2+y^2} + i\arctan\left(\frac{y}{x}\right)$$

Hence, its real part is $\mathrm{Re}(\mathrm{Ln}\, z) = \log_e\sqrt{x^2+y^2}$ and its imaginary part is $\mathrm{Im}(\mathrm{Ln}\, z) = \arctan(y/x)$. These parts are plotted over the region $-2 \le x \le 2$ and $-2 \le y \le 2$ in Figure 21 (noting that the plot of the imaginary part has some unavoidable distortions at and around the cut).
Comment: as we see, $\mathrm{Re}(\mathrm{Ln}\, z)$ has circular symmetry around the origin due to the circular symmetry of $\sqrt{x^2+y^2}$. It should be noted that the plot of $\mathrm{Im}(\mathrm{Ln}\, z)$ is based on our convention $-\pi < \theta_p \le \pi$ (where θ_p is the principal value of the argument of complex number) which we adopted earlier (see for instance § 1.8.2). As we see, $\mathrm{Im}(\mathrm{Ln}\, z)$ looks like a helical cylinder or helicoid where at each fixed θ_p a straight line emanates perpendicularly from the $\mathrm{Im}(\mathrm{Ln}\, z)$ axis[135] and it is parallel to the xy plane

[135] The $\mathrm{Im}(\mathrm{Ln}\, z)$ axis is the "z axis" noting that this z corresponds to x and y and does not mean complex number. We should also note that the $\mathrm{Im}(\mathrm{Ln}\, z)$ axis that we are talking about here is the line that is perpendicular to the xy plane at the origin of the xy plane [unlike what appears in the plot where it is displaced to the $(x, y) = (-2, 2)$ point for graphical reason].

(as well as being at $|\theta_p|$ below or above the xy plane). Accordingly, as we rotate around the $\mathrm{Im}(\mathrm{Ln}\, z)$ axis starting from $\theta_p = -\pi$ (but excluding $-\pi$ itself) and ending at $\theta_p = \pi$ we ascend on a helical stair rising from level $-\pi$ to level π.

24. Referring to the lower frame of Figure 21, try to find a characteristic property that distinguishes the branch point (which is the origin) from any other point in the complex plane.
Answer:[136] We note that the branch point is the endpoint of the branch cut and hence it is surrounded by points that belong to the (reduced) domain of the branch from all directions except the left direction (i.e. the direction of the negative real axis). So, any circle enclosing the branch point is in the domain of the branch with the exclusion of a single point on the circle (which is the point on the negative real axis). In other words, the branch point cannot be enclosed by a circle that is entirely in the domain of the branch or by a circle that is in the domain of the branch with the exclusion of more than one point. This is unlike any point P_d in the domain of the branch since P_d can be enclosed by a (potentially infinitesimal) circle that is entirely in the domain of the branch with no exclusion. This is also unlike any (other) point P_c on the branch cut because if P_c is enclosed by a sufficiently small infinitesimal circle then the circle will be in the domain of the branch but with the unavoidable exclusion of more than one point. Accordingly, if we traverse[137] any circle enclosing the branch point the branch becomes discontinuous at one (and only one) point. On the other hand, a point in the domain of the branch can be enclosed by a circle which can be traversed with no discontinuity in the branch while a point on the branch cut (other than the branch point) cannot be enclosed by an arbitrarily small infinitesimal circle that can be traversed with less than two discontinuities in the branch.
Note 1: from the above explanation (considering a branch cut that extends to infinity as it is the case in our question), we can see that a circle (of arbitrary size) centered on the branch point must have one (and only one) discontinuity point, a circle centered on a point in the domain may have zero or one or two discontinuity points, and a circle centered on a point on the branch cut (other than the branch point) can have one or two discontinuity points. So, to ease the characterization we use arbitrarily small (infinitesimal) circles centered on these points. Accordingly, we can say: the branch point is the point that all the circles centered on it have a single discontinuity point, a domain point is a point that arbitrarily small infinitesimal circles centered on it have no discontinuity point, and a point on the branch cut (other than the branch point) is a point that arbitrarily small infinitesimal circles centered on it have two discontinuity points. We may also phrase the above characterization as follows: domain point can be enclosed by circles that are entirely in the domain, branch cut point cannot be enclosed by circles that are entirely in the domain but can be enclosed by circles that are in the domain with the exclusion of more than one point, and branch point cannot be enclosed by circles that are entirely in the domain and cannot be enclosed by circles that are in the domain with the exclusion of more than one point (and hence it can be enclosed only by circles that are in the domain with the exclusion of one and only one point).
Note 2: in the above answer, we adopt a view that considers branch cut in a more natural (rather than conventional) way where a certain line or curve should specifically be a branch cut and this is determined by our definitions and conventions with regard to the function and its principal branch. In fact, this is not inline with the (seemingly common) view in the literature that makes the determination of the branch cut a rather arbitrary choice whose main purpose is to prevent crossing from one branch to another branch and hence restricting the function to a single value (or branch) so that it becomes single-valued.
25. Identify the branch cut(s) and branch point(s) of $f(z) = \sqrt{e^z + 2}$.

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[136] In this answer (whose purpose is to give a rather primitive and simple idea of branch point rather than giving a rigorous definition), we assume that the branch cuts for all the branches are represented by a single curve (as seen in Figure 21 which the question is based on) rather than by multiple curves and that is why we talk about "branch cut" rather than "branch cuts" (noting that plurality may refer to the plurality of branches or the plurality of the cut, i.e. we have multiple branch cuts represented by different curves as in Problem 25). Also, see Problem 21 of § 1.5 for the definition of branch point. We should also note that although Figure 21 belongs to the principal branch (or rather value) of $\ln z$, the question and the answer apply to every branch of $\ln z$ (as well as to similar functions like $\sqrt{z}$).

[137] It should be obvious that "traverse" here and in the following means performing a single 2π revolution starting and ending at the same point.

Answer: Let $Z = e^z + 2$. The branch cut of $\sqrt{Z}$ is obviously $Z \in \mathbb{R}$ and $Z \leq 0$. So, the branch cuts of f occur where $(e^z + 2) \in \mathbb{R}$ and $(e^z + 2) \leq 0$, i.e. $e^z \in \mathbb{R}$ and $e^z \leq -2$. Now, $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$ and hence we have $e^z \in \mathbb{R}$ only when $\sin y = 0$ (i.e. when $y = m\pi$); moreover we have $e^z \leq -2$ only when $\cos y = -1$ (considering the condition $y = m\pi$), i.e. when $y = (2n+1)\pi$. On combining these conditions (noting that the latter satisfies the former since it is more restrictive) we get $y = (2n+1)\pi$. On inserting this into the relation $e^z \leq -2$ we obtain:

$$\begin{aligned} e^{x+i(2n+1)\pi} &\leq -2 \\ -e^x &\leq -2 \\ e^x &\geq 2 \\ x &\geq \log_e 2 \end{aligned}$$

Hence, the branch cuts are the horizontal semi-lines represented by $z = x + iy = x + i(2n+1)\pi$ where $x \geq \log_e 2$ and $n = 0, \pm 1, \pm 2, \cdots$.
From the definition of branch point (see § 1.5 and Problem 21 of that section in particular), the branch points of $f(z) = \sqrt{e^z + 2}$ are $z = \log_e 2 + i(2n+1)\pi$.

26. What is the number of sheets of the Riemann surface of $\ln z$ and why?
Answer: It is infinite because $\ln z$ has infinite number of distinct branches.

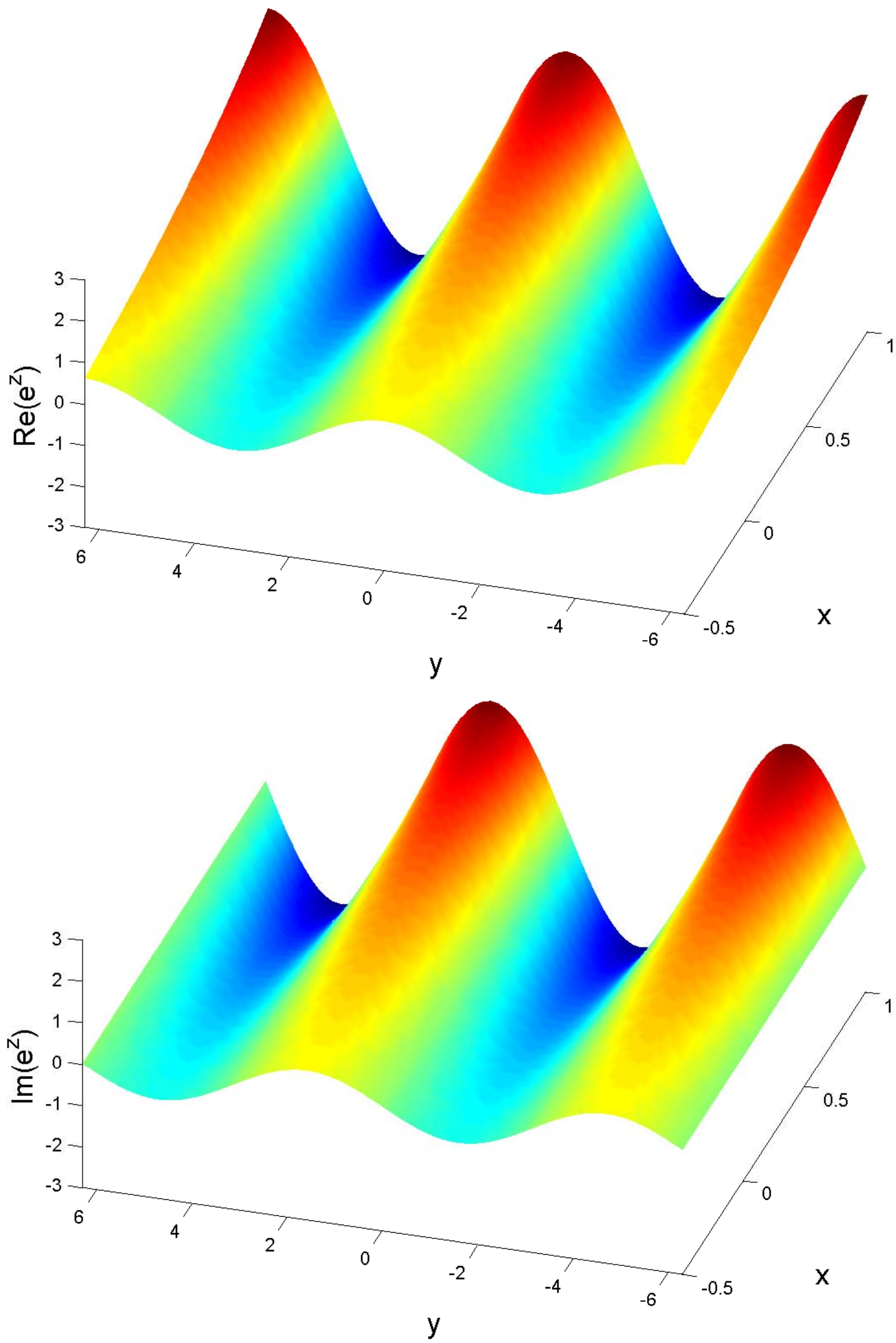

Figure 20: Graphic illustration of the real and imaginary parts of the complex function e^z over the rectangular region in the z plane defined by $-0.5 \leq x \leq 1$ and $-2\pi \leq y \leq 2\pi$. See part (a) of Problem 23 of § 2.2.

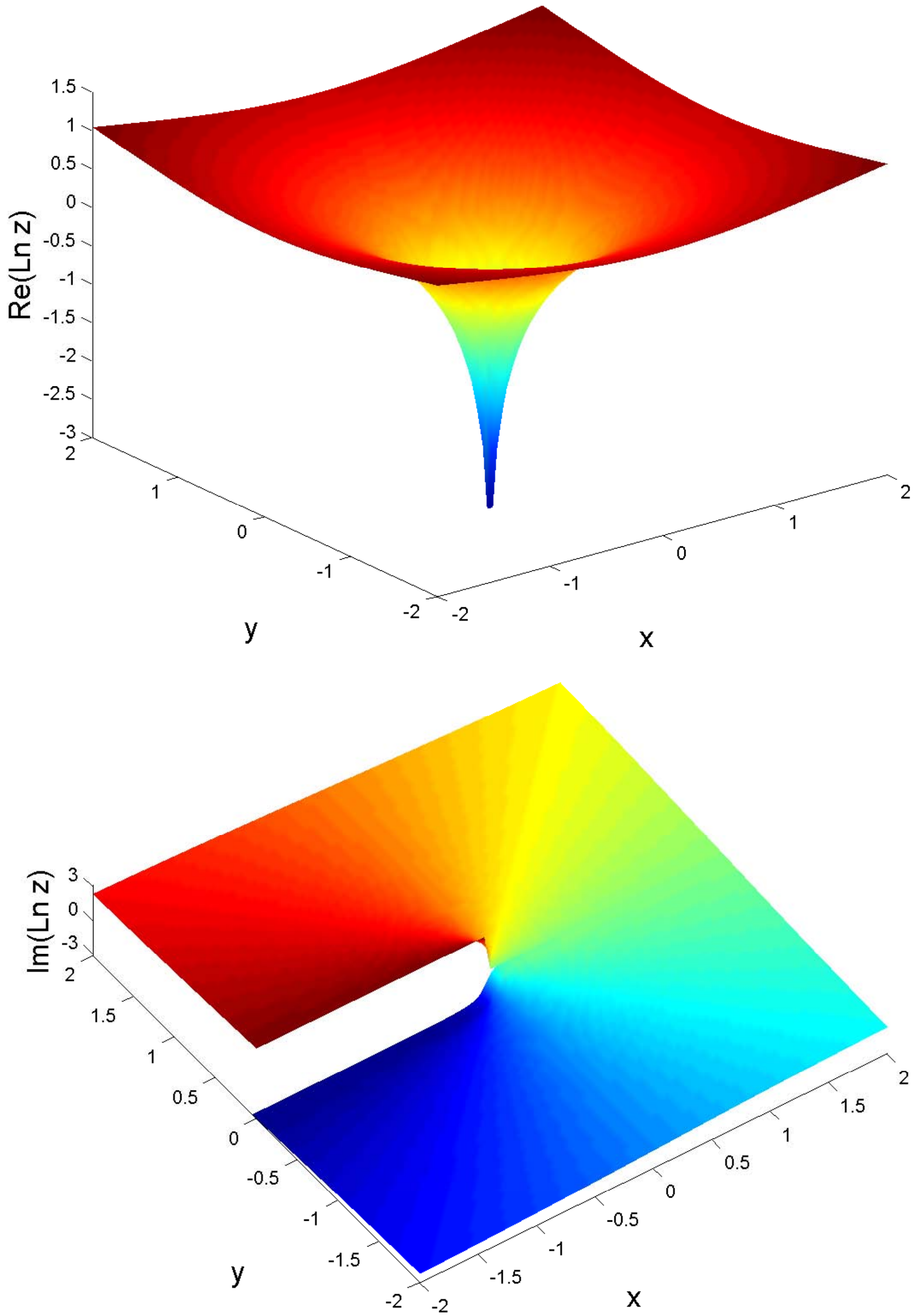

Figure 21: Graphic illustration of the real and imaginary parts of the complex function $\operatorname{Ln}(z)$ over the rectangular region in the z plane defined by $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$ (excluding the origin). As we see, $\operatorname{Im}(\operatorname{Ln} z)$ is discontinuous along the negative real line and hence $\operatorname{Ln} z$ itself is discontinuous along this line (refer to Problem 8 of § 1.11). See part (b) of Problem 23 of § 2.2.

2.3 Trigonometric and Hyperbolic Functions

The trigonometric functions of a complex number $z = x + iy$ are defined as follows:[138]

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{i2} \qquad \tan z = \frac{\sin z}{\cos z} \tag{131}$$

$$\sec z = \frac{1}{\cos z} \qquad \csc z = \frac{1}{\sin z} \qquad \cot z = \frac{1}{\tan z} \tag{132}$$

Because e^{iz} and e^{-iz} are entire functions, $\cos z$ and $\sin z$ are entire functions (see Problem 3 of § 1.5 and Problem 4 of § 3.1). However, this does not apply to the rest of the trigonometric functions since they have singularities (as can be seen from their definitions where the denominator can vanish for some values of z). Now, $\cos z$ is zero for the real numbers $z = \frac{(2n+1)\pi}{2}$ (see part a of Problem 14) and hence from the above definitions we can see that $z = \frac{(2n+1)\pi}{2}$ are singularities for $\tan z$ and $\sec z$, i.e. $\tan z$ and $\sec z$ are analytic for all z except for $z = \frac{(2n+1)\pi}{2}$. Similarly, $\sin z$ is zero for the real numbers $z = n\pi$ (see part b of Problem 14) and hence from the above definitions we can see that $z = n\pi$ are singularities for $\csc z$ and $\cot z$, i.e. $\csc z$ and $\cot z$ are analytic for all z except for $z = n\pi$.

In the same way, the hyperbolic functions of a complex number z are defined as follows:

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \qquad \sinh z = \frac{e^{z} - e^{-z}}{2} \qquad \tanh z = \frac{\sinh z}{\cosh z} \tag{133}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} \qquad \operatorname{csch} z = \frac{1}{\sinh z} \qquad \coth z = \frac{1}{\tanh z} \tag{134}$$

Because e^{z} and e^{-z} are entire functions, $\cosh z$ and $\sinh z$ are entire functions (see Problem 3 of § 1.5 and Problem 4 of § 3.1). However, this does not apply to the rest of the hyperbolic functions since they have singularities (as can be seen from their definitions where the denominator can vanish for some values of z). Now, $\cosh z$ is zero for the imaginary numbers $z = i\frac{(2n+1)\pi}{2}$ (see part c of Problem 14) and hence $z = i\frac{(2n+1)\pi}{2}$ are singularities for $\tanh z$ and $\operatorname{sech} z$, i.e. $\tanh z$ and $\operatorname{sech} z$ are analytic for all z except for $z = i\frac{(2n+1)\pi}{2}$. Similarly, $\sinh z$ is zero for the imaginary numbers $z = in\pi$ (see part d of Problem 14) and hence $z = in\pi$ are singularities for $\operatorname{csch} z$ and $\coth z$, i.e. $\operatorname{csch} z$ and $\coth z$ are analytic for all z except for $z = in\pi$.

There is a strong relationship between the trigonometric functions and the hyperbolic functions (as the above relations suggest where z in the definitions of $\cosh z$ and $\sinh z$ corresponds to iz in the definitions of $\cos z$ and $\sin z$ with an added i in the denominator of $\sin z$). This relationship will be investigated further in the upcoming Problems (see Problem 5 in particular).

Problems

1. What are the domain and range of $\cos z$, $\sin z$, $\cosh z$ and $\sinh z$?
   Answer: The domain and range of these functions are all $z \in \mathbb{C}$.
2. Verify that the following identities apply to complex variables (as to real variables):
   (**a**) $\cos^2 z + \sin^2 z = 1$.
   (**b**) $1 + \tan^2 z = \sec^2 z$.
   (**c**) $\cot^2 z + 1 = \csc^2 z$.
   (**d**) $\cos(-z) = \cos(z)$.
   (**e**) $\sin(-z) = -\sin(z)$.
   (**f**) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$.
   (**g**) $\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$.
   (**h**) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.
   (**i**) $\sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$.
   (**j**) $\cos(2z) = \cos^2 z - \sin^2 z$.
   (**k**) $\sin(2z) = 2\cos z \,\sin z$.
   (**l**) $\tan(z_1 + z_2) = \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$.

[138] As seen earlier (refer to parts g and h of Problem 1 of § 1.4), the definitions of $\cos z$ and $\sin z$ (and consequently the other functions which are based on these functions) are in fact based on our mathematical foundations (as represented here by Eqs. 8 and 10) and hence they are not really definitions in the strict sense.

Answer:
(**a**) We have:

$$\cos^2 z + \sin^2 z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{i2}\right)^2 = \frac{e^{i2z}+2+e^{-i2z}}{4} + \frac{e^{i2z}-2+e^{-i2z}}{-4} = 1$$

(**b**) We divide $\cos^2 z + \sin^2 z = 1$ (which we verified in part a) by $\cos^2 z$ ($\neq 0$) to obtain this identity.
(**c**) We divide $\cos^2 z + \sin^2 z = 1$ (which we verified in part a) by $\sin^2 z$ ($\neq 0$) to obtain this identity.
(**d**) We have:[139]

$$\cos(-z) = \frac{e^{i(-z)}+e^{-i(-z)}}{2} = \frac{e^{-iz}+e^{iz}}{2} = \frac{e^{iz}+e^{-iz}}{2} = \cos(z) \tag{135}$$

(**e**) We have:

$$\sin(-z) = \frac{e^{i(-z)}-e^{-i(-z)}}{i2} = \frac{e^{-iz}-e^{iz}}{i2} = -\frac{e^{iz}-e^{-iz}}{i2} = -\sin z$$

(**f**) We start from the right hand side to obtain the left hand side,[140] that is:

$$\begin{aligned}
\cos z_1 \cos z_2 - \sin z_1 \sin z_2 &= \left(\frac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right) - \left(\frac{e^{iz_1}-e^{-iz_1}}{i2}\right)\left(\frac{e^{iz_2}-e^{-iz_2}}{i2}\right) \\
&= \frac{e^{iz_1}e^{iz_2}+e^{iz_1}e^{-iz_2}+e^{-iz_1}e^{iz_2}+e^{-iz_1}e^{-iz_2}}{4} - \\
&\qquad \frac{e^{iz_1}e^{iz_2}-e^{iz_1}e^{-iz_2}-e^{-iz_1}e^{iz_2}+e^{-iz_1}e^{-iz_2}}{-4} \\
&= \frac{2e^{iz_1}e^{iz_2}+2e^{-iz_1}e^{-iz_2}}{4} = \frac{e^{i(z_1+z_2)}+e^{-i(z_1+z_2)}}{2} = \cos(z_1+z_2)
\end{aligned}$$

(**g**) We obtain this identity from the identity of part (f) by replacing z_2 by $-z_2$ noting that $\cos(-z) = \cos(z)$ and $\sin(-z) = -\sin(z)$, as shown in parts (d) and (e).
(**h**) We start from the right hand side to obtain the left hand side, that is:

$$\begin{aligned}
\sin z_1 \cos z_2 + \cos z_1 \sin z_2 &= \left(\frac{e^{iz_1}-e^{-iz_1}}{i2}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right) + \left(\frac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2}-e^{-iz_2}}{i2}\right) \\
&= \frac{e^{iz_1}e^{iz_2}+e^{iz_1}e^{-iz_2}-e^{-iz_1}e^{iz_2}-e^{-iz_1}e^{-iz_2}}{i4} + \\
&\qquad \frac{e^{iz_1}e^{iz_2}-e^{iz_1}e^{-iz_2}+e^{-iz_1}e^{iz_2}-e^{-iz_1}e^{-iz_2}}{i4} \\
&= \frac{2e^{iz_1}e^{iz_2}-2e^{-iz_1}e^{-iz_2}}{i4} = \frac{e^{i(z_1+z_2)}-e^{-i(z_1+z_2)}}{i2} = \sin(z_1+z_2)
\end{aligned}$$

(**i**) We obtain this identity from the identity of part (h) by replacing z_2 by $-z_2$ noting that $\cos(-z) = \cos(z)$ and $\sin(-z) = -\sin(z)$, as shown in parts (d) and (e).
(**j**) We use the identity of part (f) with $z_1 = z_2 = z$, that is:

$$\cos(2z) = \cos(z+z) = \cos z\, \cos z - \sin z\, \sin z = \cos^2 z - \sin^2 z$$

(**k**) We use the identity of part (h) with $z_1 = z_2 = z$, that is:

$$\sin(2z) = \sin(z+z) = \sin z\, \cos z + \cos z\, \sin z = 2\cos z\, \sin z$$

[139] This identity was verified earlier (see part b of Problem 1 of § 1.4) using another method. This also applies to the identity of part (e) which was verified in part (c) of Problem 1 of § 1.4.

[140] In fact, this is not necessary but it is for clarity because we can reverse the following steps (and hence we start from the left to obtain the right).

(**l**) We start from the right hand side to obtain the left hand side, that is:

$$\begin{aligned}\frac{\tan z_1+\tan z_2}{1-\tan z_1\tan z_2} &= \frac{\frac{\sin z_1}{\cos z_1}+\frac{\sin z_2}{\cos z_2}}{1-\frac{\sin z_1}{\cos z_1}\frac{\sin z_2}{\cos z_2}} = \frac{\frac{\sin z_1\cos z_2+\cos z_1\sin z_2}{\cos z_1\cos z_2}}{\frac{\cos z_1\cos z_2-\sin z_1\sin z_2}{\cos z_1\cos z_2}} = \frac{\sin z_1\cos z_2+\cos z_1\sin z_2}{\cos z_1\cos z_2-\sin z_1\sin z_2} \\ &= \frac{\sin(z_1+z_2)}{\cos(z_1+z_2)} = \tan(z_1+z_2)\end{aligned}$$

where we used the identities of parts (f) and (h).

3. Verify the following identities:

(**a**) $(\cos z)^* = \cos z^*$. (**b**) $(\sin z)^* = \sin z^*$. (**c**) $(\tan z)^* = \tan z^*$.

(**d**) $(\sec z)^* = \sec z^*$. (**e**) $(\csc z)^* = \csc z^*$. (**f**) $(\cot z)^* = \cot z^*$.

Answer: We use the rules of conjugation that we established in § 1.8.8 and elsewhere (see for example part b of Problem 3 of § 2.2) as well as rules that we establish in the present Problem.

$$\begin{aligned}(\mathbf{a})\quad (\cos z)^* &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)^* = \frac{\left(e^{iz}+e^{-iz}\right)^*}{2^*} = \frac{\left(e^{iz}\right)^*+\left(e^{-iz}\right)^*}{2} = \frac{e^{(iz)^*}+e^{(-iz)^*}}{2} \\ &= \frac{e^{-iz^*}+e^{iz^*}}{2} = \frac{e^{iz^*}+e^{-iz^*}}{2} = \cos z^* \\ (\mathbf{b})\quad (\sin z)^* &= \left(\frac{e^{iz}-e^{-iz}}{i2}\right)^* = \frac{\left(e^{iz}-e^{-iz}\right)^*}{(i2)^*} = \frac{\left(e^{iz}\right)^*-\left(e^{-iz}\right)^*}{-i2} = \frac{e^{(iz)^*}-e^{(-iz)^*}}{-i2} \\ &= \frac{e^{-iz^*}-e^{iz^*}}{-i2} = \frac{e^{iz^*}-e^{-iz^*}}{i2} = \sin z^* \\ (\mathbf{c})\quad (\tan z)^* &= \left(\frac{\sin z}{\cos z}\right)^* = \frac{(\sin z)^*}{(\cos z)^*} = \frac{\sin z^*}{\cos z^*} = \tan z^* \\ (\mathbf{d})\quad (\sec z)^* &= \left(\frac{1}{\cos z}\right)^* = \frac{1^*}{(\cos z)^*} = \frac{1}{\cos z^*} = \sec z^* \\ (\mathbf{e})\quad (\csc z)^* &= \left(\frac{1}{\sin z}\right)^* = \frac{1^*}{(\sin z)^*} = \frac{1}{\sin z^*} = \csc z^* \\ (\mathbf{f})\quad (\cot z)^* &= \left(\frac{1}{\tan z}\right)^* = \frac{1^*}{(\tan z)^*} = \frac{1}{\tan z^*} = \cot z^*\end{aligned}$$

4. Verify that the relations $e^{iz} = \cos z + i\sin z$ and $e^{-iz} = \cos z - i\sin z$ are consistent with the relations $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \frac{e^{iz}-e^{-iz}}{i2}$.

Answer: If we start from the relations $e^{iz} = \cos z + i\sin z$ and $e^{-iz} = \cos z - i\sin z$ then we have:

$$\begin{aligned}\frac{e^{iz}+e^{-iz}}{2} &= \frac{(\cos z+i\sin z)+(\cos z-i\sin z)}{2} = \frac{2\cos z}{2} = \cos z \\ \frac{e^{iz}-e^{-iz}}{i2} &= \frac{(\cos z+i\sin z)-(\cos z-i\sin z)}{i2} = \frac{i2\sin z}{i2} = \sin z\end{aligned}$$

On the other hand, if we start from the relations $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \frac{e^{iz}-e^{-iz}}{i2}$ then we have:

$$\begin{aligned}\cos z + i\sin z &= \frac{e^{iz}+e^{-iz}}{2} + i\frac{e^{iz}-e^{-iz}}{i2} = \frac{e^{iz}+e^{-iz}}{2} + \frac{e^{iz}-e^{-iz}}{2} = \frac{2e^{iz}}{2} = e^{iz} \\ \cos z - i\sin z &= \frac{e^{iz}+e^{-iz}}{2} - i\frac{e^{iz}-e^{-iz}}{i2} = \frac{e^{iz}+e^{-iz}}{2} - \frac{e^{iz}-e^{-iz}}{2} = \frac{2e^{-iz}}{2} = e^{-iz}\end{aligned}$$

So, these relations are consistent.

5. Verify the following relations between the trigonometric and hyperbolic functions:
   (**a**) $\cos(iz) = \cosh z$ and $\cosh(iz) = \cos z$.
   (**b**) $\sin(iz) = i \sinh z$ and $\sinh(iz) = i \sin z$.
   (**c**) $\tan(iz) = i \tanh z$ and $\tanh(iz) = i \tan z$.
   (**d**) $\sec(iz) = \operatorname{sech} z$ and $\operatorname{sech}(iz) = \sec z$.
   (**e**) $\csc(iz) = -i \operatorname{csch} z$ and $\operatorname{csch}(iz) = -i \csc z$.
   (**f**) $\cot(iz) = -i \coth z$ and $\coth(iz) = -i \cot z$.
   Answer: We just use the definitions of the trigonometric and hyperbolic functions (as given in the text by Eqs. 131 and 132 and Eqs. 133 and 134) and results that we establish in the present Problem.
   (**a**)
$$\begin{aligned}\cos(iz) &= \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^{z}}{2} = \cosh z \\ \cosh(iz) &= \frac{e^{iz} + e^{-iz}}{2} = \cos z\end{aligned} \tag{136}$$
   (**b**)
$$\begin{aligned}\sin(iz) &= \frac{e^{i(iz)} - e^{-i(iz)}}{i2} = \frac{e^{-z} - e^{z}}{i2} = -i\frac{e^{-z} - e^{z}}{2} = i\frac{e^{z} - e^{-z}}{2} = i \sinh z \\ \sinh(iz) &= \frac{e^{iz} - e^{-iz}}{2} = i\frac{e^{iz} - e^{-iz}}{i2} = i \sin z\end{aligned}$$
   (**c**)
$$\begin{aligned}\tan(iz) &= \frac{\sin(iz)}{\cos(iz)} = \frac{i \sinh z}{\cosh z} = i \tanh z \\ \tanh(iz) &= \frac{\sinh(iz)}{\cosh(iz)} = \frac{i \sin z}{\cos z} = i \tan z\end{aligned}$$
   (**d**)
$$\begin{aligned}\sec(iz) &= \frac{1}{\cos(iz)} = \frac{1}{\cosh z} = \operatorname{sech} z \\ \operatorname{sech}(iz) &= \frac{1}{\cosh(iz)} = \frac{1}{\cos z} = \sec z\end{aligned}$$
   (**e**)
$$\begin{aligned}\csc(iz) &= \frac{1}{\sin(iz)} = \frac{1}{i \sinh z} = -i\frac{1}{\sinh z} = -i \operatorname{csch} z \\ \operatorname{csch}(iz) &= \frac{1}{\sinh(iz)} = \frac{1}{i \sin z} = -i\frac{1}{\sin z} = -i \csc z\end{aligned}$$
   (**f**)
$$\begin{aligned}\cot(iz) &= \frac{1}{\tan(iz)} = \frac{1}{i \tanh z} = -i\frac{1}{\tanh z} = -i \coth z \\ \coth(iz) &= \frac{1}{\tanh(iz)} = \frac{1}{i \tan z} = -i\frac{1}{\tan z} = -i \cot z\end{aligned}$$

6. Summarize (in words) the relations given in Problem 5.
   Answer: These relations can be compactly stated as follows:
   The trigonometric/hyperbolic function of iz is equal to the corresponding hyperbolic/trigonometric

function of z multiplied by:
• i if the former contains (explicitly or implicitly) sin/sinh in the numerator.
• $-i$ if the former contains (explicitly or implicitly) sin/sinh in the denominator.
• 1 otherwise.

7. Verify the following identities (which are related to the hyperbolic functions and their relation to the trigonometric functions) noting that $z = x + iy$ with $x, y \in \mathbb{R}$:

(**a**) $\cosh^2 z - \sinh^2 z = 1$.
(**b**) $1 - \tanh^2 z = \operatorname{sech}^2 z$.
(**c**) $\coth^2 z - 1 = \operatorname{csch}^2 z$.
(**d**) $\cosh(-z) = \cosh(z)$.
(**e**) $\sinh(-z) = -\sinh(z)$.
(**f**) $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$.
(**g**) $\cosh(z_1 - z_2) = \cosh z_1 \cosh z_2 - \sinh z_1 \sinh z_2$.
(**h**) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$.
(**i**) $\sinh(z_1 - z_2) = \sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2$.
(**j**) $\cosh(2z) = \cosh^2 z + \sinh^2 z$.
(**k**) $\sinh(2z) = 2\cosh z \sinh z$.
(**l**) $\cos z = \cos x \cosh y - i \sin x \sinh y$.
(**m**) $\sin z = \sin x \cosh y + i \cos x \sinh y$.
(**n**) $|\cos z|^2 = \cos^2 x + \sinh^2 y$.
(**o**) $|\cos z|^2 = \cosh^2 y - \sin^2 x$.
(**p**) $|\sin z|^2 = \sin^2 x + \sinh^2 y$.
(**q**) $|\sin z|^2 = \cosh^2 y - \cos^2 x$.
(**r**) $\cosh z = \cosh x \cos y + i \sinh x \sin y$.
(**s**) $\sinh z = \sinh x \cos y + i \cosh x \sin y$.
(**t**) $\cos i = \cosh 1$.
(**u**) $\sin i = i \sinh 1$.
(**v**) $\tanh(z_1 + z_2) = \frac{\tanh z_1 + \tanh z_2}{1 + \tanh z_1 \tanh z_2}$.

Answer: Due to the relationships between the trigonometric and hyperbolic functions (which were established in Problem 5) these identities can be obtained from similar trigonometric identities (see for instance Problem 2) as we indicated in the answer of some parts of the present Problem. However, for the sake of diversity and independence we generally use different methods (also see Problems 9 and 10).

(**a**) We have:

$$\cosh^2 z - \sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2 = \frac{e^{2z} + 2 + e^{-2z}}{4} - \frac{e^{2z} - 2 + e^{-2z}}{4} = \frac{4}{4} = 1$$

(**b**) We just divide $\cosh^2 z - \sinh^2 z = 1$ (which is verified in part a) by $\cosh^2 z$ ($\neq 0$) to obtain this identity.

(**c**) We just divide $\cosh^2 z - \sinh^2 z = 1$ (which is verified in part a) by $\sinh^2 z$ ($\neq 0$) to obtain this identity.

(**d**) We have:

$$\cosh(-z) = \frac{e^{(-z)} + e^{-(-z)}}{2} = \frac{e^z + e^{-z}}{2} = \cosh(z)$$

(**e**) We have:

$$\sinh(-z) = \frac{e^{(-z)} - e^{-(-z)}}{2} = -\frac{e^z - e^{-z}}{2} = -\sinh(z)$$

(**f**) We start from the right hand side to obtain the left hand side, that is:

$$\begin{aligned}\cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 &= \left(\frac{e^{z_1} + e^{-z_1}}{2}\right)\left(\frac{e^{z_2} + e^{-z_2}}{2}\right) + \left(\frac{e^{z_1} - e^{-z_1}}{2}\right)\left(\frac{e^{z_2} - e^{-z_2}}{2}\right) \\ &= \frac{e^{z_1+z_2} + e^{z_1-z_2} + e^{-z_1+z_2} + e^{-z_1-z_2}}{4} + \\ &\qquad \frac{e^{z_1+z_2} - e^{z_1-z_2} - e^{-z_1+z_2} + e^{-z_1-z_2}}{4}\end{aligned}$$

$$= \frac{2e^{z_1+z_2} + 2e^{-z_1-z_2}}{4} = \frac{e^{z_1+z_2} + e^{-(z_1+z_2)}}{2} = \cosh(z_1 + z_2)$$

(**g**) We obtain this identity from the identity of part (f) by replacing z_2 by $-z_2$ noting that $\cosh(-z) = \cosh(z)$ and $\sinh(-z) = -\sinh(z)$, as shown in parts (d) and (e).
(**h**) We start from the right hand side to obtain the left hand side, that is:

$$\begin{aligned}
\sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2 &= \left(\frac{e^{z_1} - e^{-z_1}}{2}\right)\left(\frac{e^{z_2} + e^{-z_2}}{2}\right) + \left(\frac{e^{z_1} + e^{-z_1}}{2}\right)\left(\frac{e^{z_2} - e^{-z_2}}{2}\right) \\
&= \frac{e^{z_1+z_2} + e^{z_1-z_2} - e^{-z_1+z_2} - e^{-z_1-z_2}}{4} + \\
&\qquad \frac{e^{z_1+z_2} - e^{z_1-z_2} + e^{-z_1+z_2} - e^{-z_1-z_2}}{4} \\
&= \frac{2e^{z_1+z_2} - 2e^{-z_1-z_2}}{4} = \frac{e^{z_1+z_2} - e^{-(z_1+z_2)}}{2} = \sinh(z_1 + z_2)
\end{aligned}$$

(**i**) We obtain this identity from the identity of part (h) by replacing z_2 by $-z_2$ noting that $\cosh(-z) = \cosh(z)$ and $\sinh(-z) = -\sinh(z)$, as shown in parts (d) and (e).
(**j**) We use the identity of part (f) with $z_1 = z_2 = z$, that is:

$$\cosh(2z) = \cosh(z + z) = \cosh z \, \cosh z + \sinh z \, \sinh z = \cosh^2 z + \sinh^2 z$$

(**k**) We use the identity of part (h) with $z_1 = z_2 = z$, that is:

$$\sinh(2z) = \sinh(z + z) = \sinh z \, \cosh z + \cosh z \, \sinh z = 2 \cosh z \, \sinh z$$

(**l**) We have:

$$\begin{aligned}
\cos z &= \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{2} \\
&= \frac{(\cos x + i \sin x)e^{-y} + (\cos x - i \sin x)e^{y}}{2} = \frac{e^{-y}\cos x + ie^{-y}\sin x + e^{y}\cos x - ie^{y}\sin x}{2} \\
&= \frac{e^{-y}\cos x + e^{y}\cos x}{2} + i\frac{e^{-y}\sin x - e^{y}\sin x}{2} = \cos x \frac{e^{y} + e^{-y}}{2} - i \sin x \frac{e^{y} - e^{-y}}{2} \\
&= \cos x \, \cosh y - i \sin x \, \sinh y
\end{aligned} \tag{137}$$

Note: we may also use the identity $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$ (see part f of Problem 2) plus the identities $\cos(iz) = \cosh z$ and $\sin(iz) = i \sinh z$ (see parts a and b of Problem 5), that is:

$$\cos z = \cos(x + iy) = \cos x \, \cos(iy) - \sin x \, \sin(iy) = \cos x \, \cosh y - i \sin x \, \sinh y$$

(**m**) We have:

$$\begin{aligned}
\sin z &= \frac{e^{iz} - e^{-iz}}{i2} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{i2} = \frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{i2} \\
&= \frac{(\cos x + i \sin x)e^{-y} - (\cos x - i \sin x)e^{y}}{i2} = \frac{e^{-y}\cos x + ie^{-y}\sin x - e^{y}\cos x + ie^{y}\sin x}{i2} \\
&= \frac{-ie^{-y}\cos x + e^{-y}\sin x + ie^{y}\cos x + e^{y}\sin x}{2} = \sin x \frac{e^{y} + e^{-y}}{2} + i \cos x \frac{e^{y} - e^{-y}}{2} \\
&= \sin x \, \cosh y + i \cos x \, \sinh y
\end{aligned} \tag{138}$$

Note: we may also use the identity $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$ (see part h of Problem 2) plus the identities $\cos(iz) = \cosh z$ and $\sin(iz) = i \sinh z$ (see parts a and b of Problem 5), that is:

$$\sin z = \sin(x + iy) = \sin x \, \cos(iy) + \cos x \, \sin(iy) = \sin x \, \cosh y + i \cos x \, \sinh y$$

(**n**) We use the identities of part (l) and part (a) and the identity $\cos^2 x + \sin^2 x = 1$, that is:

$$\begin{aligned} |\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = \cos^2 x \left(1 + \sinh^2 y\right) + \sin^2 x \sinh^2 y \\ &= \cos^2 x + \left(\cos^2 x + \sin^2 x\right) \sinh^2 y = \cos^2 x + \sinh^2 y \end{aligned}$$

Note: this result indicates that $|\cos z|$ can be greater than 1 (unlike its real counterpart $|\cos x|$). In fact, $|\cos z|$ can take any real non-negative value (inline with the fact that $\cos z$ can take any complex value).
(**o**) We use the identities of part (l) and part (a) and the identity $\cos^2 x + \sin^2 x = 1$, that is:

$$\begin{aligned} |\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = \cos^2 x \cosh^2 y + \sin^2 x \left(\cosh^2 y - 1\right) \\ &= \left(\cos^2 x + \sin^2 x\right) \cosh^2 y - \sin^2 x = \cosh^2 y - \sin^2 x \end{aligned} \tag{139}$$

(**p**) We use the identities of part (m) and part (a) and the identity $\cos^2 x + \sin^2 x = 1$, that is:

$$\begin{aligned} |\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \sin^2 x \left(1 + \sinh^2 y\right) + \cos^2 x \sinh^2 y \\ &= \sin^2 x + \left(\sin^2 x + \cos^2 x\right) \sinh^2 y = \sin^2 x + \sinh^2 y \end{aligned}$$

Note: this result indicates that $|\sin z|$ can be greater than 1 (unlike its real counterpart $|\sin x|$). In fact, $|\sin z|$ can take any real non-negative value (inline with the fact that $\sin z$ can take any complex value).
(**q**) We use the identities of part (m) and part (a) and the identity $\cos^2 x + \sin^2 x = 1$, that is:

$$\begin{aligned} |\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \sin^2 x \cosh^2 y + \cos^2 x \left(\cosh^2 y - 1\right) \\ &= \left(\sin^2 x + \cos^2 x\right) \cosh^2 y - \cos^2 x = \cosh^2 y - \cos^2 x \end{aligned} \tag{140}$$

(**r**) We have:

$$\begin{aligned} \cosh z &= \frac{e^z + e^{-z}}{2} = \frac{e^{x+iy} + e^{-(x+iy)}}{2} = \frac{e^x e^{iy} + e^{-x} e^{-iy}}{2} \\ &= \frac{e^x(\cos y + i \sin y) + e^{-x}(\cos y - i \sin y)}{2} = \frac{e^x \cos y + i e^x \sin y + e^{-x} \cos y - i e^{-x} \sin y}{2} \\ &= \frac{e^x + e^{-x}}{2} \cos y + i \frac{e^x - e^{-x}}{2} \sin y = \cosh x \cos y + i \sinh x \sin y \end{aligned} \tag{141}$$

Note: we may also use the identity $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$ (see part f) plus the identities $\cosh(iz) = \cos z$ and $\sinh(iz) = i \sin z$ (see parts a and b of Problem 5), that is:

$$\cosh z = \cosh(x + iy) = \cosh x \cosh(iy) + \sinh x \sinh(iy) = \cosh x \cos y + i \sinh x \sin y$$

(**s**) We have:

$$\begin{aligned} \sinh z &= \frac{e^z - e^{-z}}{2} = \frac{e^{x+iy} - e^{-(x+iy)}}{2} = \frac{e^x e^{iy} - e^{-x} e^{-iy}}{2} \\ &= \frac{e^x(\cos y + i \sin y) - e^{-x}(\cos y - i \sin y)}{2} = \frac{e^x \cos y + i e^x \sin y - e^{-x} \cos y + i e^{-x} \sin y}{2} \\ &= \frac{e^x - e^{-x}}{2} \cos y + i \frac{e^x + e^{-x}}{2} \sin y = \sinh x \cos y + i \cosh x \sin y \end{aligned} \tag{142}$$

Note: we may also use the identity $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$ (see part h) plus the identities $\cosh(iz) = \cos z$ and $\sinh(iz) = i \sin z$ (see parts a and b of Problem 5), that is:

$$\sinh z = \sinh(x + iy) = \sinh x \cosh(iy) + \cosh x \sinh(iy) = \sinh x \cos y + i \cosh x \sin y$$

(**t**) This is an instance of the relation $\cos(iz) = \cosh z$ (with $z = 1$) which we verified in part (a) of Problem 5.
(**u**) This is an instance of the relation $\sin(iz) = i \sinh z$ (with $z = 1$) which we verified in part (b) of Problem 5.
(**v**) We start from the right hand side to obtain the left hand side, that is:

$$\begin{aligned}\frac{\tanh z_1 + \tanh z_2}{1 + \tanh z_1 \tanh z_2} &= \frac{\frac{\sinh z_1}{\cosh z_1} + \frac{\sinh z_2}{\cosh z_2}}{1 + \frac{\sinh z_1}{\cosh z_1}\frac{\sinh z_2}{\cosh z_2}} = \frac{\frac{\sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2}{\cosh z_1 \cosh z_2}}{\frac{\cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2}{\cosh z_1 \cosh z_2}} \\ &= \frac{\sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2}{\cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2} = \frac{\sinh(z_1 + z_2)}{\cosh(z_1 + z_2)} = \tanh(z_1 + z_2) \qquad (143)\end{aligned}$$

8. Try to use some of the identities of Problem 7 to justify the relations:

$$\cos(iz) = \cosh z \qquad \cosh(iz) = \cos z \qquad \sin(iz) = i \sinh z \qquad \sinh(iz) = i \sin z$$

Answer: If we inspect the relations:

$$\begin{aligned}\cos z &= \cos x \cosh y - i \sin x \sinh y \qquad (144) \\ \cosh z &= \cosh x \cos y + i \sinh x \sin y \qquad (145)\end{aligned}$$

and compare them we can see that Eq. 145 can be obtained from Eq. 144 by multiplying z in Eq. 144 with i (noting the shift of x and y to $-y$ and x as real and imaginary parts) and hence $\cos(iz) = \cosh z$, while Eq. 144 can be obtained from Eq. 145 by multiplying z in Eq. 145 with i and hence $\cosh(iz) = \cos z$. Similarly, if we inspect the relations:

$$\begin{aligned}\sin z &= \sin x \cosh y + i \cos x \sinh y \qquad (146) \\ \sinh z &= \sinh x \cos y + i \cosh x \sin y \qquad (147)\end{aligned}$$

and compare them we can see that Eq. 147 can be obtained from Eq. 146 by multiplying z in Eq. 146 with i (noting the shift of x and y to $-y$ and x as real and imaginary parts) followed by dividing the result by i and hence $\sin(iz) = i \sinh z$, while Eq. 146 can be obtained from Eq. 147 by multiplying z in Eq. 147 with i followed by dividing the result by i and hence $\sinh(iz) = i \sin z$.

9. Suggest a method for obtaining hyperbolic identities from trigonometric identities by using the relationships between trigonometric and hyperbolic functions (as given in Problem 5).
Answer: We simply take an established trigonometric identity (as a function of iz)[141] and replace the trigonometric functions in that identity by their corresponding hyperbolic functions (as given in Problem 5).
Note: the above method can be used (in reverse) to obtain trigonometric identities from hyperbolic identities.

10. Verify some of the identities of Problem 7 using this time the method suggested in Problem 9.
Answer: For example:
(**a**) From the identity $\cos^2 z + \sin^2 z = 1$ (see part a of Problem 2) plus $\cos(iz) = \cosh z$ and $\sin(iz) = i \sinh z$ (see parts a and b of Problem 5) we get:

$$\begin{aligned}\cos^2(iz) + \sin^2(iz) &= 1 \\ \cosh^2 z + i^2 \sinh^2 z &= 1 \\ \cosh^2 z - \sinh^2 z &= 1\end{aligned}$$

[141] The purpose of taking the trigonometric identity as a function of iz is to obtain a hyperbolic identity as a function of z (as can be seen from the relations of Problem 5). We should also note that some manipulation (regarding for example i) may be required to put the obtained hyperbolic identity in its final (recognized) form. These issues will be clarified in the examples of Problem 10.

which is the identity of part (a) of Problem 7.
(**b**) From the identity $1 + \tan^2 z = \sec^2 z$ (see part b of Problem 2) plus $\tan(iz) = i \tanh z$ and $\sec(iz) = \text{sech} z$ (see parts c and d of Problem 5) we get:

$$\begin{aligned} 1 + \tan^2(iz) &= \sec^2(iz) \\ 1 + i^2 \tanh^2 z &= \text{sech}^2 z \\ 1 - \tanh^2 z &= \text{sech}^2 z \end{aligned}$$

which is the identity of part (b) of Problem 7.
(**c**) From the identity $\cot^2 z + 1 = \csc^2 z$ (see part c of Problem 2) plus $\cot(iz) = -i \coth z$ and $\csc(iz) = -i\, \text{csch}\, z$ (see parts e and f of Problem 5) we get:

$$\begin{aligned} \cot^2(iz) + 1 &= \csc^2(iz) \\ (-i)^2 \coth^2 z + 1 &= (-i)^2 \text{csch}^2 z \\ -\coth^2 z + 1 &= -\text{csch}^2 z \\ \coth^2 z - 1 &= \text{csch}^2 z \end{aligned}$$

which is the identity of part (c) of Problem 7.
Note: the use of this method in reverse can also be exemplified by the above examples (in reverse), that is:
(**a**) From the identity $\cosh^2 z - \sinh^2 z = 1$ (see part a of Problem 7) plus $\cosh(iz) = \cos z$ and $\sinh(iz) = i \sin z$ (see parts a and b of Problem 5) we get:

$$\begin{aligned} \cosh^2(iz) - \sinh^2(iz) &= 1 \\ \cos^2 z - (i)^2 \sin^2 z &= 1 \\ \cos^2 z + \sin^2 z &= 1 \end{aligned}$$

(**b**) From the identity $1 - \tanh^2 z = \text{sech}^2 z$ (see part b of Problem 7) plus $\tanh(iz) = i \tan z$ and $\text{sech}(iz) = \sec z$ (see parts c and d of Problem 5) we get:

$$\begin{aligned} 1 - \tanh^2(iz) &= \text{sech}^2(iz) \\ 1 - (i)^2 \tan^2 z &= \sec^2 z \\ 1 + \tan^2 z &= \sec^2 z \end{aligned}$$

(**c**) From the identity $\coth^2 z - 1 = \text{csch}^2 z$ (see part c of Problem 7) plus $\coth(iz) = -i \cot z$ and $\text{csch}(iz) = -i \csc z$ (see parts e and f of Problem 5) we get:

$$\begin{aligned} \coth^2(iz) - 1 &= \text{csch}^2(iz) \\ (-i)^2 \cot^2 z - 1 &= (-i)^2 \csc^2 z \\ -\cot^2 z - 1 &= -\csc^2 z \\ \cot^2 z + 1 &= \csc^2 z \end{aligned}$$

11. Verify the following identities:
(**a**) $(\cosh z)^* = \cosh z^*$. (**b**) $(\sinh z)^* = \sinh z^*$. (**c**) $(\tanh z)^* = \tanh z^*$.
(**d**) $(\text{sech}\, z)^* = \text{sech}\, z^*$. (**e**) $(\text{csch}\, z)^* = \text{csch}\, z^*$. (**f**) $(\coth z)^* = \coth z^*$.
Answer: We use the rules of conjugation that we established in § 1.8.8 and elsewhere (see for example part b of Problem 3 of § 2.2) as well as rules that we establish in the present Problem.

$$(\mathbf{a}) \quad (\cosh z)^* = \left(\frac{e^z + e^{-z}}{2}\right)^* = \frac{(e^z + e^{-z})^*}{2^*} = \frac{(e^z)^* + (e^{-z})^*}{2} = \frac{e^{z^*} + e^{-z^*}}{2} = \cosh z^*$$

$$(\mathbf{b})\quad (\sinh z)^* = \left(\frac{e^z - e^{-z}}{2}\right)^* = \frac{(e^z - e^{-z})^*}{2^*} = \frac{(e^z)^* - (e^{-z})^*}{2} = \frac{e^{z^*} - e^{-z^*}}{2} = \sinh z^*$$

$$(\mathbf{c})\quad (\tanh z)^* = \left(\frac{\sinh z}{\cosh z}\right)^* = \frac{(\sinh z)^*}{(\cosh z)^*} = \frac{\sinh z^*}{\cosh z^*} = \tanh z^*$$

$$(\mathbf{d})\quad (\operatorname{sech} z)^* = \left(\frac{1}{\cosh z}\right)^* = \frac{1^*}{(\cosh z)^*} = \frac{1}{\cosh z^*} = \operatorname{sech} z^*$$

$$(\mathbf{e})\quad (\operatorname{csch} z)^* = \left(\frac{1}{\sinh z}\right)^* = \frac{1^*}{(\sinh z)^*} = \frac{1}{\sinh z^*} = \operatorname{csch} z^*$$

$$(\mathbf{f})\quad (\coth z)^* = \left(\frac{1}{\tanh z}\right)^* = \frac{1^*}{(\tanh z)^*} = \frac{1}{\tanh z^*} = \coth z^*$$

Note: the above identities can also be obtained from the trigonometric identities of Problem 3 with the aid of the relations between the trigonometric and hyperbolic functions which were established in Problem 5. For example:

$$(\cosh z)^* = [\cos(iz)]^* = \cos(iz)^* = \cos(-iz^*) = \cos(iz^*) = \cosh z^*$$

On the other hand, the identities of Problem 3 can be obtained from the identities of the present Problem with the aid of the relations between the trigonometric and hyperbolic functions which were established in Problem 5.

12. As a direct consequence of the strong relationship between the trigonometric and hyperbolic functions, it is suggested (as a practical rule) that all the identities of the real hyperbolic functions can be obtained from corresponding trigonometric identities by replacing any squared sine[142] (whether explicit or implicit) in the trigonometric identities by the negative of its hyperbolic form (as well as replacing the other trigonometric functions by their corresponding hyperbolic functions, i.e. cos to cosh, sin to sinh, tan to tanh and so on).[143] This rule similarly applies to the reverse, i.e. all the identities of the real trigonometric functions can be obtained from corresponding hyperbolic identities by replacing any squared hyperbolic sine (whether explicit or implicit) in the hyperbolic identities by the negative of its trigonometric form (as well as replacing the other hyperbolic functions by their corresponding trigonometric functions).[144]
Justify this suggestion.
Answer: This suggestion is based on (and is an instance of) the method of Problem 9 because if we look to the trigonometric functions of iz in the list of Problem 5 (i.e. those on the left of the list) we find that all those functions involving sin explicitly or implicitly are equal to i (or $-i$) times a corresponding hyperbolic function of z. Now, since we are presumably dealing with *real* hyperbolic functions then z should be real (say $z = x$) and all i's should be either in a squared form (so that they vanish)[145] or they are on both sides (so that they cancel) and this should lead to the above suggested practical rule. Regarding the functions that do not involve sin explicitly or implicitly in the trigonometric identity, they do not contribute an i factor that needs to be considered and hence they are just replaced by their corresponding hyperbolic form (without affecting the suggested rule).
This justification also applies to the reverse because if we look to the hyperbolic functions of iz in the list of Problem 5 (i.e. those on the right of the list) we find that all those functions involving sinh explicitly or implicitly are equal to i (or $-i$) times a corresponding trigonometric function of z.

[142] "Squared sine" here should mean more generally "product of sines".

[143] For example, from the trigonometric identity $\cos^2 x + \sin^2 x = 1$ we can obtain the hyperbolic identity $\cosh^2 x - \sinh^2 x = 1$ (for real x) immediately by using this rule. Similarly, from the trigonometric identity $\cos(x_1 + x_2) = \cos x_1 \cos x_2 - \sin x_1 \sin x_2$ we can obtain the hyperbolic identity $\cosh(x_1 + x_2) = \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2$ by using this rule.

[144] For example, from the hyperbolic identity $\cosh^2 x - \sinh^2 x = 1$ we can obtain the trigonometric identity $\cos^2 x + \sin^2 x = 1$ (for real x) immediately by using this rule. Similarly, from the hyperbolic identity $\cosh(x_1 + x_2) = \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2$ we can obtain the trigonometric identity $\cos(x_1 + x_2) = \cos x_1 \cos x_2 - \sin x_1 \sin x_2$ by using this rule.

[145] Or "should be raised to even powers" to be more general (noting that any even power can be split to product of squares, e.g. $i^4 = i^2 \times i^2$).

Now, since we are presumably dealing with *real* trigonometric functions then z should be real (say $z = x$) and all i's should vanish (by squaring) or cancel (from the two sides) and this should lead to the same practical rule. Regarding the functions that do not involve sinh explicitly or implicitly in the hyperbolic identity, they do not contribute an i factor and hence they are just replaced by their corresponding trigonometric form (without affecting the rule).

13. Find the values of the following:

(**a**) $\cos(-i\pi)$. (**b**) $\cos(5 - i3)$. (**c**) $\sin(i4)$. (**d**) $\sin(e^2 + i\sqrt{7})$.
(**e**) $\tan(-i8)$. (**f**) $\tan(6 - i)$. (**g**) $\sec(\sqrt{13} + i)$. (**h**) $\csc(-0.4 - i1.6)$.
(**i**) $\cot(-2.2 + i)$. (**j**) $\cosh(-i\sqrt{11})$. (**k**) $\cosh(\pi - i\pi)$. (**l**) $\sinh(ie)$.
(**m**) $\sinh(\pi^2 + ie)$. (**n**) $\tanh(0.95 + i\sqrt{e})$. (**o**) $\operatorname{sech}(1 - i\pi)$. (**p**) $\operatorname{csch}\left(5.2 + i\frac{\pi}{2}\right)$.
(**q**) $\coth(\sqrt{17} + i\pi)$. (**r**) $\cos^2(i^i)$. (**s**) $\sqrt{\sin(i3)}$. (**t**) $\coth(1 + i)$.

Answer:
(**a**) There are several ways for calculating this (as well as the other parts although we demonstrate this only for this part). For example:

$$\cos(-i\pi) = \frac{e^{i(-i\pi)} + e^{-i(-i\pi)}}{2} = \frac{e^{\pi} + e^{-\pi}}{2} = \cosh(\pi) \simeq 11.5920 \qquad \text{(using Eqs. 131 \& 133)}$$
$$\cos(-i\pi) = \cos(i\pi) = \cosh(\pi) \qquad \text{(using Eqs. 135 \& 136)}$$
$$\cos(-i\pi) = \cos(0 - i\pi) = \cos(0)\cosh(-\pi) - i\sin(0)\sinh(-\pi) = \cosh(-\pi) \qquad \text{(using Eq. 137)}$$

(**b**) We use the identity $\cos z = \cos x \cosh y - i \sin x \sinh y$ (see Eq. 137), that is:

$$\cos(5 - i3) = \cos(5)\cosh(-3) - i\sin(5)\sinh(-3) \simeq 2.8558 - i9.6064$$

(**c**) We use the identity $\sin(iz) = i \sinh z$ (see part b of Problem 5), that is:

$$\sin(i4) = i \sinh 4 \simeq i27.2899$$

(**d**) We use the identity $\sin z = \sin x \cosh y + i \cos x \sinh y$ (see Eq. 138), that is:

$$\sin(e^2 + i\sqrt{7}) = \sin(e^2)\cosh(\sqrt{7}) + i\cos(e^2)\sinh(\sqrt{7}) \simeq 6.3307 + i3.1437$$

(**e**) We use the identity $\tan(iz) = i \tanh z$ (see part c of Problem 5), that is:

$$\tan(-i8) = i\tanh(-8) \simeq -i1.0000$$

(**f**) We use the definition of $\tan z$ plus Eqs. 137 and 138, that is:

$$\begin{aligned} \tan(6 - i) &= \frac{\sin(6 - i)}{\cos(6 - i)} = \frac{\sin(6)\cosh(-1) + i\cos(6)\sinh(-1)}{\cos(6)\cosh(-1) - i\sin(6)\sinh(-1)} \\ &\simeq \frac{-0.4312 - i1.1284}{1.4816 - i0.3284} \simeq -0.1165 - i0.7874 \end{aligned}$$

(**g**) We use the definition of $\sec z$ plus Eq. 137, that is:

$$\begin{aligned} \sec(\sqrt{13} + i) &= \frac{1}{\cos(\sqrt{13} + i)} = \frac{1}{\cos(\sqrt{13})\cosh(1) - i\sin(\sqrt{13})\sinh(1)} \simeq \frac{1}{-1.3799 + i0.5259} \\ &\simeq \frac{-1.3799 - i0.5259}{2.1808} \simeq -0.6328 - i0.2411 \end{aligned}$$

(**h**) We use the definition of $\csc z$ plus Eq. 138, that is:

$$\csc(-0.4 - i1.6) = \frac{1}{\sin(-0.4 - i1.6)} = \frac{1}{\sin(-0.4)\cosh(-1.6) + i\cos(-0.4)\sinh(-1.6)}$$

$$\simeq \frac{1}{-1.0037 - i2.1880} \simeq \frac{-1.0037 + i2.1880}{5.7950} \simeq -0.1732 + i0.3776$$

(**i**) We use the definition of $\cot z$ plus Eqs. 137 and 138, that is:

$$\begin{aligned}
\cot(-2.2+i) &= \frac{\cos(-2.2+i)}{\sin(-2.2+i)} = \frac{\cos(-2.2)\cosh(1) - i\sin(-2.2)\sinh(1)}{\sin(-2.2)\cosh(1) + i\cos(-2.2)\sinh(1)} \simeq \frac{-0.9081 + i0.9501}{-1.2476 - i0.6916} \\
&\simeq \frac{(-0.9081 + i0.9501)(-1.2476 + i0.6916)}{2.0348} \simeq \frac{0.4758 - i1.8134}{2.0348} \simeq 0.2338 - i0.8912
\end{aligned}$$

(**j**) We use the identity $\cosh(iz) = \cos z$ (see part a of Problem 5), that is:

$$\cosh(-i\sqrt{11}) = \cos(-\sqrt{11}) = \cos\sqrt{11} \simeq -0.9847$$

(**k**) We use the identity $\cosh z = \cosh x \cos y + i \sinh x \sin y$ (see Eq. 141), that is:

$$\cosh(\pi - i\pi) = \cosh(\pi)\cos(-\pi) + i\sinh(\pi)\sin(-\pi) = -\cosh(\pi) \simeq -11.5920$$

(**l**) We use the identity $\sinh(iz) = i\sin z$ (see part b of Problem 5), that is:

$$\sinh(ie) = i\sin e \simeq i0.4108$$

(**m**) We use the identity $\sinh z = \sinh x \cos y + i\cosh x \sin y$ (see Eq. 142), that is:

$$\sinh(\pi^2 + ie) = \sinh(\pi^2)\cos(e) + i\cosh(\pi^2)\sin(e) \simeq -8813.5900 + i3970.9589$$

(**n**) We use the definition of $\tanh z$ plus Eqs. 141 and 142, that is:

$$\begin{aligned}
\tanh(0.95 + i\sqrt{e}) &= \frac{\sinh(0.95 + i\sqrt{e})}{\cosh(0.95 + i\sqrt{e})} = \frac{\sinh(0.95)\cos(\sqrt{e}) + i\cosh(0.95)\sin(\sqrt{e})}{\cosh(0.95)\cos(\sqrt{e}) + i\sinh(0.95)\sin(\sqrt{e})} \\
&\simeq \frac{-0.0856 + i1.4817}{-0.1157 + i1.0961} \simeq \frac{(-0.0856 + i1.4817)(-0.1157 - i1.0961)}{1.2149} \\
&\simeq \frac{1.6341 - i0.0776}{1.2149} \simeq 1.3450 - i0.0639
\end{aligned}$$

(**o**) We use the definition of $\operatorname{sech} z$ plus Eq. 141, that is:

$$\operatorname{sech}(1 - i\pi) = \frac{1}{\cosh(1 - i\pi)} = \frac{1}{\cosh(1)\cos(-\pi) + i\sinh(1)\sin(-\pi)} = \frac{1}{-\cosh(1)} \simeq -0.6481$$

(**p**) We use the definition of $\operatorname{csch} z$ plus Eq. 142, that is:

$$\begin{aligned}
\operatorname{csch}\left(5.2 + i\frac{\pi}{2}\right) &= \frac{1}{\sinh\left(5.2 + i\frac{\pi}{2}\right)} = \frac{1}{\sinh(5.2)\cos(\pi/2) + i\cosh(5.2)\sin(\pi/2)} \\
&= \frac{1}{i\cosh(5.2)} \simeq -i0.01103
\end{aligned}$$

(**q**) We use the definition of $\coth z$ plus Eqs. 141 and 142, that is:

$$\begin{aligned}
\coth(\sqrt{17} + i\pi) &= \frac{\cosh(\sqrt{17} + i\pi)}{\sinh(\sqrt{17} + i\pi)} = \frac{\cosh(\sqrt{17})\cos(\pi) + i\sinh(\sqrt{17})\sin(\pi)}{\sinh(\sqrt{17})\cos(\pi) + i\cosh(\sqrt{17})\sin(\pi)} = \frac{-\cosh(\sqrt{17})}{-\sinh(\sqrt{17})} \\
&= \frac{\cosh(\sqrt{17})}{\sinh(\sqrt{17})} = \coth(\sqrt{17}) \simeq 1.0005
\end{aligned}$$

(**r**) Considering the principal value, we have $i^i = e^{-\pi/2}$ (see Eq. 106). Hence:

$$\cos^2(i^i) = \cos^2(e^{-\pi/2}) \simeq 0.9574$$

(**s**) We use $\sin(iz) = i\sinh z$ (see part b of Problem 5) plus Eq. 102, that is:

$$\begin{aligned}\sqrt{\sin(i3)} &= \sqrt{i\sinh 3} = \sqrt{i}\sqrt{\sinh 3} = \pm\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\sqrt{\sinh 3} = \pm\left(\frac{\sqrt{\sinh 3}}{\sqrt{2}} + i\frac{\sqrt{\sinh 3}}{\sqrt{2}}\right)\\ &\simeq \pm(2.2381 + i2.2381)\end{aligned}$$

(**t**) We use the definition of coth$z$ plus Eq. 143 and $\tanh(iz) = i\tan z$ (see part c of Problem 5), that is:

$$\coth(1+i) = \frac{1}{\tanh(1+i)} = \frac{1+\tanh(1)\tanh(i)}{\tanh(1)+\tanh(i)} = \frac{1+i\tanh(1)\tan(1)}{\tanh(1)+i\tan(1)} \simeq 0.8680 - i0.2176$$

14. Solve the following equations (for $z \in \mathbb{C}$):

(**a**) $\cos z = 0$. (**b**) $\sin z = 0$. (**c**) $\cosh z = 0$.

(**d**) $\sinh z = 0$. (**e**) $\cos z = 3$. (**f**) $2\sin(iz) = i$.

Answer:

(**a**) We have $\cos z = \cos x \cosh y - i \sin x \sinh y$ (see Eq. 137) and hence if $\cos z = 0$ then both the real and imaginary parts should be zero. Now, from the real part we have $\cos x \cosh y = 0$ which leads to $\cos x = 0$ (since $\cosh y \neq 0$ noting that y is real) and hence $x = \frac{(2n+1)\pi}{2}$ (with n being integer). Also, from the imaginary part we have $\sin x \sinh y = 0$ which leads to $\sinh y = 0$ (since $\sin\frac{(2n+1)\pi}{2} \neq 0$) and hence $y = 0$. Accordingly, the solution is $z = \frac{(2n+1)\pi}{2}$ (which represents the zeros of $\cos z$ which are all real).

(**b**) We have $\sin z = \sin x \cosh y + i \cos x \sinh y$ (see Eq. 138) and hence if $\sin z = 0$ then both the real and imaginary parts should be zero. Now, from the real part we have $\sin x \cosh y = 0$ which leads to $\sin x = 0$ (since $\cosh y \neq 0$ noting that y is real) and hence $x = n\pi$ (with n being integer). Also, from the imaginary part we have $\cos x \sinh y = 0$ which leads to $\sinh y = 0$ (since $\cos n\pi \neq 0$) and hence $y = 0$. Accordingly, the solution is $z = n\pi$ (which represents the zeros of $\sin z$ which are all real).

(**c**) We have $\cosh z = \cos(iz)$ and hence $\cosh z = 0$ is equivalent to $\cos(iz) = 0$. So, from the result of part (a) we get $iz = \frac{(2n+1)\pi}{2}$ and hence the solution (which represents the zeros of $\cosh z$) is $z = -i\frac{(2n+1)\pi}{2}$ which is equivalent to $z = i\frac{(2n+1)\pi}{2}$ (noting that n is integer). So, the zeros of $\cosh z$ are all imaginary.

(**d**) We have $\sinh z = -i\sin(iz)$ and hence $\sinh z = 0$ is equivalent to $\sin(iz) = 0$. So, from the result of part (b) we get $iz = n\pi$ and hence the solution (which represents the zeros of $\sinh z$) is $z = -in\pi$ which is equivalent to $z = in\pi$ (noting that n is integer). So, the zeros of $\sinh z$ are all imaginary.

(**e**) From the definition of $\cos z$ (see Eq. 131) we have:

$$\begin{aligned}\frac{e^{iz}+e^{-iz}}{2} &= 3\\ e^{iz}+e^{-iz}-6 &= 0\\ e^{i2z}-6e^{iz}+1 &= 0 \qquad (\times e^{iz} \neq 0)\end{aligned}$$

So, from the quadratic formula we have $e^{iz} = \frac{6\pm\sqrt{36-4}}{2} = 3 \pm 2\sqrt{2}$ and hence:

$$\begin{aligned}iz &= \ln(3\pm 2\sqrt{2})\\ iz &= \log_e(3\pm 2\sqrt{2}) + i2n\pi\\ z &= -i\log_e(3\pm 2\sqrt{2}) + 2n\pi\\ z &= 2n\pi \mp i\log_e(3+2\sqrt{2})\end{aligned}$$

where the last step is based on the fact that $\log_e(3-2\sqrt{2}) = -\log_e(3+2\sqrt{2})$ (see note).

Note: we have:

$$\log_e(3-2\sqrt{2}) = \log_e\left[\frac{3-2\sqrt{2}}{1}\right] = \log_e\left[\frac{3-2\sqrt{2}}{(3+2\sqrt{2})(3-2\sqrt{2})}\right] = \log_e\left(\frac{1}{3+2\sqrt{2}}\right)$$

$$= \log_e \left(3 + 2\sqrt{2}\right)^{-1} = -\log_e \left(3 + 2\sqrt{2}\right)$$

(**f**) From the definition of $\sin z$ (see Eq. 131) we have:

$$\begin{aligned} 2 \times \frac{e^{i(iz)} - e^{-i(iz)}}{i2} &= i \\ e^{-z} - e^{z} &= -1 \\ e^{2z} - e^{z} - 1 &= 0 \qquad\qquad (\times e^{z} \neq 0) \end{aligned}$$

So, from the quadratic formula we have $e^z = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$ and hence:

$$z = \ln\left(\frac{1+\sqrt{5}}{2}\right) = \log_e\left(\frac{1+\sqrt{5}}{2}\right) + i2n\pi \qquad \text{or} \qquad z = \ln\left(\frac{1-\sqrt{5}}{2}\right) = \log_e\left|\frac{1-\sqrt{5}}{2}\right| + i(2n+1)\pi$$

15. Verify the periodicity of $\cos z$ and $\sin z$ and find the period.
Answer: This is a good example of a matter that can be verified by numerous methods. So, let demonstrate this by presenting some of these methods in the following points:
• If $\sin z$ is periodic then for a given complex constant C we should have $\sin(z + C) = \sin z$. Now, if $z = 0$ then $\sin C = 0$ and hence from the result of part (b) of Problem 14 we know that $C = m\pi$ for a given positive integer m which is the lowest positive integer that achieves periodicity (according to the definition of period). So, let try the positive integers starting from 1. From the equation $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$ (which we verified in part h of Problem 2) with $m = 1$ and hence $C = \pi$, we get:

$$\sin(z + \pi) = \sin z \cos \pi + \cos z \sin \pi = -\sin z \neq \sin z$$

in general. If we repeat this with $m = 2$ and hence $C = 2\pi$ we get:

$$\sin(z + 2\pi) = \sin z \cos 2\pi + \cos z \sin 2\pi = \sin z$$

So, $\sin z$ is periodic with a period $C = 2\pi$. The periodicity of $\cos z$ can then be inferred from the periodicity of $\sin z$ because (see part h of Problem 2):

$$\sin\left(z + \frac{\pi}{2}\right) = \sin z \cos \frac{\pi}{2} + \cos z \sin \frac{\pi}{2} = \cos z$$

which implies that $\cos z$ is an identical copy of $\sin z$ (but with a phase shift of $\pi/2$) and hence if $\sin z$ is periodic with a period $C = 2\pi$ then $\cos z$ should also be periodic with a period $C = 2\pi$.
• From Eq. 137 we have:

$$\begin{aligned} \cos z &= \cos x \, \cosh y - i \sin x \, \sinh y = \cos(x + 2\pi) \, \cosh y - i \sin(x + 2\pi) \, \sinh y \\ &= \cos(x + 2\pi + iy) = \cos(z + 2\pi) \end{aligned}$$

and hence $\cos z$ is periodic with a period of 2π. Similarly, from Eq. 138 we have:

$$\begin{aligned} \sin z &= \sin x \, \cosh y + i \cos x \, \sinh y = \sin(x + 2\pi) \, \cosh y + i \cos(x + 2\pi) \, \sinh y \\ &= \sin(x + 2\pi + iy) = \sin(z + 2\pi) \end{aligned}$$

and hence $\sin z$ is periodic with a period of 2π.
• We can also use the relations of parts (f) and (h) of Problem 2 directly, that is:

$$\begin{aligned} \cos(z + 2\pi) &= \cos z \cos 2\pi - \sin z \sin 2\pi = \cos z \\ \sin(z + 2\pi) &= \sin z \cos 2\pi + \cos z \sin 2\pi = \sin z \end{aligned}$$

Hence, $\cos z$ and $\sin z$ are periodic with a period of 2π (noting that no number smaller than 2π can achieve periodicity as can be verified for example by inspecting the graphs of these functions over the interval $[0, 2\pi]$).
• Considering the definition of $\cos z$ and $\sin z$ (see Eq. 131) and noting that e^{iz} and e^{-iz} have a period of 2π (see Problem 18 of § 2.2), we can also conclude that $\cos z$ and $\sin z$ have a period of 2π.
• The periodicity of $\cos z$ and $\sin z$ (with a period of 2π) can also be obtained as an instance of the statement of part (c) of Problem 3 of § 7.1 because $\cos x$ and $\sin x$ (for real x) are periodic on the real line with a period of 2π and hence $\cos z$ and $\sin z$ (for complex z) should be periodic on the entire complex plane with a period of 2π (noting that $\cos z$ and $\sin z$ are entire functions).
Note 1: referring to Figure 22 (in the upcoming Problem 21), we can see that the real and imaginary parts of $\sin z$ are periodic in the x direction with a period of 2π. This is in agreement (for $\sin z$) with the period 2π which we found in the present Problem. This period should also apply to $\cos z$ due to the relation $\cos(z) = \sin\left(z + \frac{\pi}{2}\right)$ which we verified earlier.
Note 2: the periodicity of $\cos z$ and $\sin z$ (with a period of 2π) means that $\cos z$ and $\sin z$ take all their possible values within the vertical strip $-\pi < x \le \pi$ and hence $\cos z$ and $\sin z$ are made of identical copies of this strip (where each strip is displaced from the strip $-\pi < x \le \pi$ by $2n\pi$ in the x direction) and therefore the strip $-\pi < x \le \pi$ may be described as a fundamental region of $\cos z$ and $\sin z$.

16. Verify the periodicity of $\cosh z$ and $\sinh z$ and find the period.
Answer: We have:

$$\begin{aligned}
&\cosh(z) = \cos(iz) && \text{(see Problem 5)}\\
& = \cos(iz + 2\pi) && \text{(see Problem 15)}\\
\text{Also:}\quad &\cosh(z + i2\pi) = \cos(i\,[z + i2\pi]) && \text{(see Problem 5)}\\
& = \cos(iz - 2\pi) &&\\
& = \cos(iz + 2\pi) && \text{(see Problem 15)}
\end{aligned}$$

On comparing these equations we get $\cosh(z) = \cosh(z + i2\pi)$ and hence $\cosh z$ is periodic with an *imaginary* period of $i2\pi$.
Similarly:

$$\begin{aligned}
\sinh(z) &= -i\sin(iz) = -i\sin(iz + 2\pi)\\
\text{Also:}\quad \sinh(z + i2\pi) &= -i\sin(i\,[z + i2\pi]) = -i\sin(iz - 2\pi) = -i\sin(iz + 2\pi)
\end{aligned}$$

On comparing these equations we get $\sinh(z) = \sinh(z + i2\pi)$ and hence $\sinh z$ is periodic with an *imaginary* period of $i2\pi$.
Note 1: referring to Figure 23 (in the upcoming Problem 21), we can see that the real and imaginary parts of $\sinh z$ are periodic in the y direction with a period of $i2\pi$. This is in agreement (for $\sinh z$) with the period $i2\pi$ which we found in the present Problem. This period should also apply to $\cosh z$ due to the relation (see part f of Problem 7 as well as parts a and b of Problem 5):

$$\cosh\left(z + \frac{i\pi}{2}\right) = \cosh z \cosh\frac{i\pi}{2} + \sinh z \sinh\frac{i\pi}{2} = \cosh z \cos\frac{\pi}{2} + i\sinh z \sin\frac{\pi}{2} = i\sinh z$$

Note 2: the periodicity of $\cosh z$ and $\sinh z$ (with a period of $i2\pi$) means that $\cosh z$ and $\sinh z$ take all their possible values within the horizontal strip $-\pi < y \le \pi$ and hence $\cosh z$ and $\sinh z$ are made of identical copies of this strip (where each strip is displaced from the strip $-\pi < y \le \pi$ by $2n\pi$ in the y direction) and therefore the strip $-\pi < y \le \pi$ may be described as a fundamental region of $\cosh z$ and $\sinh z$. It should be noted that the periodicity of the complex hyperbolic cosine and sine functions (i.e. $\cosh z$ and $\sinh z$) is one of the main differences between $\cosh z$ and $\sinh z$ and their real counterparts (i.e. $\cosh x$ and $\sinh x$) because $\cosh x$ and $\sinh x$ are not periodic.
Note 3: considering the definition of $\cosh z$ and $\sinh z$ (see Eq. 133) and noting that e^z and e^{-z} have a period of $i2\pi$ (see Problem 18 of § 2.2), we can conclude that $\cosh z$ and $\sinh z$ have a period of $i2\pi$ without the above arguments.

17. Verify that $\tan z$ and $\cot z$ have a period of π and $\tanh z$ and $\coth z$ have a period of $i\pi$.
Answer: In the following verifications we use definitions and rules that we have established before (see for example Eqs. 131, 132 and 134 and Problem 7) as well as results that we obtain in the present Problem.

$$\begin{aligned}
\tan(z+\pi) &= \frac{\sin(z+\pi)}{\cos(z+\pi)} = \frac{-\sin(z)}{-\cos(z)} = \frac{\sin(z)}{\cos(z)} = \tan(z) \\
\cot(z+\pi) &= \frac{1}{\tan(z+\pi)} = \frac{1}{\tan(z)} = \cot(z) \\
\tanh(z+i\pi) &= \frac{\tanh z + \tanh(i\pi)}{1+\tanh z \tanh(i\pi)} = \frac{\tanh z + 0}{1+\tanh z \times 0} = \tanh(z) \\
\coth(z+i\pi) &= \frac{1}{\tanh(z+i\pi)} = \frac{1}{\tanh(z)} = \coth(z)
\end{aligned}$$

18. Verify that the following differentiation rules of trigonometric functions apply to complex functions (as to real functions):
(**a**) $d\cos z/dz = -\sin z$. (**b**) $d\sin z/dz = \cos z$. (**c**) $d\tan z/dz = \sec^2 z$.
(**d**) $d\sec z/dz = \sec z \tan z$. (**e**) $d\csc z/dz = -\csc z \cot z$. (**f**) $d\cot z/dz = -\csc^2 z$.
Answer: In the following verifications we use rules and definitions that we have established before (see for example Problem 2 of § 1.10 and Problem 19 of § 2.2 as well as Eqs. 131 and 132).
(**a**) We have:

$$\frac{d\cos z}{dz} = \frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right) = \frac{ie^{iz}-ie^{-iz}}{2} = \frac{i}{i} \times \frac{ie^{iz}-ie^{-iz}}{2} = -\frac{e^{iz}-e^{-iz}}{i2} = -\sin z$$

(**b**) We have:

$$\frac{d\sin z}{dz} = \frac{d}{dz}\left(\frac{e^{iz}-e^{-iz}}{i2}\right) = \frac{ie^{iz}+ie^{-iz}}{i2} = \frac{e^{iz}+e^{-iz}}{2} = \cos z$$

(**c**) Using the definition of $\tan z$ and the quotient rule (as well as other rules and identities), we have:

$$\frac{d\tan z}{dz} = \frac{d}{dz}\left(\frac{\sin z}{\cos z}\right) = \frac{\cos^2 z + \sin^2 z}{\cos^2 z} = \frac{1}{\cos^2 z} = \sec^2 z$$

(**d**) Using the definition of $\sec z$ and the quotient rule (as well as other rules and identities), we have:

$$\frac{d\sec z}{dz} = \frac{d}{dz}\left(\frac{1}{\cos z}\right) = -\frac{-\sin z}{\cos^2 z} = \frac{1}{\cos z}\frac{\sin z}{\cos z} = \sec z \tan z$$

(**e**) Using the definition of $\csc z$ and the quotient rule (as well as other rules and identities), we have:

$$\frac{d\csc z}{dz} = \frac{d}{dz}\left(\frac{1}{\sin z}\right) = \frac{-\cos z}{\sin^2 z} = -\frac{1}{\sin z}\frac{\cos z}{\sin z} = -\csc z \cot z$$

(**f**) Using the definition of $\cot z$ and the quotient rule (as well as other rules and identities), we have:

$$\frac{d\cot z}{dz} = \frac{d}{dz}\left(\frac{\cos z}{\sin z}\right) = \frac{-\sin^2 z - \cos^2 z}{\sin^2 z} = -\frac{\sin^2 z + \cos^2 z}{\sin^2 z} = -\frac{1}{\sin^2 z} = -\csc^2 z$$

Note: it is obvious that the integration rules that correspond to the above differentiation rules are:
$\int \sin z\,dz = -\cos z + C$ $\int \cos z\,dz = \sin z + C$ $\int \sec^2 z\,dz = \tan z + C$
$\int \sec z \tan z\,dz = \sec z + C$ $\int \csc z \cot z\,dz = -\csc z + C$ $\int \csc^2 z\,dz = -\cot z + C$

19. Use the rules of differentiation of trigonometric functions (as given in Problem 18) plus the relations between the trigonometric and hyperbolic functions (as given in Problem 5) to obtain the rules of differentiation of hyperbolic functions.
Answer: In the following derivations we also use other rules that have been established earlier (e.g. those of Problem 2 of § 1.10).

$$\begin{aligned}
\frac{d\cosh z}{dz} &= \frac{d\cos(iz)}{dz} = -i\sin(iz) = -ii\sinh z = \sinh z \\
\frac{d\sinh z}{dz} &= \frac{d}{dz}\Big[-i\sin(iz)\Big] = -ii\cos(iz) = \cos(iz) = \cosh z \\
\frac{d\tanh z}{dz} &= \frac{d}{dz}\Big[-i\tan(iz)\Big] = -ii\sec^2(iz) = \sec^2(iz) = \operatorname{sech}^2 z \\
\frac{d\operatorname{sech} z}{dz} &= \frac{d\sec(iz)}{dz} = i\sec(iz)\,\tan(iz) = i\operatorname{sech} z\,\big[i\tanh z\big] = -\operatorname{sech} z\,\tanh z \\
\frac{d\operatorname{csch} z}{dz} &= \frac{d}{dz}\Big[i\csc(iz)\Big] = i\big[-i\csc(iz)\,\cot(iz)\big] = \csc(iz)\,\cot(iz) = \big[-i\operatorname{csch} z\big]\big[-i\coth z\big] \\
&= -\operatorname{csch} z\,\coth z \\
\frac{d\coth z}{dz} &= \frac{d}{dz}\Big[i\cot(iz)\Big] = i\big[-i\csc^2(iz)\big] = \csc^2(iz) = \big[-i\operatorname{csch} z\big]^2 = -\operatorname{csch}^2 z
\end{aligned}$$

Note 1: the intermediate steps in the above derivations are given for clarity; otherwise we can use the method of Problem 9 to obtain the derivatives directly by just replacement (according to that method). For example, from $d\cos(iz)/d(iz) = -\sin(iz)$ we immediately get $d\cosh z/d(iz) = -i\sinh z$ which leads to $d\cosh z/dz = \sinh z$.
Note 2: the integration rules that correspond to the above differentiation rules can be easily inferred from the fact that differentiation and integration are inverse operations (as done in Problem 18).

20. Evaluate the following complex trigonometric and hyperbolic integrals:
(**a**) $\int_i^{i5} \cosh(2z)\,dz$. (**b**) $\int_{1+i}^{3-i2} \sin^2 z\,\cos z\,dz$. (**c**) $\int_\pi^{i2} \big[\cos(z)\sinh(iz)\big]\,dz$. (**d**) $\int_{6\pi}^{i\pi} \sinh^2 z\,dz$.
Answer:

(**a**)
$$\begin{aligned}
\int_i^{i5} \cosh(2z)\,dz &= \left[\frac{\sinh(2z)}{2}\right]_i^{i5} = \frac{\sinh(i10)}{2} - \frac{\sinh(i2)}{2} = \frac{i\sin(10)}{2} - \frac{i\sin(2)}{2} \\
&\simeq -i0.7267
\end{aligned}$$

(**b**)
$$\int_{1+i}^{3-i2} \sin^2 z\,\cos z\,dz = \left[\frac{\sin^3 z}{3}\right]_{1+i}^{3-i2} = \frac{\sin^3(3-i2)}{3} - \frac{\sin^3(1+i)}{3} \simeq -7.0011 - i15.4031$$

(**c**)
$$\begin{aligned}
\int_\pi^{i2} \big[\cos(z)\sinh(iz)\big]\,dz &= i\int_\pi^{i2} \big[\cos(z)\sin(z)\big]\,dz = i\left[-\frac{\cos^2(z)}{2}\right]_\pi^{i2} = i\left[\frac{\cos^2(\pi)}{2}\right] - i\left[\frac{\cos^2(i2)}{2}\right] \\
&= i\left[\frac{\cos^2(\pi)}{2}\right] - i\left[\frac{\cosh^2(2)}{2}\right] \simeq -i6.5771
\end{aligned}$$

(**d**)
$$\begin{aligned}
\int_{6\pi}^{i\pi} \sinh^2 z\,dz &= \left[\frac{\sinh(2z) - 2z}{4}\right]_{6\pi}^{i\pi} = \left[\frac{\sinh(i2\pi) - i2\pi}{4}\right] - \left[\frac{\sinh(12\pi) - 12\pi}{4}\right] \\
&= \frac{i\sin(2\pi) - i2\pi - \sinh(12\pi) + 12\pi}{4} = \frac{12\pi - \sinh(12\pi) - i2\pi}{4} \\
&\simeq -2.9473\times 10^{15} - i\frac{\pi}{2}
\end{aligned}$$

21. Make 3D plots of the following (and comment on the plots):
(**a**) The real and imaginary parts of the trigonometric function $\sin z$ over the region $-2\pi \le x \le 2\pi$ and $-2 \le y \le 2$.
(**b**) The real and imaginary parts of the hyperbolic function $\sinh z$ over the region $-2 \le x \le 2$ and

$-2\pi \le y \le 2\pi$.
Answer:
(**a**) The function $\sin z$ is given by (see Eq. 138):

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Hence, its real part is $\mathrm{Re}(\sin z) = \sin x \cosh y$ and its imaginary part is $\mathrm{Im}(\sin z) = \cos x \sinh y$. These parts are plotted over the region $-2\pi \le x \le 2\pi$ and $-2 \le y \le 2$ in Figure 22.
Comment: as we see, $\mathrm{Re}(\sin z)$ is a superposition of a "wavy" sin function in the x direction and an "even" cosh function in the y direction and hence the sin waves in the x direction are moderated by a cosh function in the y direction where the positive peaks and the negative troughs of the waves determine how cosh concaves (i.e. up or down). Accordingly, as we move along lines of constant x we see cosh profiles that concave upward or downward (depending on the sign of $\sin x$) but they become straight lines when $\sin x = 0$, i.e. when $x = n\pi$ (with n being integer), while as we move along lines of constant y in the x direction we see ordinary sine waves whose magnitudes are scaled by the constant $\cosh y$.
Similarly, $\mathrm{Im}(\sin z)$ is a superposition of a "wavy" cos function in the x direction and an "odd" sinh function in the y direction and hence the cos waves in the x direction are moderated by a sinh function in the y direction where the positive peaks and the negative troughs of the waves determine how sinh is orientated (i.e. up-down or down-up). Accordingly, as we move along lines of constant x we see sinh profiles that descend or ascend (depending on the sign of $\cos x$) but they become straight lines when $\cos x = 0$, i.e. when $x = \left(n + \frac{1}{2}\right)\pi$, while as we move along lines of constant y in the x direction we see ordinary cosine waves whose magnitudes are scaled by the constant $\sinh y$.
(**b**) The function $\sinh z$ is given by (see Eq. 142):

$$\sinh z = \sinh x \cos y + i \cosh x \sin y$$

Hence, its real part is $\mathrm{Re}(\sinh z) = \sinh x \cos y$ and its imaginary part is $\mathrm{Im}(\sinh z) = \cosh x \sin y$. These parts are plotted over the region $-2 \le x \le 2$ and $-2\pi \le y \le 2\pi$ in Figure 23.
Comment: as we will see in Problem 22, there is a strong similarity between $\mathrm{Re}(\sin z)$ and $\mathrm{Im}(\sinh z)$ and a strong similarity between $\mathrm{Im}(\sin z)$ and $\mathrm{Re}(\sinh z)$. Hence, the comment here is similar to the comment of part (a) with the exchange of the real and imaginary and the x and y.

22. Compare $\sin z$ and $\sinh z$ using Figures 22 and 23.
Answer: On inspecting these Figures we note the following:
• We observe a strong similarity between $\mathrm{Re}(\sin z) = \sin x \cosh y$ and $\mathrm{Im}(\sinh z) = \cosh x \sin y$. This is because both are made of $\sin \times \cosh$ but with the exchange of their variables (i.e. x and y) and hence in a sense $\mathrm{Im}(\sinh z)$ is obtained from $\mathrm{Re}(\sin z)$ (or the other way around) by exchanging the xy axes (which is seen in the Figures as rotation by $-\pi/2$ or $\pi/2$). This should shed more light on the comparison made in Problem 8 (noting that multiplying by $\mp i$ is equivalent to rotation by $\mp\pi/2$; see Problem 5 of § 1.8.5). This should also provide more clarification about the nature of the relationship between the trigonometric and hyperbolic sine functions (as well as other trigonometric and hyperbolic functions) as formulated in Problem 5.
• We observe a strong similarity between $\mathrm{Im}(\sin z) = \cos x \sinh y$ and $\mathrm{Re}(\sinh z) = \sinh x \cos y$. This is because both are made of $\cos \times \sinh$ but with the exchange of their variables (i.e. x and y) and hence in a sense $\mathrm{Re}(\sinh z)$ is obtained from $\mathrm{Im}(\sin z)$ (or the other way around) by exchanging the xy axes (which is seen in the Figures as rotation by $-\pi/2$ or $\pi/2$). Again, this should shed more light on the comparison made in Problem 8 and provide more clarification about the nature of the relationship as formulated in Problem 5 (also see the second comment of Problem 21).
• The previous points should explain why the real trigonometric functions are periodic while the real hyperbolic functions are not periodic (noting that the complex trigonometric and hyperbolic functions are both periodic since both types are made in their real and imaginary parts of a superposition of a sinusoidal wave in one direction and a combination of exponentials in the other direction; also see Problems 15 and 16). This is because the wavy component in the case of trigonometric functions is

along the real axis (and hence it appears in the real trigonometric functions), while the wavy component in the case of hyperbolic functions is along the imaginary axis (and hence it does not appear in the real hyperbolic functions). This can also be seen from Eqs. 138 and 142 (for the sine functions) by setting y to zero (and similarly from Eqs. 137 and 141 for the cosine functions).

23. Outline the main properties of the (complex) trigonometric cosine and sine functions.
Answer: They are entire, unbounded and periodic (with *real* period 2π). Cosine is even and sine is odd. The zeros of cosine are real at $z = \frac{(2n+1)\pi}{2}$ and the zeros of sine are also real but at $z = n\pi$.
Note: as we see, some properties (like periodicity, parity and zeros) are the same for the complex and real forms of these functions, while other properties (like boundedness) are different. Other properties (like entirety in its complex sense) have no counterpart in the real form.
24. Outline the main properties of the (complex) hyperbolic cosine and sine functions.
Answer: They are entire, unbounded and periodic (with *imaginary* period $i2\pi$). Hyperbolic cosine is even and hyperbolic sine is odd. The zeros of hyperbolic cosine are imaginary at $z = i\frac{(2n+1)\pi}{2}$ and the zeros of hyperbolic sine are also imaginary but at $z = in\pi$.
Note: as in the case of trigonometric cosine and sine (see the note of Problem 23), the complex and real hyperbolic cosine and sine have some identical properties like unboundedness and parity, some different properties like periodicity and zeros, and some unrelated properties like entirety.

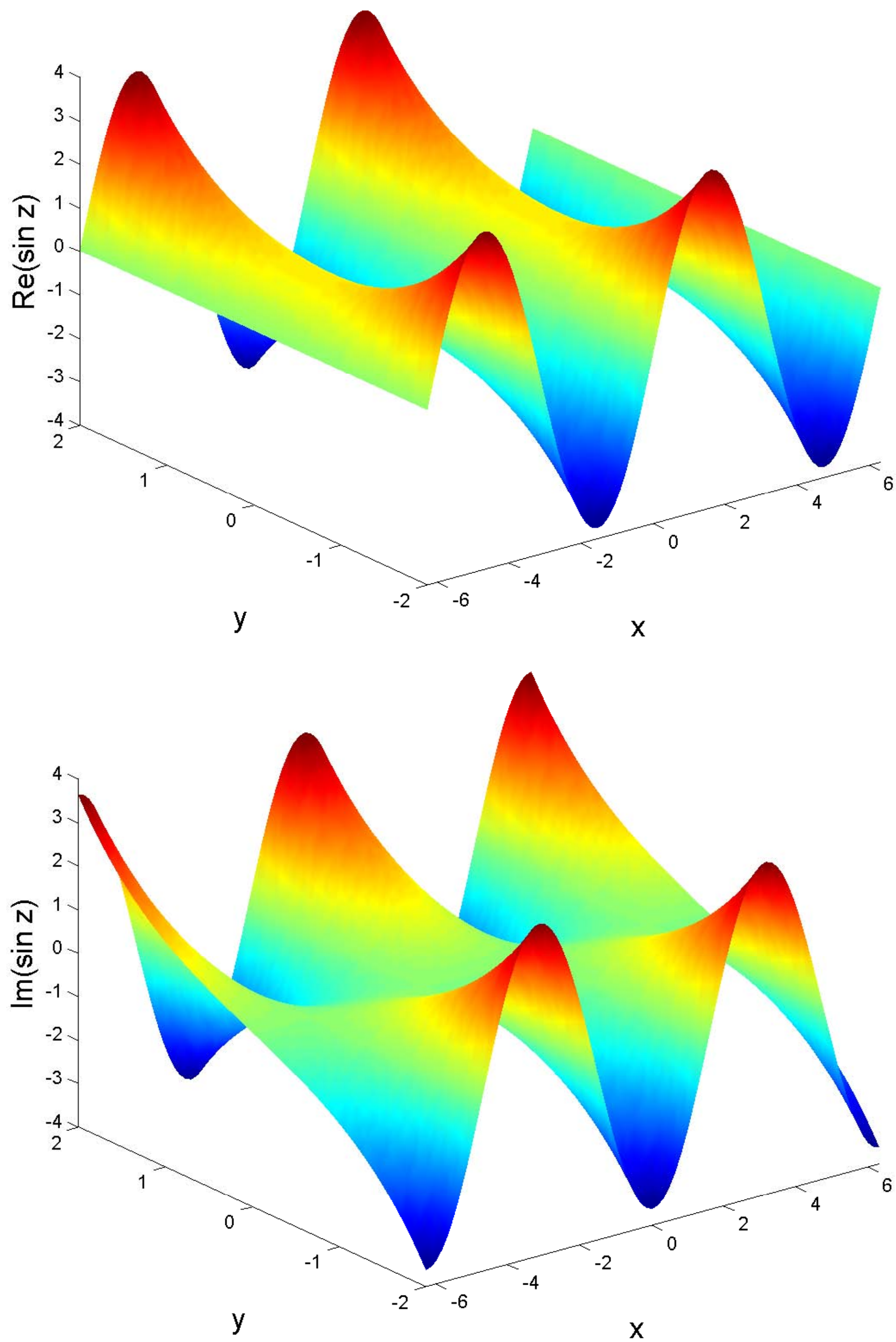

Figure 22: Graphic illustration of the real and imaginary parts of the complex function $\sin z$ over the rectangular region in the z plane defined by $-2\pi \leq x \leq 2\pi$ and $-2 \leq y \leq 2$. See part (a) of Problem 21 of § 2.3.

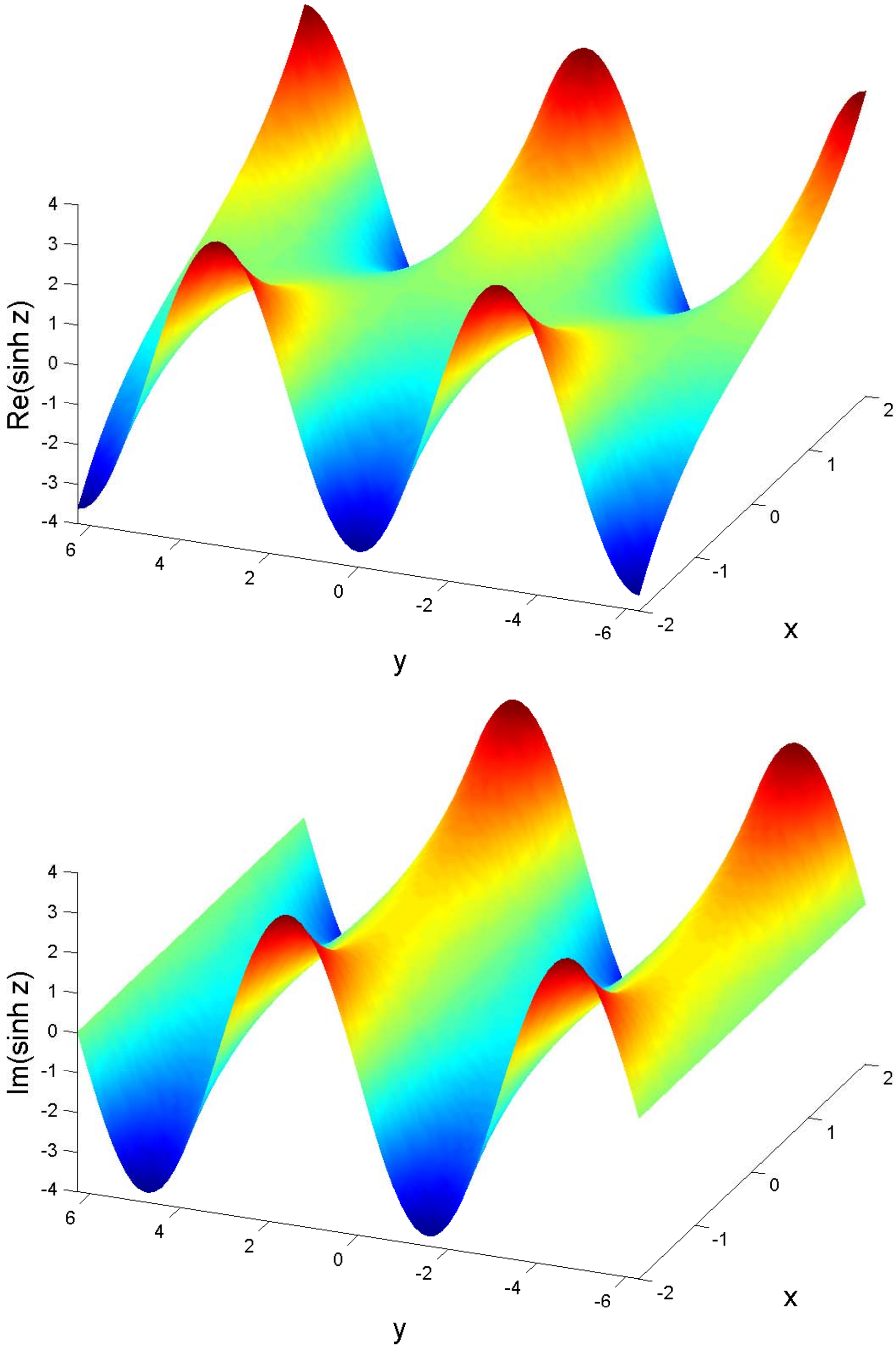

Figure 23: Graphic illustration of the real and imaginary parts of the complex function $\sinh z$ over the rectangular region in the z plane defined by $-2 \leq x \leq 2$ and $-2\pi \leq y \leq 2\pi$. See part (b) of Problem 21 of § 2.3.

2.4 Inverse Trigonometric and Hyperbolic Functions

The inverse trigonometric and hyperbolic functions of complex variables generally follow the style of their real-valued counterparts, as will be investigated and clarified in the following Problems. It should be remarked that in these Problems certain restrictions may have to be imposed on the arguments of certain functions (such as $z \neq 0$) to avoid singularities and undefined values. We did not impose these restrictions explicitly due to the presumed clarity and to avoid unnecessary elongations and complications which may cause distraction.[146]

Problems

1. Give mathematical definitions of the inverse trigonometric and hyperbolic functions of complex variables.
   Answer: A complex variable w is the inverse cosine function of a complex variable z (i.e. $w = \arccos z$) if $z = \cos w$. In other words, they are inverses of each other (as the name indicates). The other inverse trigonometric and hyperbolic functions of complex variables are defined similarly.
2. Derive analytical expressions for the inverse trigonometric functions of complex variables in terms of their argument z.
   Answer:
   • If $w = \arccos z$ then $z = \cos w$ (see Problem 1) and hence (see Eq. 131):

$$\begin{aligned} \frac{e^{iw} + e^{-iw}}{2} &= z \\ e^{iw} + e^{-iw} - 2z &= 0 \\ e^{i2w} - 2ze^{iw} + 1 &= 0 \qquad (\times e^{iw}) \end{aligned}$$

   So, from the quadratic formula we have:[147]

$$e^{iw} = \frac{2z \pm \sqrt{4z^2 - 4}}{2} = z \pm \sqrt{z^2 - 1} = z \pm \sqrt{-(1 - z^2)} = z \pm i\sqrt{1 - z^2}$$

   On taking the natural logarithm of both sides and dividing by i we get:

$$w \equiv \arccos z = -i \ln \left(z \pm i\sqrt{1 - z^2} \right) \tag{148}$$

   • If $w = \arcsin z$ then $z = \sin w$ and hence (see Eq. 131):

$$\begin{aligned} \frac{e^{iw} - e^{-iw}}{i2} &= z \\ e^{iw} - e^{-iw} - i2z &= 0 \\ e^{i2w} - i2ze^{iw} - 1 &= 0 \\ e^{iw} &= \frac{i2z \pm \sqrt{-4z^2 + 4}}{2} = iz \pm \sqrt{1 - z^2} \\ w \equiv \arcsin z &= -i \ln \left(iz \pm \sqrt{1 - z^2} \right) \qquad (149) \end{aligned}$$

   • If $w = \arctan z$ then $z = \tan w$ and hence (see Eq. 131):

$$\frac{\sin w}{\cos w} = z$$

[146] In fact, this remark should also be needed for some Problems of other (previous and upcoming) sections. We should also note that there are many details that we ignored due to limitations on scope and space.

[147] In this formula and its alike, we keep the $\pm$ sign for clarity although $\pm \sqrt[2]{}$ is implied by $\sqrt[2]{}$ (noting that a complex number has two oppositely-signed square roots). We note that in most texts only the + sign is taken and this also applies to the other occurrences of the $\pm$ sign in the upcoming expressions for other inverse trigonometric and hyperbolic functions.

$$\begin{aligned}
\frac{\left(e^{iw}-e^{-iw}\right)/i2}{\left(e^{iw}+e^{-iw}\right)/2} &= z \\
(1-iz)\,e^{iw}-(1+iz)\,e^{-iw} &= 0 \\
(1-iz)\,e^{i2w}-(1+iz) &= 0 \\
e^{i2w} &= \frac{1+iz}{1-iz} \\
w \equiv \arctan z &= -\frac{i}{2}\ln\left(\frac{1+iz}{1-iz}\right)=\frac{i}{2}\ln\left(\frac{1-iz}{1+iz}\right)=\frac{i}{2}\ln\left(\frac{i+z}{i-z}\right) \qquad (150)
\end{aligned}$$

• If $w = \operatorname{arcsec} z$ then $z = \sec w = 1/\cos w$. Accordingly, $\cos w = 1/z$ and hence $w = \arccos(1/z)$. Therefore:

$$w \equiv \operatorname{arcsec} z = \arccos\left(\frac{1}{z}\right) = -i\ln\left(\frac{1}{z}\pm i\sqrt{1-\left(\frac{1}{z}\right)^2}\right) = -i\ln\left(\frac{1\pm i\sqrt{z^2-1}}{z}\right) \qquad (151)$$

where we used the formula of arccos (with the argument $1/z$) that we obtained already (see Eq. 148).
• If $w = \operatorname{arccsc} z$ then $z = \csc w = 1/\sin w$. Accordingly, $\sin w = 1/z$ and hence $w = \arcsin(1/z)$. Therefore:

$$w \equiv \operatorname{arccsc} z = \arcsin\left(\frac{1}{z}\right) = -i\ln\left(i\frac{1}{z}\pm \sqrt{1-\left(\frac{1}{z}\right)^2}\right) = -i\ln\left(\frac{i\pm \sqrt{z^2-1}}{z}\right) \qquad (152)$$

where we used the formula of arcsin (with the argument $1/z$) that we obtained already (see Eq. 149).
• If $w = \operatorname{arccot} z$ then $z = \cot w = 1/\tan w$. Accordingly, $\tan w = 1/z$ and hence $w = \arctan(1/z)$. Therefore:

$$w \equiv \operatorname{arccot} z = \arctan\left(\frac{1}{z}\right) = \frac{i}{2}\ln\left(\frac{i+(1/z)}{i-(1/z)}\right) = \frac{i}{2}\ln\left(\frac{iz+1}{iz-1}\right) = \frac{i}{2}\ln\left(\frac{z-i}{z+i}\right) \qquad (153)$$

where we used the formula of arctan (with the argument $1/z$) that we obtained already (see Eq. 150).
Note: as we see, all the inverse trigonometric functions are defined in terms of ln. Accordingly, the principal value of these functions follows the choice of the principal value of ln. This also applies to the inverse hyperbolic functions (which are investigated in Problem 3). We should also note that the singularities of these (trigonometric and hyperbolic) functions should be excluded from their domain of definition. In fact, there are many details that we ignored here (as well as elsewhere) due to restrictions on space and scope.

3. Derive analytical expressions for the inverse hyperbolic functions of complex variables in terms of their argument z.
Answer:
• If $w = \operatorname{arccosh} z$ then $z = \cosh w$ (see Problem 1) and hence (see Eq. 133):

$$\begin{aligned}
\frac{e^w+e^{-w}}{2} &= z \\
e^{2w}-2ze^w+1 &= 0 \\
e^w &= \frac{2z\pm\sqrt{4z^2-4}}{2} = z\pm\sqrt{z^2-1} \\
w \equiv \operatorname{arccosh} z &= \ln\left(z\pm\sqrt{z^2-1}\right) \qquad (154)
\end{aligned}$$

• If $w = \operatorname{arcsinh} z$ then $z = \sinh w$ and hence (see Eq. 133):

$$\frac{e^w-e^{-w}}{2} = z$$

$$\begin{aligned} e^{2w} - 2ze^{w} - 1 &= 0 \\ e^{w} &= \frac{2z \pm \sqrt{4z^2+4}}{2} = z \pm \sqrt{z^2+1} \\ w \equiv \operatorname{arcsinh} z &= \ln\left(z \pm \sqrt{z^2+1}\right) \end{aligned} \tag{155}$$

• If $w = \operatorname{arctanh} z$ then $z = \tanh w$ and hence (see Eq. 133):

$$\begin{aligned} \frac{\sinh w}{\cosh w} &= z \\ \frac{e^{w} - e^{-w}}{e^{w} + e^{-w}} &= z \\ (1-z)\,e^{w} - (1+z)\,e^{-w} &= 0 \\ e^{2w} &= \frac{1+z}{1-z} \\ w \equiv \operatorname{arctanh} z &= \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) \end{aligned} \tag{156}$$

• If $w = \operatorname{arcsech} z$ then $z = \operatorname{sech} w = 1/\cosh w$. Accordingly, $\cosh w = 1/z$ and hence $w = \operatorname{arccosh}(1/z)$. Therefore:

$$w \equiv \operatorname{arcsech} z = \operatorname{arccosh}\left(\frac{1}{z}\right) = \ln\left(\frac{1}{z} \pm \sqrt{\left(\frac{1}{z}\right)^2 - 1}\right) = \ln\left(\frac{1 \pm \sqrt{1-z^2}}{z}\right)$$

where we used the formula of arccosh (with the argument $1/z$) that we obtained already (see Eq. 154).
• If $w = \operatorname{arccsch} z$ then $z = \operatorname{csch} w = 1/\sinh w$. Accordingly, $\sinh w = 1/z$ and hence $w = \operatorname{arcsinh}(1/z)$. Therefore:

$$w \equiv \operatorname{arccsch} z = \operatorname{arcsinh}\left(\frac{1}{z}\right) = \ln\left(\frac{1}{z} \pm \sqrt{\left(\frac{1}{z}\right)^2 + 1}\right) = \ln\left(\frac{1 \pm \sqrt{1+z^2}}{z}\right)$$

where we used the formula of arcsinh (with the argument $1/z$) that we obtained already (see Eq. 155).
• If $w = \operatorname{arccoth} z$ then $z = \coth w = 1/\tanh w$. Accordingly, $\tanh w = 1/z$ and hence $w = \operatorname{arctanh}(1/z)$. Therefore:

$$w \equiv \operatorname{arccoth} z = \operatorname{arctanh}\left(\frac{1}{z}\right) = \frac{1}{2}\ln\left(\frac{1+(1/z)}{1-(1/z)}\right) = \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)$$

where we used the formula of arctanh (with the argument $1/z$) that we obtained already (see Eq. 156).

4. Evaluate the following:
   (**a**) $\arccos(2)$. (**b**) $\arcsin\sqrt{3}$. (**c**) $\operatorname{arccsch}(i)$. (**d**) $\operatorname{arccoth}(1+i)$. (**e**) $\operatorname{arctanh}(i)$.

   Answer:[148]

   (**a**)
$$\begin{aligned} \arccos(2) &= -\,i\ln\left(2 \pm i\sqrt{1-2^2}\right) = -i\ln\left(2 \pm i\sqrt{-3}\right) = -i\ln\left(2 \mp \sqrt{3}\right) \\ &= -\,i\left[\log_e\left(2 \mp \sqrt{3}\right) + i2n\pi\right] = 2n\pi - i\log_e\left(2 \mp \sqrt{3}\right) \end{aligned}$$

   (**b**)
$$\arcsin\sqrt{3} = -\,i\ln\left(i\sqrt{3} \pm \sqrt{1-3}\right) = -i\ln\left(i\sqrt{3} \pm i\sqrt{2}\right) = -i\ln\left(i\left[\sqrt{3} \pm \sqrt{2}\right]\right)$$

[148] Similar to what we stated already in the text, we do not go through some details about these functions and their values such as their domain and range (some of which is subject to certain conventions rather than mathematical necessities). For example, some may use Ln rather than ln in the definition of these functions (which restrict them to their principal values).

$$= -i\left[\log_e\left(\sqrt{3}\pm\sqrt{2}\right) + i\left(2n+\frac{1}{2}\right)\pi\right] = \left(2n+\frac{1}{2}\right)\pi - i\log_e\left(\sqrt{3}\pm\sqrt{2}\right)$$

(**c**) $$\operatorname{arccsch}(i) = \ln\left(\frac{1\pm\sqrt{1+i^2}}{i}\right) = \ln\left(\frac{1\pm\sqrt{0}}{i}\right) = \ln(-i) = \log_e 1 + i\left(2n-\frac{1}{2}\right)\pi$$
$$= i\left(2n-\frac{1}{2}\right)\pi$$

(**d**) $$\operatorname{arccoth}(1+i) = \frac{1}{2}\ln\left(\frac{(1+i)+1}{(1+i)-1}\right) = \frac{1}{2}\ln\left(\frac{2+i}{i}\right) = \frac{1}{2}\ln(1-i2)$$
$$\simeq \frac{1}{2}\left[\log_e\sqrt{5} + i\,(2n\pi - 1.1072)\right]$$

(**e**) $$\operatorname{arctanh}(i) = \frac{1}{2}\ln\left(\frac{1+i}{1-i}\right) = \frac{1}{2}\ln\left(\frac{[1+i]^2}{2}\right) = \frac{1}{2}\ln\left(\frac{i2}{2}\right) = \frac{1}{2}\ln(i)$$
$$= \frac{1}{2}\left[\log_e 1 + i\left(2n+\frac{1}{2}\right)\pi\right] = i\left(n+\frac{1}{4}\right)\pi$$

5. Verify the following relations:
   (**a**) $\arccos z + \arcsin z = C$. (**b**) $\operatorname{arcsec} z + \operatorname{arccsc} z = C$. (**c**) $\arctan z + \operatorname{arccot} z = C$.
   Answer: We note first that C is a *generic* constant (which we use for simplicity) and hence it is generally not the same in these relations.
   (**a**) From Eqs. 148 and 149 we have:
   $$\begin{aligned}\arccos z + \arcsin z &= -i\ln\left(z\pm i\sqrt{1-z^2}\right) - i\ln\left(iz\pm\sqrt{1-z^2}\right)\\ &= -i\ln\left(\left[z\pm i\sqrt{1-z^2}\right]\times\left[iz\pm\sqrt{1-z^2}\right]\right) = -i\ln(i) = C\end{aligned}$$
   (**b**) From Eqs. 151 and 152 (as well as the result of part a) we have:
   $$\operatorname{arcsec} z + \operatorname{arccsc} z = \arccos\left(\frac{1}{z}\right) + \arcsin\left(\frac{1}{z}\right) = C$$
   (**c**) From Eqs. 150 and 153 we have:
   $$\arctan z + \operatorname{arccot} z = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right) + \frac{i}{2}\ln\left(\frac{z-i}{z+i}\right) = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\times\frac{z-i}{z+i}\right) = \frac{i}{2}\ln(-1) = C$$
6. Find the derivatives of the inverse trigonometric functions of complex variables.
   Answer: In the following formulae we exclude any value of z that causes the denominator to vanish.
   • If $w = \arccos z$ then $z = \cos w$ and hence:
   $$1 = \frac{dz}{dz} = \frac{d}{dz}\cos w = -\sin w\frac{dw}{dz} = -\sqrt{1-\cos^2 w}\,\frac{dw}{dz} = -\sqrt{1-z^2}\,\frac{dw}{dz}$$
   Hence: $$\frac{d}{dz}\arccos z \equiv \frac{dw}{dz} = \frac{-1}{\sqrt{1-z^2}}$$
   • If $w = \arcsin z$ then $z = \sin w$ and hence:
   $$1 = \frac{dz}{dz} = \frac{d}{dz}\sin w = \cos w\frac{dw}{dz} = \sqrt{1-\sin^2 w}\,\frac{dw}{dz} = \sqrt{1-z^2}\,\frac{dw}{dz}$$
   Hence: $$\frac{d}{dz}\arcsin z \equiv \frac{dw}{dz} = \frac{1}{\sqrt{1-z^2}}$$
   We may also obtain this result from the previous result by noting that $\arcsin z = C - \arccos z$ (see part a of Problem 5) and hence:
   $$\frac{d}{dz}\arcsin z = 0 - \frac{d}{dz}\arccos z = \frac{1}{\sqrt{1-z^2}}$$

• If $w = \arctan z$ then $z = \tan w$ and hence:

$$\begin{aligned} 1 = \frac{dz}{dz} &= \frac{d}{dz}\tan w = \sec^2 w \,\frac{dw}{dz} = \left(1+\tan^2 w\right)\frac{dw}{dz} = \left(1+z^2\right)\frac{dw}{dz} \\ \text{Hence:} \qquad \frac{d}{dz}\arctan z &\equiv \frac{dw}{dz} = \frac{1}{1+z^2} \end{aligned}$$

• If $w = \operatorname{arcsec} z$ then $z = \sec w$ and hence:

$$\begin{aligned} 1 = \frac{dz}{dz} &= \frac{d}{dz}\sec w = \sec w \tan w \,\frac{dw}{dz} = \sec w\sqrt{\sec^2 w - 1}\,\frac{dw}{dz} = z\sqrt{z^2-1}\,\frac{dw}{dz} \\ \text{Hence:} \qquad \frac{d}{dz}\operatorname{arcsec} z &\equiv \frac{dw}{dz} = \frac{1}{z\sqrt{z^2-1}} \end{aligned}$$

• If $w = \operatorname{arccsc} z$ then $z = \csc w$ and hence:

$$\begin{aligned} 1 = \frac{dz}{dz} &= \frac{d}{dz}\csc w = -\csc w \cot w \,\frac{dw}{dz} = -\csc w\sqrt{\csc^2 w - 1}\,\frac{dw}{dz} = -z\sqrt{z^2-1}\,\frac{dw}{dz} \\ \text{Hence:} \qquad \frac{d}{dz}\operatorname{arccsc} z &\equiv \frac{dw}{dz} = \frac{-1}{z\sqrt{z^2-1}} \end{aligned}$$

We may also use $\operatorname{arccsc} z = C - \operatorname{arcsec} z$ (see part b of Problem 5) and hence:

$$\frac{d}{dz}\operatorname{arccsc} z = 0 - \frac{d}{dz}\operatorname{arcsec} z = \frac{-1}{z\sqrt{z^2-1}}$$

• If $w = \operatorname{arccot} z$ then $z = \cot w$ and hence:

$$\begin{aligned} 1 = \frac{dz}{dz} &= \frac{d}{dz}\cot w = -\csc^2 w \,\frac{dw}{dz} = -\left(1+\cot^2 w\right)\frac{dw}{dz} = -\left(1+z^2\right)\frac{dw}{dz} \\ \text{Hence:} \qquad \frac{d}{dz}\operatorname{arccot} z &\equiv \frac{dw}{dz} = \frac{-1}{1+z^2} \end{aligned}$$

We may also use $\operatorname{arccot} z = C - \arctan z$ (see part c of Problem 5) and hence:

$$\frac{d}{dz}\operatorname{arccot} z = 0 - \frac{d}{dz}\arctan z = \frac{-1}{1+z^2}$$

Note: the integration rules that correspond to the above differentiation rules can be easily inferred from the fact that differentiation and integration are inverse operations.

7. Find the derivatives of the inverse hyperbolic functions of complex variables.
Answer:
• If $w = \operatorname{arccosh} z$ then $z = \cosh w$ and hence:

$$\begin{aligned} 1 = \frac{dz}{dz} &= \frac{d}{dz}\cosh w = \sinh w \,\frac{dw}{dz} = \sqrt{\cosh^2 w - 1}\,\frac{dw}{dz} = \sqrt{z^2-1}\,\frac{dw}{dz} \\ \text{Thus:} \qquad \frac{d}{dz}\operatorname{arccosh} z &\equiv \frac{dw}{dz} = \frac{1}{\sqrt{z^2-1}} \end{aligned}$$

• If $w = \operatorname{arcsinh} z$ then $z = \sinh w$ and hence:

$$\begin{aligned} 1 = \frac{dz}{dz} &= \frac{d}{dz}\sinh w = \cosh w \,\frac{dw}{dz} = \sqrt{\sinh^2 w + 1}\,\frac{dw}{dz} = \sqrt{z^2+1}\,\frac{dw}{dz} \\ \text{Thus:} \qquad \frac{d}{dz}\operatorname{arcsinh} z &\equiv \frac{dw}{dz} = \frac{1}{\sqrt{z^2+1}} \end{aligned}$$

• If $w = \operatorname{arctanh} z$ then $z = \tanh w$ and hence:

$$1 = \frac{dz}{dz} = \frac{d}{dz}\tanh w = \operatorname{sech}^2 w \,\frac{dw}{dz} = \left(1-\tanh^2 w\right)\frac{dw}{dz} = \left(1-z^2\right)\frac{dw}{dz}$$

$$\text{Thus:} \quad \frac{d}{dz}\operatorname{arctanh} z \equiv \frac{dw}{dz} = \frac{1}{1-z^2}$$

• If $w = \operatorname{arcsech} z$ then $z = \operatorname{sech} w$ and hence:

$$1 = \frac{dz}{dz} = \frac{d}{dz}\operatorname{sech} w = -\operatorname{sech} w \tanh w \frac{dw}{dz} = -\operatorname{sech} w \sqrt{1-\operatorname{sech}^2 w}\frac{dw}{dz} = -z\sqrt{1-z^2}\frac{dw}{dz}$$

$$\text{Thus:} \quad \frac{d}{dz}\operatorname{arcsech} z \equiv \frac{dw}{dz} = \frac{-1}{z\sqrt{1-z^2}}$$

• If $w = \operatorname{arccsch} z$ then $z = \operatorname{csch} w$ and hence:

$$1 = \frac{dz}{dz} = \frac{d}{dz}\operatorname{csch} w = -\operatorname{csch} w \coth w \frac{dw}{dz} = -\operatorname{csch} w \sqrt{\operatorname{csch}^2 w + 1}\frac{dw}{dz} = -z\sqrt{z^2+1}\frac{dw}{dz}$$

$$\text{Thus:} \quad \frac{d}{dz}\operatorname{arccsch} z \equiv \frac{dw}{dz} = \frac{-1}{z\sqrt{z^2+1}}$$

• If $w = \operatorname{arccoth} z$ then $z = \coth w$ and hence:

$$1 = \frac{dz}{dz} = \frac{d}{dz}\coth w = -\operatorname{csch}^2 w \frac{dw}{dz} = \left(1-\coth^2 w\right)\frac{dw}{dz} = \left(1-z^2\right)\frac{dw}{dz}$$

$$\text{Thus:} \quad \frac{d}{dz}\operatorname{arccoth} z \equiv \frac{dw}{dz} = \frac{1}{1-z^2}$$

Note: the integration rules that correspond to the above differentiation rules can be easily inferred from the fact that differentiation and integration are inverse operations.

8. Find the derivatives of the following complex functions $f(z)$ at the given points:
   (**a**) $f(z) = i\pi \arccos z$ at point $z = 3 - i$. (**b**) $f(z) = \arctan(3z)$ at point $z = 8 + i2$.
   (**c**) $f(z) = -\operatorname{arcsinh}(iz)$ at point $z = 4 + i9$. (**d**) $f(z) = i\operatorname{arcsech}(5z)$. at point $z = i7$.
   Answer:

(**a**) $$\left.\frac{df}{dz}\right|_{z=3-i} = i\pi\left.\frac{-1}{\sqrt{1-z^2}}\right|_{z=3-i} = \frac{-i\pi}{\sqrt{1-(3-i)^2}} = \frac{-i\pi}{\sqrt{-7+i6}} \simeq -0.9704 - i0.3590$$

(**b**) $$\left.\frac{df}{dz}\right|_{z=8+i2} = \left.\frac{3}{1+(3z)^2}\right|_{z=8+i2} = \frac{3}{1+(24+i6)^2} = \frac{3}{541+i288} \simeq 0.004321 - i0.002300$$

(**c**) $$\left.\frac{df}{dz}\right|_{z=4+i9} = \left.\frac{-i}{\sqrt{(iz)^2+1}}\right|_{z=4+i9} = \frac{-i}{\sqrt{(-9+i4)^2+1}} = \frac{-i}{\sqrt{66-i72}} \simeq 0.04074 - i0.09262$$

(**d**) $$\left.\frac{df}{dz}\right|_{z=i7} = \left.\frac{-i5}{(5z)\sqrt{1-(5z)^2}}\right|_{z=i7} = \frac{-1}{7\sqrt{1-(i35)^2}} = \frac{-1}{7\sqrt{1226}} \simeq -0.004080$$

9. How do you classify the inverse trigonometric and hyperbolic functions of complex variables from the perspective of being single-valued or multi-valued?
   Answer: As we saw, all these functions are defined in terms of the natural logarithm function (which is infinitely multi-valued) and hence they are infinitely multi-valued.
10. Verify the results of Problem 14 of § 2.3 by using the formulae developed in the present section.
   Answer:
   (**a**) If $\cos z = 0$ then $z = \arccos(0)$. Hence, from Eq. 148 (as well as the result of part e of Problem 9 of § 2.2) we have:

$$z = \arccos(0) = -i\ln\left(0 \pm i\sqrt{1-0^2}\right) = -i\ln(\pm i) = -i\ln(i)^{\pm 1} = \mp i\ln(i)$$

$$= \mp i\left[i\left(\frac{\pi}{2}+2n\pi\right)\right] = \pm\left(2n+\frac{1}{2}\right)\pi = \frac{(2n+1)\pi}{2}$$

(**b**) If $\sin z = 0$ then $z = \arcsin(0)$. Hence, from Eq. 149 (as well as the results of parts a and c of Problem 9 of § 2.2) we have:

$$z = \arcsin(0) = -i\ln\left(i0 \pm \sqrt{1-0^2}\right) = -i\ln(\pm 1) = -i\left\{\begin{array}{c} i2n\pi \\ i(2n+1)\pi \end{array}\right\} = \left\{\begin{array}{c} 2n\pi \\ (2n+1)\pi \end{array}\right\} = n\pi$$

(**c**) If $\cosh z = 0$ then $z = \operatorname{arccosh}(0)$. Hence, from Eq. 154 (as well as the result of part e of Problem 9 of § 2.2) we have:

$$z = \operatorname{arccosh}(0) = \ln\left(0 \pm \sqrt{0^2-1}\right) = \ln\left(\pm\sqrt{-1}\right) = \ln(\pm i) = \ln(i)^{\pm 1} = \pm\ln(i) = i\frac{(2n+1)\pi}{2}$$

(**d**) If $\sinh z = 0$ then $z = \operatorname{arcsinh}(0)$. Hence, from Eq. 155 (as well as the results of parts a and c of Problem 9 of § 2.2) we have:

$$z = \operatorname{arcsinh}(0) = \ln\left(0 \pm \sqrt{0^2+1}\right) = \ln(\pm 1) = \left\{\begin{array}{c} i2n\pi \\ i(2n+1)\pi \end{array}\right\} = in\pi$$

(**e**) If $\cos z = 3$ then $z = \arccos(3)$. Hence, from Eq. 148 we have:

$$\begin{aligned} z &= \arccos(3) = -i\ln\left(3 \pm i\sqrt{1-3^2}\right) = -i\ln\left(3 \pm i\sqrt{-8}\right) = -i\ln\left(3 \mp 2\sqrt{2}\right) \\ &= -i\left[\log_e(3 \mp 2\sqrt{2}) + i2n\pi\right] = -i\log_e(3 \mp 2\sqrt{2}) + 2n\pi = 2n\pi \pm i\log_e(3+2\sqrt{2}) \end{aligned}$$

(**f**) From $2\sin(iz) = i$ we get $i2\sinh(z) = i$ (see part b of Problem 5) and hence $\sinh z = 1/2$. Now, if $\sinh z = 1/2$ then $z = \operatorname{arcsinh}(1/2)$ and hence from Eq. 155 we have:

$$z = \operatorname{arcsinh}\left(\frac{1}{2}\right) = \ln\left[\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + 1}\right] = \ln\left(\frac{1 \pm \sqrt{5}}{2}\right) = \left\{\begin{array}{l} \log_e\left(\frac{1+\sqrt{5}}{2}\right) + i2n\pi \\ \log_e\left|\frac{1-\sqrt{5}}{2}\right| + i(2n+1)\pi \end{array}\right\}$$

Chapter 3 Analyticity

In this chapter we investigate a number of aspects and properties of analytic functions as well as some of the applications related to these aspects and properties. In fact, the investigation of analytic functions (and hence analyticity) is the main subject of complex analysis and hence this investigation is vital to any other investigation in complex analysis. As we will see, analytic functions have very favorable properties and behavior and they provide (through their analyticity) the dynamics and spirit of the entire subject of complex analysis. In fact, analyticity is what makes complex analysis complex analysis and gives it its fascinating and attractive characteristic features among other branches of mathematical analysis and mathematics in general.

It should be remarked that when we talk about the analyticity in general (as it is the case in many parts and sections of this chapter as well as other chapters) we generally assume a proper domain in which the given function is analytic (e.g. the complex plane excluding the origin) which could be the entire complex plane (i.e. when the analytic function is entire). The reliance on this understanding rather than explicit definition and identification is to avoid overcrowding the text which may cause distraction and confusion (as well as taking precious space). Moreover, in such contexts the focus is mainly the analyticity and hence the domain is not required to be specified since any proper domain will achieve the purpose of the investigation. Yes, when we investigate particular regions or neighborhoods or contours or points, for instance, then associating the analyticity with a specific domain becomes unavoidable and hence we provide full specification accordingly.

3.1 Cauchy-Riemann Equations

If a complex function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point $z = x + iy$ in the z plane then at point z the first order partial derivatives of u and v with respect to x and y do exist and they satisfy the following relationships which are called the Cauchy-Riemann equations:[149]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{157}$$

The converse of this statement is also true that is: if $u(x, y)$ and $v(x, y)$ are real functions (of x and y) that are defined and have partial derivatives (with respect to x and y) in a neighborhood N of a point z and these partial derivatives are continuous and satisfy the Cauchy-Riemann equations at z then $f(z) = u(x, y) + iv(x, y)$ is analytic at z. The latter statement can be used as a basis for analyticity criterion and test (see Problem 8).[150] Apart from their use in the analyticity test, the Cauchy-Riemann equations have many direct and indirect applications (as we will see in the Problems of the present section and in the coming sections and chapters). In fact, the Cauchy-Riemann equations are at the heart of complex analysis and they represent one of its pillars. This is because the essence of complex analysis is the investigation of analytic functions and their distinguished properties which they enjoy through their analyticity.

Problems

[149] They may also be called "Cauchy-Riemann conditions" or "d'Alembert-Euler conditions". We should also note that these are the *Cartesian form* of the Cauchy-Riemann equations and we will obtain their *polar form* in Problem 20.

[150] It should be obvious that the Cauchy-Riemann equations can also be used as a direct or indirect test for entirety (noting that failure of analyticity at a point means failure of entirety).

1. Verify the Cauchy-Riemann equations.
Answer: Because f is analytic we have:

$$\begin{aligned}
\frac{df}{dz} &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} \\
&= \lim_{\Delta x, \Delta y \to 0} \frac{\big[u(x+\Delta x, y+\Delta y)+iv(x+\Delta x, y+\Delta y)\big]-\big[u(x,y)+iv(x,y)\big]}{\Delta x + i\Delta y} \qquad (158)
\end{aligned}$$

Now, because this limit does exist (since f is analytic), Δz can approach zero from any direction in the complex plane (see § 1.9). In particular, if Δz approaches zero horizontally then $\Delta y = 0$ and $\Delta z = \Delta x$ and hence Eq. 158 becomes:

$$\begin{aligned}
\frac{df}{dz} &= \lim_{\Delta x \to 0} \frac{\big[u(x+\Delta x, y)+iv(x+\Delta x, y)\big]-\big[u(x,y)+iv(x,y)\big]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y)-u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y)-v(x,y)}{\Delta x} \\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad (159)
\end{aligned}$$

Similarly, if Δz approaches zero vertically then $\Delta x = 0$ and $\Delta z = i\Delta y$ and hence Eq. 158 becomes:

$$\begin{aligned}
\frac{df}{dz} &= \lim_{\Delta y \to 0} \frac{\big[u(x, y+\Delta y)+iv(x, y+\Delta y)\big]-\big[u(x,y)+iv(x,y)\big]}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y)-u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y)-v(x,y)}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y)-v(x,y)}{\Delta y} - i \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y)-u(x,y)}{\Delta y} \\
&= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad (160)
\end{aligned}$$

On comparing the real and imaginary parts of Eqs. 159 and 160 we get the Cauchy-Riemann equations (see Eq. 157).
Note 1: as we know, the (total) derivative of a complex function (assumed to be differentiable) can be obtained from its definition (see Eq. 81) and can be obtained from the general differentiation rules (such as the product and quotient rules; see Problem 2 of § 1.10) in association with the function-specific rules (like $de^z/dz = e^z$). Now, Eqs. 159 and 160 show that the (total) derivative of a complex (analytic) function can also be calculated from the partial derivatives of its real and imaginary parts (also see the upcoming Eq. 161 and Problem 3). In fact, the former two represent the lump approach while the latter represents the composite approach (see § 1.11).[151]
Note 2: from Eqs. 159 and 160 plus the Cauchy-Riemann equations we can obtain the following (compact) expressions for the (total) derivative of a complex analytic function $f(z)$ in terms of the partial derivatives of its real and imaginary parts:

$$f'(z) = u_x + iv_x = v_y - iu_y = u_x - iu_y = v_y + iv_x \qquad (161)$$

where the prime represents derivative with respect to z while the subscripts mean partial derivatives with respect to the variables symbolized by these subscripts.
2. Merge the Cauchy-Riemann equations into a single equation.
Answer: By the Cauchy-Riemann equations we have $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$ and hence:

$$\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 + i0$$

[151] The subject here is obviously analytic functions.

$$\begin{aligned}
\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y} &= 0 \\
\frac{\partial}{\partial x}(u+iv)+i\frac{\partial}{\partial y}(u+iv) &= 0 \\
\frac{\partial}{\partial x}f+i\frac{\partial}{\partial y}f &= 0 \\
\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f &= 0 \qquad (162)
\end{aligned}$$

3. Find the derivatives (with respect to z) of the following analytic functions which are given as $f(x,y)$ instead of $f(z)$:

   (**a**) $f(x,y)=2x+i2y$.

   (**b**) $f(x,y)=\left(3x^3+2x^2+x-9xy^2-2y^2-4\right)+i\left(9x^2y-3y^3+4xy+y\right)$.

   Answer: We can use Eq. 159 or Eq. 160 (which should produce identical results).[152]

   (**a**)
$$\frac{df}{dz}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=2+i0=2 \qquad \text{(from Eq. 159)}$$
$$\frac{df}{dz}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}=2-i0=2 \qquad \text{(from Eq. 160)}$$

   (**b**)
$$\frac{df}{dz}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\left(9x^2+4x+1-9y^2\right)+i\left(18xy+4y\right) \qquad \text{(from Eq. 159)}$$
$$\frac{df}{dz}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}=\left(9x^2-9y^2+4x+1\right)-i\left(-18xy-4y\right) \qquad \text{(from Eq. 160)}$$

   Note: the above results can be easily verified by comparison to the lump approach noting that for part (a) we have $f(z)=2z$ and hence $\frac{df}{dz}=2$ while for part (b) we have $f(z)=3z^3+2z^2+z-4$ and hence $\frac{df}{dz}=9z^2+4z+1=(9x^2+4x+1-9y^2)+i(18xy+4y)$.

4. Show that the polynomial, exponential, and trigonometric and hyperbolic cosine and sine functions are entire.

   Answer: First, let accept that products, algebraic sums (including infinite series) and compositions of entire functions are entire (see § 1.5). We also note that quotients of entire functions are entire, if the denominator never vanish such as $\frac{\sinh z}{e^z}$, where this can be justified by the fact that quotients are actually products (i.e. numerator times the reciprocal of denominator noting that since the denominator is always analytic and never vanish its reciprocal is always defined and analytic).

   Regarding the **polynomial** function, a complex constant function $f(z)=a=\alpha+i\beta=u+iv$ (with a being a complex constant and α and β being real constants) is obviously entire since it satisfies the Cauchy-Riemann equations over the entire complex plane, i.e. $\frac{\partial\alpha}{\partial x}=0=\frac{\partial\beta}{\partial y}$ and $\frac{\partial\alpha}{\partial y}=0=-\frac{\partial\beta}{\partial x}$. Similarly, the complex function $f(z)=z=x+iy=u+iv$ is entire since $\frac{\partial u}{\partial x}=1=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=0=-\frac{\partial v}{\partial x}$ over the entire complex plane. So, az^n must be entire since it is a product of entire functions. Accordingly a polynomial (which is no more than an algebraic sum of terms like az^n which may be called monomials) must be entire.

   Regarding the **exponential** function, it is like the polynomial function in this regard because it is made of a series of terms like $z^n/n!$ (see Eq. 6) and hence by the same logic it must be entire. However, let use the Cauchy-Riemann equations directly to verify this formally. A general form of an exponential function is $e^z=e^{x+iy}=e^x(\cos y+i\sin y)=e^x\cos y+ie^x\sin y$. Now, if we use the Cauchy-Riemann equations then we have:

$$\frac{\partial u}{\partial x}=e^x\cos y=\frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y}=-e^x\sin y=-\frac{\partial v}{\partial x}$$

[152] From Eq. 161, we can see that we actually have two more choices for calculating these derivatives, i.e. $f'(z)=u_x-iu_y$ and $f'(z)=v_y+iv_x$.

Accordingly, it is analytic everywhere (noting that these equations are valid over the entire complex plane) and hence it is entire.
Regarding the **trigonometric and hyperbolic cosine and sine** functions, they are no more than algebraic sums of exponential functions (see Eqs. 131 and 133) and hence they must be entire.[153] However, let again use the Cauchy-Riemann equations directly to verify this formally. We use Eqs. 137, 138, 141 and 142 respectively to verify the analyticity (and hence entirety) of $\cos z$, $\sin z$, $\cosh z$ and $\sinh z$ respectively, that is:

$$\frac{\partial u}{\partial x} = -\sin x \cosh y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = +\cos x \sinh y = -\frac{\partial v}{\partial x} \qquad (\text{for } \cos z)$$

$$\frac{\partial u}{\partial x} = +\cos x \cosh y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = +\sin x \sinh y = -\frac{\partial v}{\partial x} \qquad (\text{for } \sin z)$$

$$\frac{\partial u}{\partial x} = +\sinh x \cos y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\cosh x \sin y = -\frac{\partial v}{\partial x} \qquad (\text{for } \cosh z)$$

$$\frac{\partial u}{\partial x} = +\cosh x \cos y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\sinh x \sin y = -\frac{\partial v}{\partial x} \qquad (\text{for } \sinh z)$$

Accordingly, these functions are analytic everywhere (noting that these equations are valid over the entire complex plane) and hence they are entire.
Note 1: any variations of the basic forms (i.e. involving z not z^*) of the above functions (e.g. e^{-i2z} or $\sin^3 z$ or $\cosh z^2$ or $az \cos 2z$) are no more than combinations (e.g. composition or sum or product) of functions of the above basic forms and hence they must also be entire.
Note 2: when we talk about the entirety (and even analyticity) of these functions (i.e. polynomial, exponential, etc.) and their alike we should restrict this to the functions of strictly complex variables, i.e. excluding those which are defined on the real line or imaginary line (for instance) and hence their arguments by definition are real or imaginary (see Problem 3 of § 1.5). This is because such functions by definition cannot be entire since they are defined only on the real or imaginary axis of the complex plane and hence their entirety is meaningless because they are not defined (let alone be analytic or not) on the entire complex plane. In fact, they are not even analytic in the sense of complex analysis since any point in their domain of definition cannot have a neighborhood and hence it cannot be approached from different directions. To be more formal, let investigate (as examples) polynomials $P_n(x)$ of real variable x and the exponentials of real and imaginary variables (i.e. e^x and e^{iy} with x and y being real) using the Cauchy-Riemann equations.
Regarding $P_n(x)$, its real part u is not zero and the derivative of u with respect to x is not zero (in general) while its imaginary part v is zero (as well as the derivatives of v). Therefore, the first of the Cauchy-Riemann equations is violated, that is:

$$0 \neq \frac{\partial u}{\partial x} \quad \neq \quad \frac{\partial v}{\partial y} = 0$$

Accordingly, $P_n(x)$ cannot be analytic (in the sense of complex analysis) or entire.
Regarding e^x, we have $e^x = e^{x+i0} = e^x(\cos 0 - i \sin 0) = e^x(1 - i0) = e^x - i0$. Therefore, the first of the Cauchy-Riemann equations is violated, that is:

$$e^x = \frac{\partial u}{\partial x} \quad \neq \quad \frac{\partial v}{\partial y} = 0$$

Accordingly, e^x cannot be analytic (in the sense of complex analysis) or entire.
Regarding e^{iy}, we have $e^{iy} = e^{0+iy} = e^0(\cos y + i \sin y) = \cos y + i \sin y$. Therefore, the Cauchy-Riemann equations are violated, that is:

$$0 = \frac{\partial u}{\partial x} \quad \neq \quad \frac{\partial v}{\partial y} = \cos y \qquad \text{and} \qquad -\sin y = \frac{\partial u}{\partial y} \quad \neq \quad -\frac{\partial v}{\partial x} = 0$$

[153] In fact, to establish this we need the upcoming note 1 about the variations of the basic forms since the trigonometric and hyperbolic cosine and sine are made of algebraic sums of variants of the exponential function.

Accordingly, e^{iy} cannot be analytic (in the sense of complex analysis) or entire.

5. Show that the principal branch of the complex natural logarithm function is analytic over its domain.
Answer: We have (noting that $\operatorname{Ln} z$ here represents the principal branch):

$$\begin{aligned}\operatorname{Ln} z &= \log_e |z| + i\operatorname{Arg}(z) \qquad\qquad (z \in \mathbb{C},\ z \neq -|z|)\\ &= \log_e \sqrt{x^2+y^2} + i \arctan\left(\frac{y}{x}\right)\end{aligned}$$

Hence, from the Cauchy-Riemann equations (noting the domain of $\operatorname{Ln} z$) we get:

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2} = -\frac{\partial v}{\partial x}$$

Accordingly, $\operatorname{Ln} z$ is analytic over its domain. This should also apply to the other branches of the complex natural logarithm function.

6. Verify the Cauchy-Riemann equations for the following complex functions:
(**a**) $f(z) = 3z + 2$. (**b**) $f(z) = z^2 + 4z - 1$. (**c**) $f(z) = z^3 - z$.
(**d**) $f(z) = \sin 3z$. (**e**) $f(z) = \cosh z^2$.
Answer: We note first that all these functions are entire because the first three are polynomials (see § 2.1 and Problem 4) while the last two are compositions of polynomials with trigonometric sine and hyperbolic cosine (see § 2.3 and Problem 4) and hence they should satisfy the Cauchy-Riemann equations over the entire complex plane.
(**a**)

$$f = 3z + 2 = 3(x+iy) + 2 = 3x + i3y + 2 = (3x+2) + i3y$$

and hence $u = 3x + 2$ and $v = 3y$. Therefore:

$$\frac{\partial u}{\partial x} = 3 = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$$

(**b**)

$$f = z^2 + 4z - 1 = (x+iy)^2 + 4(x+iy) - 1 = \left(x^2 - y^2 + 4x - 1\right) + i(2xy + 4y)$$

and hence $u = x^2 - y^2 + 4x - 1$ and $v = 2xy + 4y$. Therefore:

$$\frac{\partial u}{\partial x} = 2x + 4 = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

(**c**)

$$f = z^3 - z = (x+iy)^3 - (x+iy) = \left(x^3 - 3xy^2 - x\right) + i\left(3x^2y - y^3 - y\right)$$

and hence $u = x^3 - 3xy^2 - x$ and $v = 3x^2y - y^3 - y$. Therefore:

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 - 1 = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -6xy = -\frac{\partial v}{\partial x}$$

(**d**) We have $3z = 3x + i3y$ and hence:

$$f = \sin 3z = \sin 3x \cosh 3y + i \cos 3x \sinh 3y \qquad \text{(see Eq. 138)}$$

Thus, $u = \sin 3x \cosh 3y$ and $v = \cos 3x \sinh 3y$. Accordingly:

$$\frac{\partial u}{\partial x} = 3\cos 3x \cosh 3y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = 3\sin 3x \sinh 3y = -\frac{\partial v}{\partial x}$$

(**e**) We have $z^2 = (x^2 - y^2) + i2xy$ and hence:

$$f = \cosh z^2 = \cosh(x^2 - y^2)\cos(2xy) + i\sinh(x^2 - y^2)\sin(2xy) \qquad \text{(see Eq. 141)}$$

Thus, $u = \cosh(x^2 - y^2)\cos(2xy)$ and $v = \sinh(x^2 - y^2)\sin(2xy)$. Accordingly:

$$\begin{aligned} \frac{\partial u}{\partial x} &= +2x\sinh(x^2 - y^2)\cos(2xy) - 2y\cosh(x^2 - y^2)\sin(2xy) = +\frac{\partial v}{\partial y} \\ \text{and} \quad \frac{\partial u}{\partial y} &= -2y\sinh(x^2 - y^2)\cos(2xy) - 2x\cosh(x^2 - y^2)\sin(2xy) = -\frac{\partial v}{\partial x} \end{aligned}$$

7. Show that the following complex functions are not analytic:
   (**a**) $f = x^3 + iy^3$. (**b**) $f = (x^2 + y) + i(x + y^2)$. (**c**) $f = (e^x y + e^y x^2) + i(x^2 \sin y - \tan x)$.
   (**d**) $f = z^*$. (**e**) $f = \cosh(z^*)$. (**f**) $f = e^{z^*}$.
   (**g**) $f = |z|$. (**h**) $f = z^* - |z|^2$.

   Answer: We show that these functions are not analytic (anywhere) by showing that they do not satisfy the Cauchy-Riemann equations.[154]
   (**a**)
   $$\frac{\partial u}{\partial x} = 3x^2 \neq 3y^2 = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$$
   So, although f satisfies the second of the Cauchy-Riemann equations it does not satisfy the first and hence it is not analytic because analyticity requires the satisfaction of both equations.
   (**b**)
   $$\frac{\partial u}{\partial x} = 2x \neq 2y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = 1 \neq -1 = -\frac{\partial v}{\partial x}$$
   (**c**)
   $$\frac{\partial u}{\partial x} = e^x y + 2e^y x \neq x^2 \cos y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = e^x + e^y x^2 \neq -2x\sin y + \sec^2 x = -\frac{\partial v}{\partial x}$$
   (**d**) We have $f = z^* = x - iy$ and hence $u = x$ and $v = -y$. Therefore:
   $$\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$$
   So, although f satisfies the second of the Cauchy-Riemann equations it does not satisfy the first and hence it is not analytic because analyticity requires the satisfaction of both equations.
   (**e**) We have $\cosh z = \cosh x \cos y + i\sinh x \sin y$ (see Eq. 141) and hence $\cosh(z^*) = \cosh x \cos(-y) + i\sinh x \sin(-y) = \cosh x \cos y - i\sinh x \sin y$. Therefore:
   $$\frac{\partial u}{\partial x} = \sinh x \cos y \neq -\sinh x \cos y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\cosh x \sin y \neq \cosh x \sin y = -\frac{\partial v}{\partial x}$$
   (**f**) We have $e^{z^*} = e^{x-iy} = e^x(\cos y - i\sin y) = e^x \cos y - ie^x \sin y$. Therefore:
   $$\frac{\partial u}{\partial x} = e^x \cos y \neq -e^x \cos y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -e^x \sin y \neq e^x \sin y = -\frac{\partial v}{\partial x}$$
   (**g**) We have $|z| = \sqrt{x^2 + y^2} = \sqrt{x^2 + y^2} + i0$. Therefore:
   $$\frac{\partial u}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \neq 0 = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} \neq 0 = -\frac{\partial v}{\partial x}$$
   (**h**) We have $z^* - |z|^2 = (x - iy) - (x^2 + y^2) = (x - x^2 - y^2) - iy$. Therefore:
   $$\frac{\partial u}{\partial x} = 1 - 2x \neq -1 = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -2y \neq 0 = -\frac{\partial v}{\partial x}$$

[154] In fact, we are using the contrapositive of the statement "if they are analytic then they should satisfy the Cauchy-Riemann equations".

Note: the use of "$\neq$" in some parts above means "in general" noting that the occasional satisfaction of the Cauchy-Riemann equations (on points or curves) does not qualify the functions to be analytic at these locations because they do not extend to a neighborhood.

8. Give a formal criterion for analyticity based on the Cauchy-Riemann equations.
Answer: If the real functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first order partial derivatives (with respect to x and y) over a given domain D in the z plane, then if u and v satisfy the Cauchy-Riemann equations at all points of D then the complex function $w = f(z) = u(x, y) + iv(x, y)$ is analytic in D.
Note: we may also propose a "criterion for non-analyticity" by reversing the above criterion, that is: if $f(z)$ does not satisfy the Cauchy-Riemann equations, then it is not analytic. In fact, because the satisfaction of the Cauchy-Riemann equations (within the given conditions) is a necessary and sufficient condition for analyticity we may propose criteria for analyticity/non-analyticity and satisfaction/non-satisfaction of Cauchy-Riemann equations by taking the *iff* statement and its two contrapositives.

9. Show that the following complex functions are analytic:
(**a**) $f = \frac{y}{x^2+y^2} + i\frac{x}{x^2+y^2} \quad (x^2 + y^2 \neq 0)$. (**b**) $f = 2e^z$. (**c**) $f = ze^z$.
(**d**) $f = \sinh x \cos y + i \cosh x \sin y$. (**e**) $f = \frac{z^*}{|z|^2} \quad (z \neq 0)$.
Answer: We use the analyticity criterion which we stated in Problem 8, i.e. we verify analyticity by showing that the functions satisfy the Cauchy-Riemann equations (noting that all these functions are continuous and have continuous first order partial derivatives over their domain).
(**a**) We have $u = \frac{y}{x^2+y^2}$ and $v = \frac{x}{x^2+y^2}$ and hence:

$$\frac{\partial u}{\partial x} = -\frac{2xy}{(x^2+y^2)^2} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}$$

So, f is analytic over the entire z plane excluding the origin $z = 0$ (since $x^2 + y^2 \neq 0$). In fact, the analyticity of f is fairly obvious because:

$$\frac{y}{x^2+y^2} + i\frac{x}{x^2+y^2} = \frac{y+ix}{x^2+y^2} = \frac{i(x-iy)}{(x+iy)(x-iy)} = \frac{iz^*}{zz^*} = \frac{i}{z}$$

and hence the analyticity of f is based on the analyticity of i/z which is analytic everywhere excluding $z = 0$.
(**b**) We have $f = 2e^z = 2e^{x+iy} = 2e^x e^{iy} = 2e^x(\cos y + i \sin y) = 2e^x \cos y + i2e^x \sin y$ and hence $u = 2e^x \cos y$ and $v = 2e^x \sin y$. Therefore:

$$\frac{\partial u}{\partial x} = 2e^x \cos y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -2e^x \sin y = -\frac{\partial v}{\partial x}$$

So, f is analytic (and entire). In fact, the analyticity (and entirety) of f should be obvious because it is a product of polynomial (i.e. 2) and exponential functions both of which are analytic (and entire) as seen in Problem 4.
(**c**) We have:

$$\begin{aligned} f &= ze^z = (x+iy)e^{x+iy} = xe^{x+iy} + iye^{x+iy} = x(e^x \cos y + ie^x \sin y) + iy(e^x \cos y + ie^x \sin y) \\ &= xe^x \cos y + ixe^x \sin y + iye^x \cos y - ye^x \sin y = (xe^x \cos y - ye^x \sin y) + i\,(xe^x \sin y + ye^x \cos y) \end{aligned}$$

and hence $u = xe^x \cos y - ye^x \sin y$ and $v = xe^x \sin y + ye^x \cos y$. Therefore:

$$\frac{\partial u}{\partial x} = e^x \cos y + xe^x \cos y - ye^x \sin y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -xe^x \sin y - e^x \sin y - ye^x \cos y = -\frac{\partial v}{\partial x}$$

So, f is analytic (and entire). In fact, the analyticity (and entirety) of f should be obvious because it is a product of polynomial (i.e. z) and exponential functions both of which are analytic (and entire)

as seen in Problem 4.
(**d**) We have $u = \sinh x \cos y$ and $v = \cosh x \sin y$ and hence:

$$\frac{\partial u}{\partial x} = \cosh x \cos y = \frac{\partial v}{\partial y} \quad\text{and}\quad \frac{\partial u}{\partial y} = -\sinh x \sin y = -\frac{\partial v}{\partial x}$$

So, f is analytic (and entire). In fact, f is the hyperbolic sine function (i.e. $f = \sinh z$; see Eq. 142) whose analyticity (and entirety) was established in Problem 4.
(**e**) We have:

$$f = \frac{z^*}{|z|^2} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} \qquad (z \neq 0)$$

and hence $u = \frac{x}{x^2+y^2}$ and $v = \frac{-y}{x^2+y^2}$. Therefore:

$$\frac{\partial u}{\partial x} = \frac{-x^2 + y^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y} \quad\text{and}\quad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x}$$

So, f is analytic everywhere except at $z = 0$ where it is not defined. In fact, the analyticity of f everywhere except at $z = 0$ can be easily concluded by noting that $f = z^*/|z|^2 = 1/z$ which is analytic everywhere except at $z = 0$.

10. Test the analyticity and entirety of the following functions using the merged form of the Cauchy-Riemann equations (see Eq. 162):
(**a**) $f = |z|$. (**b**) $f = |z^2|$. (**c**) $f = ix^2 - 2xy - iy^2$. (**d**) $f = \cos(y - ix)$.
(**e**) $f = \sinh^2(x^2 - y^2)$. (**f**) $f = \cos(2x + iy)$. (**g**) $f = e^{x+y^2}$. (**h**) $f = \sin\sqrt{x^2 + y^2}$.
(**i**) $f = e^{iz}$. (**j**) $f = e^{iz^*}$.
Answer:
(**a**) f is neither analytic (anywhere) or entire because we have $f = |z| = \sqrt{x^2 + y^2}$ and hence from Eq. 162 we get:

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\sqrt{x^2 + y^2} = \frac{x}{\sqrt{x^2 + y^2}} + i\frac{y}{\sqrt{x^2 + y^2}} \neq 0$$

(**b**) f is neither analytic (anywhere) or entire because $f = \sqrt{(x^2 - y^2)^2 + 4x^2y^2} = \sqrt{x^4 + 2x^2y^2 + y^4}$ and hence from Eq. 162 we get:

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\sqrt{x^4 + 2x^2y^2 + y^4} = \frac{2x^3 + 2xy^2}{\sqrt{x^4 + 2x^2y^2 + y^4}} + i\frac{2x^2y + 2y^3}{\sqrt{x^4 + 2x^2y^2 + y^4}} \neq 0$$

(**c**) f is analytic (everywhere) and entire because from Eq. 162 we have:

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(ix^2 - 2xy - iy^2\right) = (i2x - 2y) + i\,(-2x - i2y) = i2x - 2y - i2x + 2y = 0$$

In fact, $f = iz^2$ whose analyticity and entirety are obvious (see Problem 4).
(**d**) f is analytic (everywhere) and entire because from Eq. 162 we have:

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\cos(y - ix) = \big[i\sin(y - ix)\big] + i\big[-\sin(y - ix)\big] = i\sin(y - ix) - i\sin(y - ix) = 0$$

In fact, $f = \cos(-iz)$ whose analyticity and entirety are obvious (see Problem 4).
(**e**) f is neither analytic (anywhere) or entire because from Eq. 162 we have:

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\sinh^2(x^2 - y^2) = 4x\sinh(x^2 - y^2)\cosh(x^2 - y^2) - i4y\sinh(x^2 - y^2)\cosh(x^2 - y^2) \neq 0$$

(**f**) f is neither analytic (anywhere) or entire because from Eq. 162 we have:

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\cos(2x+iy)=\big[-2\sin(2x+iy)\big]+i\big[-i\sin(2x+iy)\big]=-\sin(2x+iy)\neq 0$$

(**g**) f is neither analytic (anywhere) or entire because from Eq. 162 we have:

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)e^{x+y^2}=e^{x+y^2}+i2ye^{x+y^2}\neq 0$$

(**h**) f is neither analytic (anywhere) or entire because from Eq. 162 we have:

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\sin\sqrt{x^2+y^2}=\frac{x}{\sqrt{x^2+y^2}}\cos\sqrt{x^2+y^2}+i\frac{y}{\sqrt{x^2+y^2}}\cos\sqrt{x^2+y^2}\neq 0$$

(**i**) f is analytic (everywhere) and entire because we have $f=e^{iz}=e^{-y+ix}$ and hence from Eq. 162 we get:

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)e^{-y+ix}=\left(ie^{-y+ix}\right)+i\left(-e^{-y+ix}\right)=ie^{-y+ix}-ie^{-y+ix}=0$$

In fact, the analyticity and entirety of f should be obvious from the result of Problem 4.
(**j**) f is neither analytic (anywhere) or entire because we have $f=e^{iz^*}=e^{y+ix}$ and hence from Eq. 162 we get:

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)e^{y+ix}=\left(ie^{y+ix}\right)+i\left(e^{y+ix}\right)=i2e^{y+ix}\neq 0$$

Note: the use of "$\neq 0$" in some parts above means "in general" noting that the occasional vanishing (on points or lines) does not qualify the functions to be analytic at these locations because they do not extend to a neighborhood.

11. Show that a non-constant real function cannot be analytic.
Answer: A non-constant real function has the form $f=u(x,y)+i0$ where either $\partial u/\partial x\neq 0$ or $\partial u/\partial y\neq 0$ (or both) while both derivatives of its imaginary part are zero, i.e. $\partial v/\partial x=\partial v/\partial y=0$. Therefore, at least one of the Cauchy-Riemann equations are not satisfied and hence it is not analytic.
12. Show that a non-constant imaginary function cannot be analytic.
Answer: A non-constant imaginary function has the form $f=0+iv(x,y)$ where either $\partial v/\partial x\neq 0$ or $\partial v/\partial y\neq 0$ (or both) while both derivatives of its real part are zero, i.e. $\partial u/\partial x=\partial u/\partial y=0$. Therefore, at least one of the Cauchy-Riemann equations are not satisfied and hence it is not analytic.
13. Give an example of a complex function that is continuous everywhere (in the complex plane) but analytic nowhere.
Answer: An example is $f(z)=z\,|z|^2$ because both z and $|z|$ are continuous everywhere (in the complex plane) and hence their product (i.e. $z\times|z|\times|z|$) should also be continuous everywhere (see Problems 2 and 12 of § 1.5). However, $f=(x+iy)\left(x^2+y^2\right)=\left(x^3+xy^2\right)+i\left(x^2y+y^3\right)$ and hence from the merged form of the Cauchy-Riemann equations (see Eq. 162) we get:

$$\begin{aligned}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f &= \left[\left(3x^2+y^2\right)+i2xy\right]+i\left[2xy+i\left(x^2+3y^2\right)\right]\\ &= 2x^2+i4xy-2y^2=2\left(x^2+i2xy-y^2\right)=2\left(x+iy\right)^2=2z^2\end{aligned}$$

As we see, $2z^2$ vanishes nowhere (except at the origin) and hence f is analytic nowhere in the complex plane (except possibly at the origin). However, for a function to be analytic at a point it should be differentiable on a neighborhood of that point (see § 1.5). Therefore, f is not analytic even at the origin. Accordingly, f is continuous everywhere but analytic nowhere. In fact, there are many other examples of functions that are continuous everywhere in the complex plane but analytic nowhere, e.g. $f(z)=z^*$.

14. Verify that the following differentiation rules apply to complex functions (as to real functions):
(**a**) $\frac{de^z}{dz} = e^z$. (**b**) $\frac{d\ln z}{dz} = \frac{1}{z}$ $(z \neq 0)$.
Answer:[155] We use Eq. 159 or Eq. 160 in this verification.
(**a**) We have $e^z = e^{x+iy} = e^x(\cos y + i \sin y) = e^x \cos y + ie^x \sin y$ and hence $u = e^x \cos y$ and $v = e^x \sin y$. Now, if we use Eq. 159 then we have:

$$\frac{de^z}{dz} = \frac{\partial}{\partial x}(e^x \cos y) + i\frac{\partial}{\partial x}(e^x \sin y) = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = e^{x+iy} = e^z$$

Similarly, if we use Eq. 160 then we have:

$$\frac{de^z}{dz} = \frac{\partial}{\partial y}(e^x \sin y) - i\frac{\partial}{\partial y}(e^x \cos y) = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = e^{x+iy} = e^z$$

(**b**) We have $\ln z = \log_e r + i(\theta_p + 2n\pi)$ and hence $u = \log_e r = \log_e \sqrt{x^2+y^2}$ and $v = \theta_p + 2n\pi = \arctan(y/x) + 2n\pi$. Now, if we use Eq. 159 then we have:

$$\begin{aligned}\frac{d\ln z}{dz} &= \frac{\partial}{\partial x}\left(\log_e \sqrt{x^2+y^2}\right) + i\frac{\partial}{\partial x}\left(\arctan\left[\frac{y}{x}\right] + 2n\pi\right) \\ &= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} = \frac{z^*}{zz^*} = \frac{1}{z}\end{aligned}$$

Similarly, if we use Eq. 160 then we have:

$$\begin{aligned}\frac{d\ln z}{dz} &= \frac{\partial}{\partial y}\left(\arctan\left[\frac{y}{x}\right] + 2n\pi\right) - i\frac{\partial}{\partial y}\left(\log_e \sqrt{x^2+y^2}\right) \\ &= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} = \frac{z^*}{zz^*} = \frac{1}{z}\end{aligned}$$

Note: Eqs. 159 and 160 are based on the assumption of analyticity which is obvious for e^z since it is entire (see Problem 4). Regarding $\ln z$, we repeat what we said in Problem 19 of § 2.2 about the applicability to individual branches with the removal of the branch cut and hence $\ln z$ is analytic within these conditions (see Problem 5). We should also remark that this Problem is essentially about testing the consistency (noting that no circularity should be suspected).

15. Make a simple argument in support of the proposal that if a complex function $f(z)$ satisfies the Cauchy-Riemann equations then $\left[f(z)\right]^n$ (where n is a positive integer) should also satisfy these equations.
Answer: If f satisfies the Cauchy-Riemann equations then it is analytic and hence its n^{th} power f^n (which is no more than a product $f \times f \times \cdots \times f$) should also be analytic and therefore it should also satisfy the Cauchy-Riemann equations (noting that we are using the *iff* statement in this argument).

16. Show that if $f = u + iv$ is analytic and $|f|$ is constant, then f is constant.[156]
Answer: $|f|$ is constant and hence $|f|^2 = u^2 + v^2 = C^2$ (where C is a real constant). On partial-differentiating this with respect to x and y respectively (and canceling the common factor 2) we get:

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} &= 0 \qquad \text{and} \qquad & u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} &= 0 \\ u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} &= 0 \qquad \text{and} \qquad & u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} &= 0 \\ u^2\frac{\partial u}{\partial x} - uv\frac{\partial u}{\partial y} &= 0 \qquad \text{and} \qquad & uv\frac{\partial u}{\partial y} + v^2\frac{\partial u}{\partial x} &= 0 \\ uv\frac{\partial u}{\partial x} - v^2\frac{\partial u}{\partial y} &= 0 \qquad \text{and} \qquad & u^2\frac{\partial u}{\partial y} + uv\frac{\partial u}{\partial x} &= 0 \end{aligned}$$

[155] The reader should also refer to parts (a) and (b) of Problem 19 of § 2.2 for a different method of verification.
[156] As indicated earlier, we are assuming a proper domain over which such a statement applies.

where in line 2 we used the Cauchy-Riemann equations (see Eq. 157), and in line 3 we multiplied the left equation (of line 2) with u and the right equation (of line 2) with v while in line 4 we multiplied the left equation (of line 2) with v and the right equation (of line 2) with u. Now, if we add the equations in line 3 side by side we get $\left(u^2+v^2\right)\frac{\partial u}{\partial x}=0$ and if we subtract the equations in line 4 side by side we get $\left(u^2+v^2\right)\frac{\partial u}{\partial y}=0$. So, if $u=v=0$ then $f=0$ and hence f is constant. Otherwise, $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=0$ and hence u is constant. Also, on using the Cauchy-Riemann equations with the last equation (i.e. $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=0$) we get $\frac{\partial v}{\partial y}=-\frac{\partial v}{\partial x}=0$ and hence v is also constant. So, both u and v are constants and hence f is constant.

17. Show that if $f=u+iv$ is an analytic function then $|\nabla u|=|\nabla v|$.
Answer: Noting that $\nabla u=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$ and $\nabla v=\left(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right)$, we have:

$$\begin{aligned}|\nabla u| &= \sqrt{\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\cdot\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)}=\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2}=\sqrt{\left(\frac{\partial v}{\partial y}\right)^2+\left(-\frac{\partial v}{\partial x}\right)^2}\\ &= \sqrt{\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2}=\sqrt{\left(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right)\cdot\left(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right)}=|\nabla v|\end{aligned}$$

where we took the dot product in the first equality and used the Cauchy-Riemann equations in the third equality.

18. Show that if $f=u+iv$ is an analytic function then the gradients of u and v (at any point in the domain of f) are orthogonal.
Answer: The gradients of u and v are $\nabla u=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$ and $\nabla v=\left(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right)$. The orthogonality of these gradients can be easily shown by showing that their dot product is zero, that is:

$$\nabla u\cdot\nabla v=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)\cdot\left(\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right)=\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}=\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}-\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}=0$$

where in the third equality we used the first of the Cauchy-Riemann equations (see Eq. 157) to replace $\frac{\partial u}{\partial x}$ by $\frac{\partial v}{\partial y}$ in the first term and the second of the Cauchy-Riemann equations to replace $\frac{\partial u}{\partial y}$ by $-\frac{\partial v}{\partial x}$ in the second term.

19. Show that if f is a function of z^* $(=x-iy)$ then f is not analytic.
Answer: We prove this statement by proving its contrapositive, i.e. if f is analytic then it is not a function of z^*. So, let $f=u(x,y)+iv(x,y)$ be an analytic function and we take its partial derivative with respect to z^* using the chain rule, that is:

$$\begin{aligned}\frac{\partial f}{\partial z^*} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*}\\ &= \left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\frac{\partial x}{\partial z^*}+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\frac{\partial y}{\partial z^*}\\ &= \left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\frac{1}{2}+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\frac{-1}{i2} && \left(x=\frac{z+z^*}{2}\text{ and }y=\frac{z-z^*}{i2}\right)\\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+i\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\\ &= 0\end{aligned}$$

where the last step is because f is analytic and hence it should satisfy the Cauchy-Riemann equations (see Eq. 157). The vanishing of this partial derivative means that f is independent of z^*, i.e. f is not a function of z^*. The significance of this result is that an analytic function of z (where z here may include its conjugate) can only be a function of the combination $x+iy$ (i.e. z strictly) but not of the combination $x-iy$ (i.e. z^*).

20. Derive the polar form of the Cauchy-Riemann equations of an analytic function $f(z) = u(r, \theta) + iv(r, \theta)$.
Answer: We have $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right) + 2n\pi$. Hence, from (the Cartesian form of) the Cauchy-Riemann equations (see Eq. 157) we get:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = -\frac{\partial v}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial r}\frac{x}{\sqrt{x^2+y^2}} = \frac{\partial v}{\partial \theta}\frac{1/x}{1+\left(\frac{y}{x}\right)^2} \quad \text{and} \quad \frac{\partial u}{\partial \theta}\frac{1/x}{1+\left(\frac{y}{x}\right)^2} = -\frac{\partial v}{\partial r}\frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial r}\frac{1}{\sqrt{x^2+y^2}} = \frac{\partial v}{\partial \theta}\frac{1}{x^2+y^2} \quad \text{and} \quad \frac{\partial u}{\partial \theta}\frac{1}{x^2+y^2} = -\frac{\partial v}{\partial r}\frac{1}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial r} = \frac{1}{\sqrt{x^2+y^2}}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -\sqrt{x^2+y^2}\,\frac{\partial v}{\partial r}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

21. List some of the properties of analytic functions (as defined over a given region in the complex plane).
Answer: For example:[157]
• A function that is analytic at a point z_0 is differentiable at z_0 and at every point in some neighborhood of z_0.
• It is infinitely differentiable and its derivatives of all orders are analytic over the region of analyticity (see Problem 6 of § 4.3).
• It possesses a (Taylor) power series around the point of analyticity (see § 5.1).[158]
• It does not possess a local maximum or (non-zero) minimum inside the region of analyticity (see § 4.6).
• The value of the function at any point inside the region of analyticity is determined by its value on the boundary of this region (see § 4.3).

3.2 Contour Integration

Contour integration means line integration (or path integration) in the complex plane.[159] Accordingly, if C is a given curve in the complex plane and f is a complex function defined on a region that contains C and it is represented mathematically by the relation $w = f(z)$ then the contour integration of f along C is the line integral $\int_C w\,dz$. In this regard, C can be a closed curve (e.g. the circle $|z - i| = 5$) or an open curve connecting two distinct points (e.g. the upper half of the circle $|z| = 2$ between the points $z_1 = 2$ and $z_2 = -2$). Also, a closed curve can be tracked in a clockwise sense or in an anticlockwise sense, where the common convention is to consider the former as negative and the latter as positive.[160] Similarly, an open curve can be tracked in one direction or another where the direction is indicated by determining the start and end points (e.g. from point $z_1 = 0$ to point $z_2 = 1 + i$ rather than the opposite).[161]

[157] We note that most of these properties will be investigated thoroughly later on. Also, the following is just a sketch of these properties and hence restrictions and conditions generally apply (although they are ignored here).
[158] It is commonly accepted that a function $f(z)$ is analytic at a point z_0 *iff* it can be represented by a power series at z_0 (where the series converges to f in a neighborhood of z_0).
[159] "Contour" may be defined more technically and specifically as: directed, closed and smooth (at least piecewise) curve in the complex plane (and contour integration is then defined accordingly). However, in this book we adopt a more loose and general definition (as indicated above).
[160] In this book, all the contour integrals of closed curves (and fragments of closed curves) are assumed to be in the positive sense (i.e. anticlockwise) unless stated otherwise. Anyway, the sense of tracking closed curves in contour integrals will be indicated by using the integral symbol $\oint$ for clockwise sense of tracking and the integral symbol $\oint$ for anticlockwise sense of tracking.
[161] As indicated earlier (see footnote [22] on page 16), it is sensible to identify the sense of tracking of some open curves by "clockwise" and "anticlockwise".

It is noteworthy that contour integration in complex analysis is very similar to (2D) multi-variable line integration in real analysis (but usually with a big advantage for the contour integration since many operations and manipulations can be done in one go using compact expressions and formulations as well as exploiting the amazing properties of analyticity and providing more fundamental understanding of the mechanisms of these operations).

It is important to notice that contour integrals can be calculated using three main approaches:
• Contour integrals may be calculated by using a parameterization approach where the curve (and hence the integral) is represented and formulated mathematically by using a scalar parameter like t (see Problem 1). According to this approach we can define the contour integral of a complex function $w(z)$ over a t-parameterized curve C $\big[$i.e. C is defined as $z = z(t)\big]$ by the following equation:

$$\int_C w(z)\,dz \equiv \int_{t_1}^{t_2} w(z(t))\,\dot{z}\,dt \tag{163}$$

where t_1 and t_2 are the values of t corresponding to the end points of C and the overdot means d/dt.
• They may also be calculated by using a real and imaginary variables (i.e. x and y) approach where Eq. 80 facilitates the formulation and calculation (see Problem 2).[162]
• They may also be calculated by using ordinary integration, i.e. by the use of indefinite integration (to get a primitive) followed by substitution of limits as done for the calculation of definite integrals in real analysis (see Problem 3).[163]
However, we should note that not every one of these approaches is employable in each contour integration problem. Also, each approach has its advantages and disadvantages although they should lead to the same result (assuming they are employable). More about these issues will follow (see for example Problem 4).

Problems

1. Calculate the following contour integrals over the given (anticlockwise oriented) curves C using a parameterization approach:
   (**a**) $\int_C 3z\,dz$ where C is the upper half of the circle $|z-1+i| = 4$.[164]
   (**b**) $\int_C iz\,dz$ where C is the quarter of the ellipse $\frac{x^2}{4} + y^2 = 1$ in the first quadrant.
   (**c**) $\oint_C z^*\,dz$ where C is the square with vertices at $z_1 = -1-i$, $z_2 = 1-i$, $z_3 = 1+i$ and $z_4 = -1+i$.
   (**d**) $\oint_C \frac{z^3+\pi}{z}\,dz$ where C is the origin-centered unit circle.
   Answer:
   (**a**) The curve C is the upper half of the circle centered on the point $1-i$ with radius 4 and hence it can be parameterized as $z = 1-i+4e^{i\theta}$ (where $0 \le \theta \le \pi$). Therefore, $dz = i4e^{i\theta}d\theta$ and we have:

$$\begin{aligned}\int_C 3z\,dz &= \int_0^\pi 3(1-i+4e^{i\theta})\,i4e^{i\theta}d\theta = 12\int_0^\pi (ie^{i\theta} + e^{i\theta} + i4e^{i2\theta})\,d\theta \\ &= 12\Big[e^{i\theta} - ie^{i\theta} + 2e^{i2\theta}\Big]_0^\pi = 12\Big[-1+i+2\Big] - 12\Big[1-i+2\Big] = -24 + i24\end{aligned}$$

   (**b**) The curve C is the quarter (in the first quadrant) of the origin-centered ellipse with semi-major axis $a = 2$ and semi-minor axis $b = 1$ and hence it can be parameterized as $z = 2\cos\theta + i\sin\theta$ (where $0 \le \theta \le \pi/2$). Therefore, $dz = (-2\sin\theta + i\cos\theta)\,d\theta$ and we have:

$$\int_C iz\,dz = \int_0^{\frac{\pi}{2}} i\,(2\cos\theta + i\sin\theta)\,(-2\sin\theta + i\cos\theta)\,d\theta$$

[162] It is straightforward to show that Eq. 80 and Eq. 163 (with regard to contour integration) are equivalent and hence the real and imaginary variables approach and the parameterization approach are essentially the same (although they usually lead to very different formulations and technical manipulations as will be seen in the Problems).

[163] In fact, this is not really a contour integration (in the strict sense) but it works when the integral is independent of the path, i.e. when the integral depends only on the two end points which requires the integrand to be analytic (see Problem 4). This means that this method is restricted to analytic functions (i.e. the integrand is analytic) while the other two methods are general and apply to all (continuous) functions.

[164] Being "anticlockwise oriented" means from the start point $5-i$ on the right to the end point $-3-i$ on the left. Similarly, for part (b) it means from the start point on the positive real axis to the end point on the positive imaginary axis.

$$
\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \Big[\big(2\sin^2\theta - 2\cos^2\theta\big) - i5\cos\theta\sin\theta\Big]d\theta = \left[\left(\theta - \frac{\sin 2\theta}{2} - \frac{\sin 2\theta}{2} - \theta\right) + i\frac{5}{2}\cos^2\theta\right]_0^{\frac{\pi}{2}} \\
&= \left[-\sin 2\theta + i\frac{5}{2}\cos^2\theta\right]_0^{\frac{\pi}{2}} = \Big[0 + i0\Big] - \left[0 + i\frac{5}{2}\right] = -i\frac{5}{2}
\end{aligned}
$$

(**c**) The square is made of 4 straight line segments that can be parameterized as follows:
C_1 from z_1 to z_2: $z = t - i$ $(-1 \le t \le 1)$ and hence $dz = dt$.
C_2 from z_2 to z_3: $z = 1 + it$ $(-1 \le t \le 1)$ and hence $dz = idt$.
C_3 from z_3 to z_4: $z = -t + i$ $(-1 \le t \le 1)$ and hence $dz = -dt$.
C_4 from z_4 to z_1: $z = -1 - it$ $(-1 \le t \le 1)$ and hence $dz = -idt$.
Therefore, we have:

$$
\begin{aligned}
\oint_C z^*\,dz &= \int_{C_1} z^*\,dz + \int_{C_2} z^*\,dz + \int_{C_3} z^*\,dz + \int_{C_4} z^*\,dz \\
&= \int_{-1}^{+1} (t+i)\,dt + \int_{-1}^{+1} (1-it)\,idt + \int_{-1}^{+1} (-t-i)\,(-dt) + \int_{-1}^{+1} (-1+it)\,(-idt) \\
&= \int_{-1}^{+1} \Big[(t+i) + (i+t) + (t+i) + (i+t)\Big]dt = \int_{-1}^{+1} \Big[4t + i4\Big]dt \\
&= \Big[2t^2 + i4t\Big]_{-1}^{+1} = \Big[2 + i4\Big] - \Big[2 - i4\Big] = i8
\end{aligned}
$$

(**d**) The curve C is the origin-centered unit circle and hence it can be parameterized as $z = e^{i\theta}$ (where $0 \le \theta < 2\pi$). Therefore, $dz = ie^{i\theta}d\theta$ and we have:

$$
\begin{aligned}
\oint_C \frac{z^3+\pi}{z}\,dz &= \int_0^{2\pi} \frac{e^{i3\theta}+\pi}{e^{i\theta}}\,ie^{i\theta}d\theta = \int_0^{2\pi} \left(ie^{i3\theta} + i\pi\right)\,d\theta = \left[\frac{e^{i3\theta}}{3} + i\pi\theta\right]_0^{2\pi} \\
&= \left[\frac{1}{3} + i2\pi^2\right] - \left[\frac{1}{3} + i0\right] = i2\pi^2
\end{aligned}
$$

2. Calculate the following contour integrals over the given curves C using a real and imaginary variables approach:
   (**a**) As part (a) of Problem 1.
   (**b**) As part (b) of Problem 1.
   (**c**) $\int_C (2z^2 - z + 5)\,dz$ where C is the straight line segment $y = x$ from $z_1 = 0$ to $z_2 = 3 + i3$.
   (**d**) As part (c) of the present Problem but with C being the parabola $y = x^2 - 2x$.
   Answer:
   (**a**) The circle can be represented as $(x-1)^2 + (y+1)^2 = 16$ and hence the curve C (which is the upper half of this circle) can be represented as $y = \sqrt{16 - (x-1)^2} - 1$ (where $5 \ge x \ge -3$). Accordingly, $dy = \frac{(1-x)}{\sqrt{16-(x-1)^2}}dx$. We also have:

$$
w = u + iv = 3z = 3x + i3y = 3x + i\left(3\sqrt{16 - (x-1)^2} - 3\right)
$$

So, from Eq. 80 we have:

$$
\begin{aligned}
\int_C 3z\,dz &= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx) \\
&= \int_5^{-3} \left[3x\,dx - \left(3\sqrt{16-(x-1)^2} - 3\right)\frac{(1-x)}{\sqrt{16-(x-1)^2}}dx\right] +
\end{aligned}
$$

$$\begin{aligned}
& i\int_{5}^{-3}\left[3x\frac{(1-x)}{\sqrt{16-(x-1)^2}}dx+\left(3\sqrt{16-(x-1)^2}-3\right)dx\right] \\
= & \int_{5}^{-3}\left[6x-3+\frac{3(1-x)}{\sqrt{16-(x-1)^2}}\right]dx+ \\
& i\int_{5}^{-3}\left[\frac{3x\,(1-x)}{\sqrt{16-(x-1)^2}}+3\sqrt{16-(x-1)^2}-3\right]dx \\
= & \left[\left(3x^2-3x+3\sqrt{16-(x-1)^2}\right)+i\left(3x\sqrt{16-(x-1)^2}-3x\right)\right]_5^{-3} \\
= & \left[36+i9\right]-\left[60-i15\right]=-24+i24
\end{aligned}$$

which is identical to the result of part (a) of Problem 1.
(**b**) The ellipse is represented as $\frac{x^2}{4}+y^2=1$ and hence the curve C (which is the quarter of this ellipse in the first quadrant) can be represented as $y=\sqrt{1-(x/2)^2}$ (where $2\ge x\ge 0$). Accordingly, $dy=\frac{-x}{4\sqrt{1-(x/2)^2}}dx$. We also have:

$$w=u+iv=iz=-y+ix=-\sqrt{1-(x/2)^2}+ix$$

So, from Eq. 80 we have:

$$\begin{aligned}
\int_C iz\,dz &= \int_C (u\,dx-v\,dy)+i\int_C (u\,dy+v\,dx) \\
&= \int_2^0\left[-\sqrt{1-(x/2)^2}\,dx-x\frac{-x}{4\sqrt{1-(x/2)^2}}\,dx\right]+ \\
&\quad i\int_2^0\left[-\sqrt{1-(x/2)^2}\frac{-x}{4\sqrt{1-(x/2)^2}}\,dx+x\,dx\right] \\
&= \int_2^0\left[-\sqrt{1-(x/2)^2}+\frac{x^2}{4\sqrt{1-(x/2)^2}}\right]dx+i\int_2^0\frac{5x}{4}\,dx \\
&= \left[\left(-x\sqrt{1-(x/2)^2}\right)+i\frac{5x^2}{8}\right]_2^0=\left[0+i0\right]-\left[0+i\frac{5}{2}\right]=-i\frac{5}{2}
\end{aligned}$$

which is identical to the result of part (b) of Problem 1.
(**c**) The curve C is represented by $y=x$ $(0\le x\le 3)$ and hence $dy=dx$. We also have:

$$\begin{aligned}
w &= u+iv=2z^2-z+5=2(x^2-y^2+i2xy)-x-iy+5=(2x^2-2y^2-x+5)+i(4xy-y) \\
&= (2x^2-2x^2-x+5)+i(4x^2-x)=(5-x)+i(4x^2-x)
\end{aligned}$$

So, from Eq. 80 we have:

$$\begin{aligned}
\int_C (2z^2-z+5)\,dz &= \int_C (u\,dx-v\,dy)+i\int_C (u\,dy+v\,dx) \\
&= \int_0^3\left[(5-x)\,dx-\left(4x^2-x\right)dx\right]+i\int_0^3\left[(5-x)\,dx+\left(4x^2-x\right)dx\right] \\
&= \int_0^3\left(5-4x^2\right)dx+i\int_0^3\left(5-2x+4x^2\right)dx \\
&= \left[\left(5x-\frac{4}{3}x^3\right)+i\left(5x-x^2+\frac{4}{3}x^3\right)\right]_0^3=\left[-21+i42\right]-\left[0+i0\right]
\end{aligned}$$

$$= \quad -21 + i42$$

(**d**) The curve C is represented by $y = x^2 - 2x$ $(0 \le x \le 3)$ and hence $dy = (2x - 2)\,dx$. We also have:

$$\begin{aligned}
w &= u + iv = 2z^2 - z + 5 = 2(x^2 - y^2 + i2xy) - x - iy + 5 = (2x^2 - 2y^2 - x + 5) + i(4xy - y) \\
&= \left(2x^2 - 2\left[x^2 - 2x\right]^2 - x + 5\right) + i\left(4x\left[x^2 - 2x\right] - \left[x^2 - 2x\right]\right) \\
&= \left(-2x^4 + 8x^3 - 6x^2 - x + 5\right) + i\left(4x^3 - 9x^2 + 2x\right)
\end{aligned}$$

So, from Eq. 80 we have:

$$\begin{aligned}
\int_C (2z^2 - z + 5)\,dz &= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx) \\
&= \int_0^3 \Big[\left(-2x^4 + 8x^3 - 6x^2 - x + 5\right)dx - \left(4x^3 - 9x^2 + 2x\right)(2x - 2)dx\Big] + \\
&\qquad i\int_0^3 \Big[\left(-2x^4 + 8x^3 - 6x^2 - x + 5\right)(2x - 2)dx + \left(4x^3 - 9x^2 + 2x\right)dx\Big] \\
&= \int_0^3 \left(-10x^4 + 34x^3 - 28x^2 + 3x + 5\right)dx + \\
&\qquad i\int_0^3 \left(-4x^5 + 20x^4 - 24x^3 + x^2 + 14x - 10\right)dx \\
&= \Bigg[\left(-2x^5 + \frac{17}{2}x^4 - \frac{28}{3}x^3 + \frac{3}{2}x^2 + 5x\right) + \\
&\qquad i\left(-\frac{2}{3}x^6 + 4x^5 - 6x^4 + \frac{x^3}{3} + 7x^2 - 10x\right)\Bigg]_0^3 \\
&= \big[-21 + i42\big] - \big[0 + i0\big] = -21 + i42
\end{aligned}$$

which is identical to the result of part (c).

3. Calculate the following contour integrals over the given curves C using an ordinary integration approach:
(**a**) As part (a) of Problem 1.
(**b**) As part (b) of Problem 1.
(**c**) As part (c) of Problem 2 (or as part d of Problem 2).
Answer:
(**a**) The two end points of C are $z_1 = 5 - i$ and $z_2 = -3 - i$ and hence:

$$\int_C 3z\,dz = \int_{5-i}^{-3-i} 3z\,dz = \left[\frac{3}{2}z^2\right]_{5-i}^{-3-i} = \left[\frac{3}{2}(-3 - i)^2\right] - \left[\frac{3}{2}(5 - i)^2\right] = -24 + i24$$

which is identical to the result of part (a) of Problem 1 (as well as part a of Problem 2).
(**b**) The two end points of C are $z_1 = 2$ and $z_2 = i$ and hence:

$$\int_C iz\,dz = \int_2^i iz\,dz = \left[i\frac{z^2}{2}\right]_2^i = \left[i\frac{-1}{2}\right] - \left[i\frac{4}{2}\right] = -i\frac{5}{2}$$

which is identical to the result of part (b) of Problem 1 (as well as part b of Problem 2).
(**c**) The two end points of C are $z_1 = 0$ and $z_2 = 3 + i3$ and hence:

$$\begin{aligned}
\int_C (2z^2 - z + 5)\,dz &= \int_0^{3+i3} (2z^2 - z + 5)\,dz = \left[\frac{2z^3}{3} - \frac{z^2}{2} + 5z\right]_0^{3+i3} \\
&= \left[\frac{2}{3}(3 + i3)^3 - \frac{(3 + i3)^2}{2} + 5(3 + i3)\right] - \big[0 - 0 + 0\big] = -21 + i42
\end{aligned}$$

which is identical to the result of part (c) of Problem 2 (as well as part d of Problem 2).

4. Discuss the approach of using ordinary integration to calculate contour integrals.
Answer: As it is known, line integrals are generally path dependent and hence using ordinary integration for calculating contour integrals should not work (in general) because in ordinary integration the value of the integral depends on the end points only as the notation itself indicates, i.e. $\int_{z_1}^{z_2} f(z)dz = \left[F(z)\right]_{z_1}^{z_2}$ which has no indication to the path of integration (except its end points). However, in a simply-connected region (see § 1.5) line integral becomes path independent if it is of the form $\int (P\,dx + Q\,dy)$ where $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout that region. For example, in part (c) of Problem 2 the real part integral (which is in a simply-connected region) is of the from $\int (P\,dx + Q\,dy)$ with $P = u = 2x^2 - 2y^2 - x + 5$ and $Q = -v = y - 4xy$ where $\frac{\partial P}{\partial y} = -4y = \frac{\partial Q}{\partial x}$ and hence this integral is independent of the path. Similarly, the imaginary part integral is of the from $\int (P\,dx + Q\,dy)$ with $P = v = 4xy - y$ and $Q = u = 2x^2 - 2y^2 - x + 5$ where $\frac{\partial P}{\partial y} = 4x - 1 = \frac{\partial Q}{\partial x}$ and hence this integral is also independent of the path.[165] The path independence of the contour integral (in its real and imaginary parts) of part (c) of Problem 2 was confirmed in part (c) of Problem 3 where we used ordinary integration (which is based on path independence) to calculate this line integral and we obtained the same result. The path independence is also confirmed in a specific case in part (d) of Problem 2 where we obtained the same result as in part (c) of Problem 2 despite the fact that the curve C is different (i.e. the parabola $y = x^2 - 2x$) although this does not establish the path independence in general (i.e. along any curve).
We should finally note that by comparing the condition $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ to the form of Eq. 80 (where in the real part integral $P \equiv u$ and $Q \equiv -v$ while in the imaginary part integral $P \equiv v$ and $Q \equiv u$) we can see that the condition $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ for the real part is no more than $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (which is the second of the Cauchy-Riemann equations) and for the imaginary part is no more than $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ (which is the first of the Cauchy-Riemann equations). Now, if we note that the Cauchy-Riemann equations represent a criterion for analyticity, we can then conclude that path independence of contour integrals is conditioned by the analyticity of their integrands.[166] This issue will be investigated further later on (see for example § 4.1 and § 4.2).

3.3 Singularities

As indicated earlier (see § 1.5), a function is singular at a given point in the complex plane if it is problematic in some way at that point, e.g. by having a vanishing denominator at that point and hence it blows up. We also indicated that a given point in the complex plane may be described as singular or singularity of a given complex function $f(z)$ if f is singular at that point. In fact, there is some specific terminology in the literature of complex analysis about singularities (which is based on their mathematical nature) that the reader should be aware of. The essentials of this terminology will be briefly discussed in the following paragraphs and it will be investigated further in the Problems of this section (noting that the literature here, as elsewhere, is not entirely consistent and hence the reader should be aware of the specific convention of each particular author and sometimes the apparent or genuine contradiction of some authors in different contexts and locations).

A point z_0 in the complex plane is an **isolated singular point** or an **isolated singularity** of a complex function $f(z)$ if f is analytic in some neighborhood of z_0 except at the point z_0 (also see § 1.5).[167] Isolated singularities are generally divided into three main types (noting that these types may occasionally be used in the literature, correctly or incorrectly, in reference to non-isolated singularities):
(**A**) **Removable singularity** which is a point z_0 at which the function $f(z)$ is not analytic (i.e. in a strict technical sense) but f has a limit at z_0 and hence f is bounded at z_0 and can be defined sensibly

[165] It should be obvious that the path independence of the contour integral in its real and imaginary parts means path independence of the contour integral (as a complex integral).

[166] In brief, the integral of a given function $f(z)$ is path-independent in a given (simply-connected) region R *iff* f is analytic in the entire R.

[167] "Non-isolated singular point" (or non-isolated singularity) is defined accordingly, i.e. there is no deleted neighborhood of z_0 on which f is entirely analytic since any such neighborhood contains some singularities.

at z_0 to make it analytic there.
(**B**) **Pole of order** $\boldsymbol{n}$ (where n is a finite positive integer) which is a singularity at which the limit $\lim_{z\to z_0}(z-z_0)^n f(z)$ does exist where n is the smallest integer for which this limit exists.[168]
(**C**) **Essential singularity** which is a point z_0 at which the limit $\lim_{z\to z_0}(z-z_0)^n f(z)$ does not exist for any (non-negative) finite integer n.[169] In more simple terms, essential singularity is an isolated singularity that is neither removable nor pole.
As we will see (refer to Problem 6 of § 5.2), these three types of isolated singularity can be defined in a simpler way by using the Laurent series expansion of the function around the singularity (and hence these types of isolated singularity may be attributed to the series that represents the function as well as to the function).[170]

We present in the following bullet points a number of useful remarks about singularities and their terminology:
• A pole of order 1 is commonly known as **simple pole**. Similarly, a pole of order 2 may be called **double pole**, a pole of order 3 may be called **triple pole**, and so on.
• Essential singularity may be described as a **pole of infinite order** or **essential pole**.
• As we will see (refer to § 5.2 and Problem 6 of § 5.2 in particular), removable singularity is *effectively* "analytic point" (although *formally* it is not) since the Laurent series expansion at that point has no principal part and hence it is actually a Taylor series. In fact, this should explain why it is "removable". It is noteworthy that being removable is due to the boundedness of the function (since it has a limit) at this type of singularity and hence it is possible to redefine the function to make it analytic at the singular point and hence remove the singularity.
• Whether the function at its removable singularities (according to our definition) can be described as analytic (considering, for instance, it has a Taylor-like series) or not (considering, for instance, it is not defined there formally by the function in its original form) seems to be a contentious issue. However, this is a trivial matter (as long as we know the actual properties and behavior of the function and its series at such points) since it is a matter of labeling with no substantial consequences. In our view (which we indicated in the previous point), removable singularities are *effectively* "analytic points" although *formally* they are singular points.
• Although the above terminology about the types of isolated singularities is the common one in the literature of complex analysis, there seem to be other conventions (or abuses or mistakes) beside the above convention. For example, regarding the terms "removable" and "essential" some authors seem to use "removable" for (A) specifically and use "essential" for (B) and (C), while other authors seem to use "removable" for (A) and (B) and use "essential" for (C) specifically. In fact, a careful inspection of the literature may reveal other conventions (or mistakes or sloppiness). In this regard, we should also consider the terms "non-removable" and "non-essential" in this context although they should depend in their ultimate meaning on the adopted meaning of the terms "removable" and "essential" and hence "non-removable" or "non-essential" for each author should mean not "removable" or not "essential" (according to the convention and definition of "removable" and "essential" for that author). Anyway, these differences are generally trivial (as long as we know what is going on and we are aware of the definitions and conventions of each author).

[168] In fact, there are some differences in the definition of "pole of order n". For example, some define it as: a point z_0 at which the function $f(z)$ is not analytic but the function $g(z)=(z-z_0)^n f(z)$ is analytic where n is the smallest number that makes g analytic, i.e. $(z-z_0)^m f(z)$ is not analytic for $m<n$. Some of these differences are superficial while others reflect real difference in the concept. The latter definition suggests that removable singularity does not affect analyticity.

[169] It may also be defined (following the definition of pole in the previous footnote) as: a point z_0 at which neither the function $f(z)$ is analytic nor the function $(z-z_0)^n f(z)$ for any finite n (where n is a positive integer).

[170] The investigation of the singularities of functions is intimately and naturally linked to their series representation and hence a natural approach is to investigate the subject of singularities within or following the investigation of series representation of functions (i.e. Laurent series specifically). However, for structural necessities we assigned the present section and section 5.2 to their current positions, and this dictates the deferral of the investigation of some aspects of singularities (which are the subject of the present section) to section 5.2 since they are linked to the subject of Laurent series.

Problems

1. Give an example of a function that has one removable singularity and one non-removable singularity (e.g. simple pole).
Answer: For example, the following function:
$$f(z) = \frac{z^2 + z - 2}{z^2 - 1} = \frac{(z-1)(z+2)}{(z-1)(z+1)} \tag{164}$$
has one removable singularity at $z = 1$ and one non-removable singularity at $z = -1$ (which is a simple pole). The reason is that although f is not defined at both points (since its denominator vanishes at both points),[171] it has a limit (which is 3/2) at $z = 1$ and hence it can be re-defined at $z = 1$ as $f(z) = \frac{z+2}{z+1}$ and the singularity is removed, but it has no limit (since it blows up by going to infinity) at $z = -1$ and hence it cannot be defined there in a way that removes the singularity.
2. Are branch points of multi-valued functions singular points, and if so to what type of singularity they belong?
Answer: Yes, they are singular points because the function is discontinuous at these points (and hence it cannot be analytic). Moreover, due to their connection to the branch cut (at which the function is also discontinuous and hence non-analytic) they are not isolated and hence they are non-isolated singularities. More technically, there is no deleted neighborhood of a branch point on which the function is entirely analytic since any such neighborhood contains some singular points from the branch cut.
3. Identify the singularities of the following functions and determine if they are removable, poles or essential (or non-isolated):
(**a**) $f = z^3 + iz^{-1}$. (**b**) $f = \frac{1}{z^3-1}$. (**c**) $f = \frac{z}{(z^2+5)^2}$. (**d**) $f = \frac{z}{z^4+i8z}$.
(**e**) $f = \frac{2-i-z}{z^2-4z+5}$. (**f**) $f = \frac{\sin z}{z}$. (**g**) $f = \tanh z$. (**h**) $f = e^{1/z}$.
(**i**) $f = \cot z$. (**j**) $f = \ln(z-3)$. (**k**) $f = (z+3-i)^{1/3}$. (**l**) $f = \frac{z^2}{(z-i)^2(z^2+2)}$.
Answer:
(**a**) f has only one singularity at $z_1 = 0$ since the denominator of $z^{-1} = 1/z$ vanishes at z_1. This singularity is a simple pole.
(**b**) f has three singularities (which are the roots of $z^3 - 1 = 0$): at $z_1 = 1$, at $z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and at $z_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ (see part b of Problem 1 of § 1.8.11 noting the difference in labeling). These singularities are simple poles.
(**c**) f has two singularities (which are the roots of $z^2 + 5 = 0$): at $z_1 = -i\sqrt{5}$ and at $z_2 = +i\sqrt{5}$. These singularities are double poles.
(**d**) f has four singularities (which are the roots of $z^4 + i8z = 0$): at $z_1 = 0$, at $z_2 = i2$, at $z_3 = \sqrt{3} - i$ and at $z_4 = -\sqrt{3} - i$. The singularity at z_1 is removable (since it can be removed by defining f by its limit at z_1 which is $-i0.125$) while the other three singularities are simple poles.
(**e**) f has two singularities (which are the roots of $z^2 - 4z + 5 = 0$): at $z_1 = 2 + i$ and at $z_2 = 2 - i$. The singularity at z_1 is a simple pole while the singularity at z_2 is removable (since it can be removed by defining f by its limit at z_2 which is $-i0.5$).
(**f**) f has only one singularity at $z_1 = 0$ since the numerator and denominator of f vanish at z_1 and hence f reduces to the indeterminate form $0/0$ at z_1. This singularity is removable (since it can be removed by defining f by its limit at z_1 which is 1).[172]
(**g**) We have $\tanh z = \frac{\sinh z}{\cosh z}$ and hence the singularities of $\tanh z$ occur where $\cosh z = 0$, i.e. at $z = i\frac{(2n+1)\pi}{2}$ (see part c of Problem 14 of § 2.3). So, we have an infinite number of singularities. These singularities are simple poles because the zeros of $\cosh z$ are simple since the derivative of $\cosh z$ (which is $\sinh z$) does not vanish at these points (see part d of Problem 14 of § 2.3; also see § 1.5).

[171] In fact, at point $z = 1$ it reduces to the indeterminate form $0/0$.
[172] Using the series of $\sin z$ (see Eq. 8) we have $\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$ which obviously has a limit ($= 1$) at $z = 0$. This limit can also be obtained from L'Hospital's rule.

(**h**) As we will see later (refer to Problem 6 of § 5.2), f has an essential singularity at the origin.
(**i**) We have $\cot z = \frac{\cos z}{\sin z}$ and hence the singularities of $\cot z$ occur where $\sin z = 0$, i.e. at $z = n\pi$ (see part b of Problem 14 of § 2.3). So, we have an infinite number of singularities. These singularities are simple poles because the zeros of $\sin z$ are simple since the derivative of $\sin z$ (which is $\cos z$) does not vanish at these points (see part a of Problem 14 of § 2.3; also see § 1.5).
(**j**) $\ln z$ has a branch point at $z = 0$ which is a non-isolated singularity (see Problem 2). Hence, $\ln(z-3)$ has a non-isolated singularity at $z = 3$ (which is its branch point). In fact, this should apply to all points on the branch cut of this function.
(**k**) $z^{1/3}$ has a branch point at $z = 0$ which is a non-isolated singularity (see Problem 2). Hence, $(z + 3 - i)^{1/3}$ has a non-isolated singularity at $z = -3 + i$ (which is its branch point). In fact, this should apply to all points on the branch cut of this function.
(**l**) f has three singularities: a double pole at $z = i$, a simple pole at $z = i\sqrt{2}$, and a simple pole at $z = -i\sqrt{2}$.

4. Give some examples of isolated and non-isolated singularities.
Answer: The singularities of rational functions are isolated (see Problem 7 of § 2.1 and Problem 5 of § 7.1). Also, the singularities of $\tanh z$ and $\cot z$ are isolated (see parts g and i of Problem 3).
The singularity of $\ln z$ at $z = 0$ is not isolated since the non-positive real axis (which is the branch cut) is a continuous line and hence any neighborhood of the point $z = 0$ contains singularities (i.e. on the branch cut) other than $z = 0$. This similarly applies to $\sqrt{z}$ which has a non-isolated singularity at $z = 0$ which is its branch point.

3.4 Harmonic Functions

Harmonic function (in a given domain D) is a real function $\phi(x, y)$ of two real variables x and y that possesses continuous second order partial derivatives in D and it satisfies Laplace's equation in D. It can be shown that if a complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D then both its real part u and imaginary part v are harmonic functions in D (see Problem 1). Noting the link between analyticity and the Cauchy-Riemann equations (see § 3.1) as well as the link between being harmonic and satisfying Laplace's equation, we can say that harmonic functions (within complex functions) provide a link between the Cauchy-Riemann equations and Laplace's equation (in 2D). It is noteworthy that if $u(x, y)$ and $v(x, y)$ are harmonic functions in a given domain D and $f = u(x, y) + iv(x, y)$ is analytic in D then u and v are commonly described as "harmonic conjugate functions" or "harmonic conjugates" (i.e. of each other).

We should now draw the attention to the following points:

• The real and imaginary parts of a complex analytic function in polar form satisfy Laplace's equation in polar form (see Problem 20 of § 3.1). So, the above statements and symbolism (which suggest Cartesian form) are not restricted to Cartesian form.
• Harmonic conjugates may also be called "conjugate harmonic functions" or "conjugate harmonics".
• This "harmonic *conjugate*" should not be confused with "conjugate" of complex numbers and variables (see for instance § 1.8.8). The two are completely different.
• It is important to note "and" in the sentence above [i.e. "... are harmonic functions in a given domain D *and* $f = u(x, y) + iv(x, y)$ is analytic ..."] because not every two harmonic functions are harmonic conjugates. The reason is that two arbitrary harmonic functions may not form the real and imaginary parts of an analytic function. In other words, if u and v are two arbitrary harmonic functions then $f = u + iv$ is not necessarily analytic (i.e. f may or may not be analytic since u and v may or may not satisfy the Cauchy-Riemann equations). So in brief, every real and imaginary parts of an analytic function are harmonic (i.e. harmonic conjugates) but not every two harmonic functions are the real and imaginary parts of an analytic function (i.e. harmonic conjugates). Accordingly, harmonic conjugates are a proper subset of (paired) harmonic functions where each one in the pair has a special relation with the other.
• As we will see (refer to Problem 3), the harmonic conjugate of a given harmonic function is unique (within an arbitrary constant and within a ± sign difference considering the position of the harmonic

function as real or imaginary noting that if $f = u + iv$ then $if = iu - v$).[173]

Problems

1. Prove that if $f(z) = u(x, y) + iv(x, y)$ is analytic then u and v are harmonics (read the complete statement in the text).
   Answer: Since f is analytic then according to the Cauchy-Riemann equations (see Eq. 157) we have $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$. On taking the partial derivatives of these equations with respect to x and y (respectively) we get:[174]
   $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$
   Now, if we note that $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ and we add the last two equations side by side then we get $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ which is the 2D Laplace equation. This means that u is harmonic.
   Similarly, on taking the partial derivatives of these equations with respect to y and x (respectively) we get:
   $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}$$
   Now, if we note that $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ and we subtract the last two equations side by side then we get $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ which is the 2D Laplace equation. This means that v is also harmonic.
   So, both u and v are harmonic functions (as required).
2. Which of the following real functions is harmonic:
   (**a**) $u(x, y) = x^2 + x - 3y$. (**b**) $u(x, y) = x^3 y - xy^3 + xy + 6y$.
   (**c**) $u(x, y) = \log_e \sqrt{x^2 - y^2} \quad (x^2 > y^2)$. (**d**) $u(x, y) = x^3 - 3xy^2$.
   Answer:
   (**a**)
   $$\frac{\partial u}{\partial x} = 2x + 1 \qquad \frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial u}{\partial y} = -3 \qquad \frac{\partial^2 u}{\partial y^2} = 0$$
   So, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + 0 = 2 \neq 0$ and hence the function is not harmonic.
   (**b**)
   $$\frac{\partial u}{\partial x} = 3x^2 y - y^3 + y \qquad \frac{\partial^2 u}{\partial x^2} = 6xy \qquad \frac{\partial u}{\partial y} = x^3 - 3xy^2 + x + 6 \qquad \frac{\partial^2 u}{\partial y^2} = -6xy$$
   So, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6xy - 6xy = 0$ and hence the function is harmonic (over the entire complex plane).
   (**c**)
   $$\frac{\partial u}{\partial x} = \frac{x}{x^2 - y^2} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{-x^2 - y^2}{(x^2 - y^2)^2} \qquad \frac{\partial u}{\partial y} = \frac{-y}{x^2 - y^2} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{-x^2 - y^2}{(x^2 - y^2)^2}$$
   So, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{-x^2 - y^2}{(x^2 - y^2)^2} + \frac{-x^2 - y^2}{(x^2 - y^2)^2} = \frac{-2(x^2 + y^2)}{(x^2 - y^2)^2} \neq 0$ and hence the function is not harmonic.
   (**d**)
   $$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial^2 u}{\partial x^2} = 6x \qquad \frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial^2 u}{\partial y^2} = -6x$$
   So, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$ and hence the function is harmonic (over the entire complex plane).

[173] So, if we fix the sign then we should talk about *ordered* pair.

[174] The existence and continuity of partial derivatives of all orders should be guaranteed by the fact that analytic functions are infinitely differentiable (see Problem 6 of § 4.3).

3. Find the harmonic conjugates of the following harmonic functions and verify the results:
   (**a**) $x^2 - y^2$. (**b**) $3xy^2 - x^3 + 2x - 5$. (**c**) $3x^3y - x^2 - 3xy^3 + y^2 + x$.
   Answer:
   (**a**) Because $x^2 - y^2$ is harmonic we can assume that it is the real part u of an analytic function f and hence its harmonic conjugate should be the imaginary part v of f. Accordingly, u and v should satisfy the Cauchy-Riemann equations (as given by Eq. 157), that is:

$$\frac{\partial v}{\partial y} = +\frac{\partial u}{\partial x} = 2x \tag{165}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y \tag{166}$$

On integrating Eq. 165 (with respect to y) we get $v = 2xy + h(x)$ where h is a function of x. Now, if we partial-differentiate this expression of v with respect to x and compare it to Eq. 166 we get:

$$\begin{aligned} 2y + \frac{\partial h(x)}{\partial x} &= 2y \\ \frac{\partial h(x)}{\partial x} &= 0 \\ h(x) &= C \qquad (C \text{ is constant}) \end{aligned}$$

Accordingly, $v = 2xy + C$.
Verification: both u and v are harmonic because $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 + 0 = 0$. Moreover, because they are supposed to be the real and imaginary parts of an analytic function f (as required by being harmonic conjugates of each other) they should satisfy the Cauchy-Riemann equations (as given by Eq. 157), that is:

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

(**b**) Because $3xy^2 - x^3 + 2x - 5$ is harmonic we can assume that it is the real part u of an analytic function f and hence its harmonic conjugate should be the imaginary part v of f. Accordingly, u and v should satisfy the Cauchy-Riemann equations (as given by Eq. 157), that is:

$$\frac{\partial v}{\partial y} = +\frac{\partial u}{\partial x} = 3y^2 - 3x^2 + 2 \tag{167}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -6xy \tag{168}$$

On integrating Eq. 168 (with respect to x) we get $v = -3x^2y + h(y)$ where h is a function of y. Now, if we partial-differentiate this expression of v with respect to y and compare it to Eq. 167 we get:

$$\begin{aligned} -3x^2 + \frac{\partial h(y)}{\partial y} &= 3y^2 - 3x^2 + 2 \\ \frac{\partial h(y)}{\partial y} &= 3y^2 + 2 \\ h(y) &= y^3 + 2y + C \qquad (C \text{ is constant}) \end{aligned}$$

Accordingly, $v = -3x^2y + y^3 + 2y + C$.
Verification: both u and v are harmonic because $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -6y + 6y = 0$. Moreover, because they are supposed to be the real and imaginary parts of an analytic function f (as required by being harmonic conjugates of each other) they should satisfy the Cauchy-Riemann equations (as given by Eq. 157), that is:

$$\frac{\partial u}{\partial x} = 3y^2 - 3x^2 + 2 = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = 6xy = -\frac{\partial v}{\partial x}$$

(**c**) Because $3x^3y - x^2 - 3xy^3 + y^2 + x$ is harmonic we can assume that it is the imaginary part v of an analytic function f and hence its harmonic conjugate should be the real part u of f. Accordingly, u and v should satisfy the Cauchy-Riemann equations (as given by Eq. 157), that is:

$$\frac{\partial u}{\partial x} = +\frac{\partial v}{\partial y} = 3x^3 - 9xy^2 + 2y \tag{169}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -9x^2y + 2x + 3y^3 - 1 \tag{170}$$

On integrating Eq. 169 (with respect to x) we get $u = \frac{3}{4}x^4 - \frac{9}{2}x^2y^2 + 2xy + h(y)$ where h is a function of y. Now, if we partial-differentiate this expression of u with respect to y and compare it to Eq. 170 we get:

$$\begin{aligned} -9x^2y + 2x + \frac{\partial h(y)}{\partial y} &= -9x^2y + 2x + 3y^3 - 1 \\ \frac{\partial h(y)}{\partial y} &= 3y^3 - 1 \\ h(y) &= \frac{3}{4}y^4 - y + C \qquad (C \text{ is constant}) \end{aligned}$$

Accordingly, $u = \frac{3}{4}x^4 - \frac{9}{2}x^2y^2 + 2xy + \frac{3}{4}y^4 - y + C$.
Verification: both u and v are harmonic because $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(9x^2 - 9y^2\right) + \left(-9x^2 + 9y^2\right) = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = (18xy - 2) + (-18xy + 2) = 0$. Moreover, because they are supposed to be the real and imaginary parts of an analytic function f (as required by being harmonic conjugates of each other) they should satisfy the Cauchy-Riemann equations (as given by Eq. 157), that is:

$$\frac{\partial u}{\partial x} = 3x^3 - 9xy^2 + 2y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -9x^2y + 2x + 3y^3 - 1 = -\frac{\partial v}{\partial x}$$

4. Analyze Problem 3 and its results.
Answer: We note first that the question in Problem 3 is rather ambiguous because we are asked to find "the" harmonic conjugates without specifying the position of the given harmonic functions as the real or imaginary parts (i.e. u or v). Therefore, we made our choices rather arbitrarily (i.e. by choosing u in parts a and b and choosing v in part c). This is inline with the aforementioned $\pm$ sign difference. Accordingly, if we reverse our choices (i.e. by choosing v in parts a and b and choosing u in part c) then the harmonic conjugates will be (respectively):

$$u = -2xy + C \qquad u = 3x^2y - y^3 - 2y + C \qquad v = -\frac{3}{4}x^4 + \frac{9}{2}x^2y^2 - 2xy - \frac{3}{4}y^4 + y + C$$

5. Suggest a complex analysis approach for solving Laplace's equations in 2D.
Answer: Let have a Laplace's equation in 2D of the form $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$. From our previous investigations we know that there is a real function $\phi(x, y)$ $\big[$that is the real or imaginary part of a complex function $f(z)\big]$ which satisfies this equation and hence this $\phi(x, y)$ is the solution of our Laplace's equation.[175] So, what we need to do is to search for a complex function $f(z)$ whose real or imaginary part is ϕ (where we use in this search the given data and information about the problem).
6. Give some examples of known analytic functions and verify that their real and imaginary parts are harmonic.
Answer: Examples are:

[175] In fact, this may be understood from our previous investigations although it is not stated (or proved) explicitly. To be explicit we say: any real function that satisfies a 2D Laplace's equation in a simply-connected region is the real part or the imaginary part of a complex analytic function.

• $f(z) = z^2 - z + 1$ which is a (quadratic) polynomial and hence it is obviously analytic (see § 2.1; also see Problem 4 of § 3.1). Now:

$$f(z) = (x + iy)^2 - (x + iy) + 1 = x^2 - y^2 + i2xy - x - iy + 1 = \left(x^2 - y^2 - x + 1\right) + i\,(2xy - y)$$

and hence $u = x^2 - y^2 - x + 1$ and $v = 2xy - y$. Accordingly:

$$\frac{\partial u}{\partial x} = 2x - 1 \qquad \frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial u}{\partial y} = -2y \qquad \frac{\partial^2 u}{\partial y^2} = -2$$
$$\frac{\partial v}{\partial x} = 2y \qquad \frac{\partial^2 v}{\partial x^2} = 0 \qquad \frac{\partial v}{\partial y} = 2x - 1 \qquad \frac{\partial^2 v}{\partial y^2} = 0$$

As we see, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ and hence both u and v are harmonic.

• $f(z) = e^{-z}$ which is an exponential function and hence it is obviously analytic (see § 2.2; also see Problem 4 of § 3.1). Now:

$$f(z) = e^{-z} = e^{-x-iy} = e^{-x}\left(\cos[-y] + i\sin[-y]\right) = e^{-x}\cos y - ie^{-x}\sin y$$

and hence $u = e^{-x}\cos y$ and $v = -e^{-x}\sin y$. Accordingly:

$$\frac{\partial u}{\partial x} = -e^{-x}\cos y \qquad \frac{\partial^2 u}{\partial x^2} = +e^{-x}\cos y \qquad \frac{\partial u}{\partial y} = -e^{-x}\sin y \qquad \frac{\partial^2 u}{\partial y^2} = -e^{-x}\cos y$$
$$\frac{\partial v}{\partial x} = +e^{-x}\sin y \qquad \frac{\partial^2 v}{\partial x^2} = -e^{-x}\sin y \qquad \frac{\partial v}{\partial y} = -e^{-x}\cos y \qquad \frac{\partial^2 v}{\partial y^2} = +e^{-x}\sin y$$

As we see, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ and hence both u and v are harmonic.

• $f(z) = \cos z$ which is a trigonometric cosine function and hence it is obviously analytic (see § 2.3; also see Problem 4 of § 3.1). Now, according to Eq. 137 we have:

$$f(z) = \cos z = \cos x\,\cosh y - i\sin x\,\sinh y$$

and hence $u = \cos x\,\cosh y$ and $v = -\sin x\,\sinh y$. Accordingly:

$$\frac{\partial u}{\partial x} = -\sin x\,\cosh y \qquad \frac{\partial^2 u}{\partial x^2} = -\cos x\,\cosh y \qquad \frac{\partial u}{\partial y} = +\cos x\,\sinh y \qquad \frac{\partial^2 u}{\partial y^2} = +\cos x\,\cosh y$$
$$\frac{\partial v}{\partial x} = -\cos x\,\sinh y \qquad \frac{\partial^2 v}{\partial x^2} = +\sin x\,\sinh y \qquad \frac{\partial v}{\partial y} = -\sin x\,\cosh y \qquad \frac{\partial^2 v}{\partial y^2} = -\sin x\,\sinh y$$

As we see, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ and hence both u and v are harmonic.

• $f(z) = \sinh z$ which is a hyperbolic sine function and hence it is obviously analytic (see § 2.3; also see Problem 4 of § 3.1). Now, according to Eq. 142 we have:

$$f(z) = \sinh z = \sinh x\,\cos y + i\cosh x\,\sin y$$

and hence $u = \sinh x\,\cos y$ and $v = \cosh x\,\sin y$. Accordingly:

$$\frac{\partial u}{\partial x} = \cosh x\,\cos y \qquad \frac{\partial^2 u}{\partial x^2} = \sinh x\,\cos y \qquad \frac{\partial u}{\partial y} = -\sinh x\,\sin y \qquad \frac{\partial^2 u}{\partial y^2} = -\sinh x\,\cos y$$
$$\frac{\partial v}{\partial x} = \sinh x\,\sin y \qquad \frac{\partial^2 v}{\partial x^2} = \cosh x\,\sin y \qquad \frac{\partial v}{\partial y} = +\cosh x\,\cos y \qquad \frac{\partial^2 v}{\partial y^2} = -\cosh x\,\sin y$$

As we see, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ and hence both u and v are harmonic.

7. Justify briefly why the following functions are harmonic (without obtaining their partial derivatives and Laplace's equation):
   (**a**) $f = e^{-y}\cos x$. (**b**) $f = \cos x \sinh y$. (**c**) $f = x^2 - y^2$.
   (**d**) $f = \sinh x \sin y$. (**e**) $f = xe^{-x}\cos y + ye^{-x}\sin y$. (**f**) $f = \cos(\pi x)\cosh(\pi y)$.
   Answer:
   (**a**) This is the real part of $e^{iz} = e^{-y}\cos x + ie^{-y}\sin x$ which is analytic (because it is a composition of analytic functions) and hence f is harmonic.
   (**b**) This is the imaginary part of $\sin z = \sin x \cosh y + i\cos x \sinh y$ (see Eq. 138) which is analytic (see § 2.3; also see Problem 4 of § 3.1) and hence f is harmonic.
   (**c**) This is the real part of $z^2 = x^2 - y^2 + i2xy$ which is analytic (because it is a polynomial function) and hence f is harmonic.
   (**d**) This is the imaginary part of $\cosh z = \cosh x \cos y + i\sinh x \sin y$ (see Eq. 141) which is analytic (see § 2.3; also see Problem 4 of § 3.1) and hence f is harmonic.
   (**e**) This is the real part of $ze^{-z} = (xe^{-x}\cos y + ye^{-x}\sin y) + i(ye^{-x}\cos y - xe^{-x}\sin y)$ which is analytic (because it is a product of a polynomial function and a composition of exponential function both of which are analytic) and hence f is harmonic.
   (**f**) This is the real part of $\cos(\pi z) = \cos(\pi x)\cosh(\pi y) - i\sin(\pi x)\sinh(\pi y)$ (see Eq. 137) which is analytic (because it is a composition of analytic functions) and hence f is harmonic.
8. Given that $f(x, y)$ is a real C^2 harmonic function,[176] which of the following complex functions is analytic (where subscript means partial derivative with respect to the variable of subscript):
   (**a**) $f_x - if_y$. (**b**) $f_x + if_y$. (**c**) $f_y + if_x$. (**d**) $(f_x - if_y)^2$.
   Answer:
   (**a**) From the Cauchy-Riemann equations we have:
   $$\frac{\partial u}{\partial x} = f_{xx} \stackrel{?}{=} -f_{yy} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = f_{xy} \stackrel{?}{=} f_{yx} = -\frac{\partial v}{\partial x}$$
   Now, since f is harmonic then it satisfies Laplace's equation $f_{xx} + f_{yy} = 0$ and hence $f_{xx} = -f_{yy}$, i.e. the first Cauchy-Riemann equation is satisfied. Also, since f is a C^2 function then its mixed derivatives are equal (i.e. $f_{xy} = f_{yx}$) and hence the second Cauchy-Riemann equation is also satisfied. Accordingly, $f_x - if_y$ is analytic.
   (**b**) From the Cauchy-Riemann equations we have:
   $$\frac{\partial u}{\partial x} = f_{xx} \stackrel{?}{=} f_{yy} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = f_{xy} \stackrel{?}{=} -f_{yx} = -\frac{\partial v}{\partial x}$$
   Referring to the analysis of part (a), $f_x + if_y$ violates both Cauchy-Riemann equations and hence it is not analytic.
   (**c**) From the Cauchy-Riemann equations we have:
   $$\frac{\partial u}{\partial x} = f_{yx} \stackrel{?}{=} f_{xy} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = f_{yy} \stackrel{?}{=} -f_{xx} = -\frac{\partial v}{\partial x}$$
   Referring to the analysis of part (a), $f_y + if_x$ satisfies both Cauchy-Riemann equations and hence it is analytic.
   (**d**) We have $(f_x - if_y)^2 = (f_x - if_y) \times (f_x - if_y)$ and because $f_x - if_y$ is analytic (according to the result of part a) then the product is analytic, i.e. $(f_x - if_y)^2$ is analytic. However, let verify this by using the Cauchy-Riemann equations (as we did in the previous parts). Now, $(f_x - if_y)^2 = \left(f_x^2 - f_y^2\right) - i2f_xf_y = u + iv$ and hence from the Cauchy-Riemann equations we have:
   $$\frac{\partial u}{\partial x} = 2f_xf_{xx} - 2f_yf_{yx} \quad \stackrel{?}{=} \quad -2f_{xy}f_y - 2f_xf_{yy} = +\frac{\partial v}{\partial y}$$

[176] Being C^2 function means it possesses continuous second order partial derivatives, i.e. $f_{xx}, f_{yy}, f_{xy}, f_{yx}$.

$$\text{and} \qquad \frac{\partial u}{\partial y} = 2f_x f_{xy} - 2f_y f_{yy} \quad \stackrel{?}{=} \quad +2f_{xx}f_y + 2f_x f_{yx} = -\frac{\partial v}{\partial x}$$

Referring to the analysis of part (a), $(f_x - if_y)^2$ satisfies both Cauchy-Riemann equations and hence it is analytic.

9. Give an example of two harmonic functions which are not harmonic conjugates.
Answer: The functions $f_1 = x^3y - xy^3 + xy + 6y$ and $f_2 = x^3 - 3xy^2$ are harmonic (see parts b and d of Problem 2) but they are not harmonic conjugates because neither $f_1 + if_2$ or $f_2 + if_1$ is analytic (as can be easily concluded from the failure of these complex functions to satisfy the Cauchy-Riemann equations).

Chapter 4
Important Theorems

We dedicate this chapter to the investigation of very important theorems in complex analysis. In fact, a very large part (and possibly the largest part) of complex analysis is based on these theorems and their applications. These theorems do not only facilitate the application and employment of complex analysis and enrich its techniques, but they also come with some novel results and methods that cannot be obtained (at least easily) without them.

4.1 The Fundamental Theorem of the Calculus of Complex Variables

This theorem may be seen as an extension or generalization of the fundamental theorem of the calculus of real variables to include complex variables. The theorem states: if $f(z)$ is an analytic function in a simply-connected region R of the complex plane containing a curve C that connects two points z_1 and z_2 and $F(z)$ is an antiderivative (or primitive) of f (i.e. $dF/dz = f$) in R then the integral of f from z_1 to z_2 along C is given by:[177]

$$\int_C f\,dz = \int_{z_1}^{z_2} f\,dz = \big[F(z)\big]_{z_1}^{z_2} = F(z_2) - F(z_1) \tag{171}$$

In fact, this theorem has been anticipated earlier during our investigation of contour integration (see Problem 4 of § 3.2).

The significance and power of the fundamental theorem lies largely in the fact that it severs the connection between the value of the integral and the curve C and hence it makes the integral independent of the path. In other words, the value of the integral is a function of the end points z_1 and z_2 but not of the curve C that connects z_1 and z_2. In fact, this can be seen clearly from the notation in the last three steps of Eq. 171 (particularly in the last two steps) where there is an indication to z_1 and z_2 but not to C. As indicated above, this theorem may be seen as an extension to its real counterpart (considering for instance that the proof of the complex version may employ the real version; see Problem 1).

Problems

1. Discuss the proof of the fundamental theorem of the calculus of complex variables.
   Answer: The existing proofs are essentially demonstrations of the logic of this theorem more than proofs. The following is an example of one of these demonstrations (where we assume the validity of the fundamental theorem of the calculus of *real* variables). So, let parameterize C as $z(t)$ where $t_1 \le t \le t_2$ is a continuous real variable with t_1 and t_2 corresponding to z_1 and z_2, i.e. $z_1 = z(t_1)$ and $z_2 = z(t_2)$. Accordingly, $f(z)$ becomes $f(z(t))$ and $F(z)$ becomes $F(z(t))$ with $dz = \frac{dz}{dt}dt$ and hence we have:

$$\begin{aligned}
\int_C f(z)\,dz &= \int_{t_1}^{t_2} f(z(t))\,\frac{dz}{dt}\,dt \\
&= \int_{t_1}^{t_2} \frac{dF(z(t))}{dz}\,\frac{dz}{dt}\,dt \\
&= \int_{t_1}^{t_2} \frac{dF(z(t))}{dt}\,dt && \text{(chain rule)} \\
&= \big[F(z(t))\big]_{t_1}^{t_2} && \text{(fundamental theorem of calculus of real variables)}
\end{aligned}$$

[177] The intermediate steps in this equation are no more than alternative notations with no more substance and hence they are given for additional clarity.

$$= F(z(t_2)) - F(z(t_1))$$
$$= F(z_2) - F(z_1)$$

We note that a split into real and imaginary parts may be considered in some of the intermediate steps (noting that this will not add any actual substance to the above demonstration although it may clarify the reliance on the calculus of real variables).

2. Outline the significance of the generic equation $\int_C f\,dz = F(z_2) - F(z_1)$.
Answer: We note the following:
• This equation reflects the path-independence nature of this integral because on the left we have a reference to a path (i.e. C) while on the right we have no reference to the path (apart from its end points z_1 and z_2).
• This equation provides the link between the paradigm of line (or path or contour) integral and the paradigm of definite integral (where the paradigm of analyticity provides the underlying justification for this link).
3. In § 3.2 we outlined three approaches for evaluating contour integrals: parameterization, using a real and imaginary variables approach and ordinary integration. Investigate this matter in the light of the fundamental theorem of the calculus of complex variables and the issue of path-dependence and path-independence.
Answer: Evaluating a line integral analytically as it stands (i.e. as a line integral) is impossible. Hence, we need to convert the line integral to a definite integral where it can be evaluated analytically. Now, if the line integral is path-dependent (i.e. when the integrand is not analytic) then we have no choice but to embed the information of the path in the integral through parameterization or using a real and imaginary variables approach and this embedding leads to a "path-independent" definite integral that can be evaluated analytically (at least in principle). On the other hand, if the line integral is path-independent (i.e. when the integrand is analytic) then we have another (and usually better) choice that is using ordinary integration which (thanks to the fundamental theorem of the calculus of complex variables) exploits path-independence immediately and directly to evaluate the integral analytically (noting that we still have the choice of using parameterization or real and imaginary variables approach although they usually do not offer any advantage).
4. Outline the link between the path-independence of a line integral and the analyticity of its integrand.
Answer: Noting that complex functions are defined on regions in the complex plane, the path-independence of a line integral in a given (simply-connected) region R requires the integrand f to be analytic over the entire R (i.e. R does not contain any singularity of f). This means that f is analytic over the entirety of any curve in R. So, if f is singular at a given point in R (i.e. there are curves in R on which f is not analytic over their entirety) then the line integral is not path-independent inside R in general (although f can still be path-independent on a sub-region of R, i.e. if the sub-region does not contain a singularity of f as illustrated in Figure 24). In fact, this issue will lead us to the subject of the next section, i.e. Cauchy's theorem (see § 4.2).
Note: the above answer (including the explanations in Figure 24) should indicate the local nature of path-independence, i.e. a given contour integral can be path-independent in a given region but not in another region. This obviously is a consequence of the local nature of analyticity, i.e. a function can be analytic over a given region but not over another region (due to having a singularity in the latter).
5. Referring to Figure 25, let the complex function $f(z)$ be analytic over the entire region R except at the points s_1, s_2, s_3. On which of the closed curves C_k $(k = 1, \cdots, 6)$ the line integral of f is zero and on which it is not (i.e. in general).
Answer: Based on the results of Problem 4, the line integral is zero on the curves C_1, C_5 and C_7 since they do not enclose singularities, and it is not zero (i.e. in general) on the other curves since they enclose singularities.

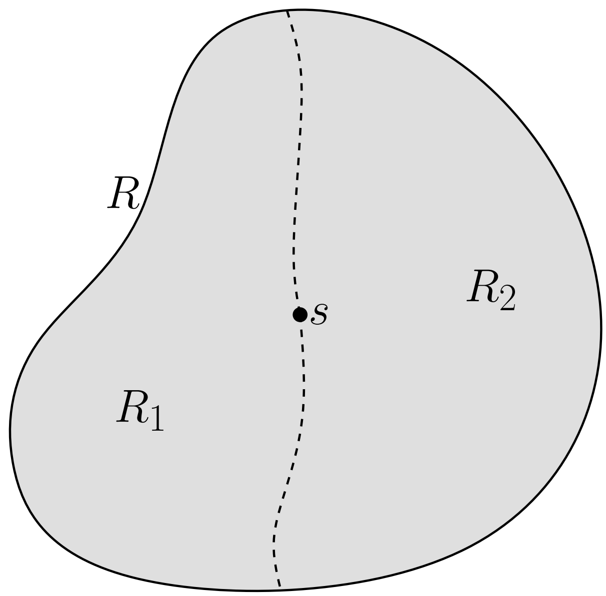

Figure 24: Graphic illustration of the analyticity of a function f over a region and its link to the path-independence of a line integral of f on a curve inside that region where if f is analytic over the entire region (like R_1 and R_2 which do not contain any singularity of f) then the line integral is path-independent inside that region, while if f is not analytic over the entire region (like $R = R_1 \cup R_2$ which contains a singularity s of f) then the line integral is not path-independent inside that region in general. Accordingly, if a closed path C inside R does not enclose s then (as a result of path-independence) the line integral of f over C is zero while if C does enclose s then (as a result of path-dependence) the line integral of f over C is not zero in general (and in fact this is the essence of Cauchy's theorem which is investigated in § 4.2). See Problem 4 of § 4.1.

4.2 Cauchy's Theorem

Cauchy's theorem (which is one of the pillars of complex analysis and possibly its main pillar) states that if $w = f(z)$ is a function analytic over a piecewise-smooth and closed curve C and the entire simply-connected region surrounded by C in the complex plane then the contour integral of w over C is zero, i.e. $\oint_C w\,dz = 0$.[178] In fact, this theorem (which has a rather simple formal proof; see Problem 1) has been anticipated earlier during our investigation of contour integration (see Problem 4 of § 3.2 as well as § 4.1) because the independence of path leads to the vanishing of the contour integral over the closed curve.[179] To be more clear, C can be split into two sub-curves C_1 and C_2 connecting two given points z_1 and z_2 and hence the value of the integral from z_2 to z_1 along C_2 will cancel the value of the integral from z_1 to z_2 along C_1 (noting that the value of the two integrals from z_1 to z_2 is the same due to the path independence and the value of a contour integral in one direction is the negative of its value in the opposite direction; see Problem 5 of § 1.10).

We should now draw the attention to the following useful remarks:

• As we will see in the upcoming Problems and sections, this theorem (which underlies the entire complex analysis and in reality it lies behind many of its characteristic features) has numerous applications in complex analysis (and even in real analysis). For example, it is very useful in evaluating many integrals

[178] Extension of this theorem to multiply-connected regions will be investigated later (see § 4.2.1). We should also note that since this integral (and its alike) is zero the sense of tracking the curve is irrelevant (noting that it is oriented) although we generally keep the anticlockwise sense to emphasize our convention and to avoid any potential ambiguity.

[179] Depending on our start, the above reasoning may be reversed and hence path independence may be obtained as a consequence of Cauchy's theorem. Anyway, in this context path independence and Cauchy's theorem are equivalent. In fact, they are also equivalent to having a primitive (or antiderivative) of f.

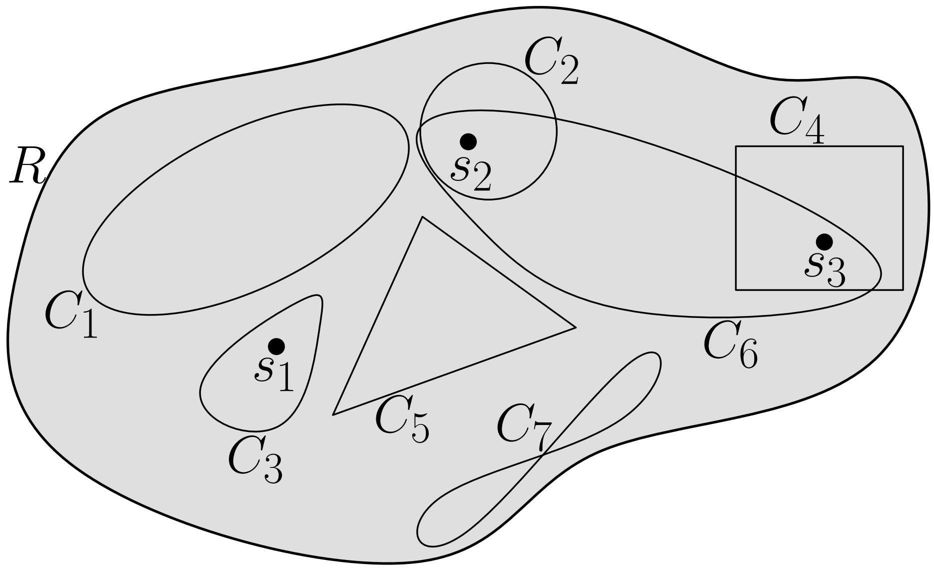

Figure 25: Graphic illustration of the setting of Problem 5 of § 4.1. We note that we did not identify the orientation of these curves (i.e. clockwise or anticlockwise) because it is irrelevant to our purpose.

(real as well as complex) as it facilitates the calculations and provides some short cuts. It also provides the theoretical basis and motive for many theorems and theoretical results of complex analysis.

• There are two main proofs of this theorem: one is based on using Green's theorem (see Problem 1) and the other is not. While the former proof requires the real and imaginary parts of f $\big[$i.e. $u(x, y)$ and $v(x, y)\big]$ to have continuous first order partial derivatives (since it is based on Green's theorem; see the note of Problem 1), the latter proof (which is attributed to Goursat) does not. Hence, the Goursat proof establishes a more general version of this theorem. Therefore, Cauchy's theorem is commonly known as the Cauchy-Goursat theorem (thanks to this generalization).[180]

• Cauchy's theorem (or Cauchy-Goursat theorem) is not the same as Cauchy's integral formula theorem (which will be investigated in § 4.3) and hence the reader should be aware of this to avoid confusion by the similarity of their names.[181] In fact, to distinguish the two theorems very clearly we call the former "Cauchy's theorem" and call the latter "integral formula theorem" (without "Cauchy").

• Cauchy's theorem states that $\oint_C w\,dz = 0$ if the stated conditions are satisfied, and this does not mean $\oint_C w\,dz \neq 0$ when these conditions are not satisfied. In other words, we can have $\oint_C w\,dz = 0$ even when the conditions of the theorem are not satisfied (and hence the theorem is not applicable). So, the above conditions are sufficient but not necessary. For example, if C is the origin-centered unit circle then $\oint_C \frac{dz}{z^3} = 0$ although the integrand is not analytic at $z = 0$.

Problems

1. Prove Cauchy's theorem using Green's theorem.
   Answer:[182] From Eq. 80 (as applied to the contour integral over the closed curve C) we have:

$$\oint_C w\,dz = \oint_C (u\,dx - v\,dy) + i\oint_C (u\,dy + v\,dx)$$

[180] It should be noted that some authors use "Cauchy's theorem" for the version with the continuity condition, and "Cauchy-Goursat theorem" for the version without the continuity condition (as if they are two theorems).

[181] Actually, there are many differences in the labeling and naming of these theorems and hence the reader should be vigilant and fully aware of the terminology of each author.

[182] This proof is provided for the sake of more formality. Otherwise, the independence of path (as required by the fundamental theorem of the calculus of complex variables) should be sufficient for establishing Cauchy's theorem.

On applying Green's theorem (see the upcoming note) on the real part integral we get:

$$\oint_C (u\,dx - v\,dy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

where R is the region enclosed by C. Now, since w is analytic then it should satisfy the second of the Cauchy-Riemann equations (i.e. $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$) which leads to $-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ and hence the real part integral is zero, i.e. $\oint_C (u\,dx - v\,dy) = 0$.
Likewise, on applying Green's theorem on the imaginary part integral we get:

$$\oint_C (u\,dy + v\,dx) = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Now, since w is analytic then it should satisfy the first of the Cauchy-Riemann equations (i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$) which leads to $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$ and hence the imaginary part integral is zero, i.e. $\oint_C (u\,dy + v\,dx) = 0$.
As both the real and imaginary parts of this contour integral are zero then the contour integral is zero, as claimed by Cauchy's theorem.
Note: Green's theorem (which is commonly used in real analysis for converting line integrals to area integrals and vice versa) essentially states that if C is a piecewise-smooth closed simple planar positively-oriented curve enclosing a simply-connected region R in the xy plane, and P and Q are differentiable functions of x and y defined on C and R then:[183]

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

The reader is referred to the literature of multi-variable calculus for more details about Green's theorem. We should also note that Green's theorem is a special case of Stokes theorem (and hence the above proof is ultimately based on Stokes theorem).

2. Let a given function $f(z)$ be defined on a region R in the z plane and assume f to be analytic in R (except possibly at some isolated points inside R). Moreover, let z_1 and z_2 be two (distinct) points inside R and C_1 and C_2 are two (distinct and non-intersecting) curves inside R both of which connect z_1 to z_2. Give criteria for the following:
(**a**) The validity of the equation $\int_{C_1} f\,dz = \int_{C_2} f\,dz$.
(**b**) The path-independence of the contour integral $\int_C f\,dz$ for any curve C that connects z_1 to z_2 and it is confined in the region R_1 between C_1 and C_2.
Answer:
(**a**) From Cauchy's theorem, $\int_{C_1} f\,dz = \int_{C_2} f\,dz$ if f has no singularity on C_1 and C_2 and in the region R_1 between them. This is because in this case the contour integral of f on the closed curve made of one of these curves and the opposite of the other curve (i.e. $C_1 \cup C_2^-$ or $C_1^- \cup C_2$ where the minus sign in the superscript indicates traversing the curve in the opposite direction, i.e. from z_2 to z_1) is zero and hence the two line integrals (i.e. $\int_{C_1} f\,dz$ and $\int_{C_2} f\,dz$) should be equal, that is:

$$\int_{C_1 \cup C_2^-} f\,dz = \int_{C_1} f\,dz + \int_{C_2^-} f\,dz = \int_{C_1} f\,dz - \int_{C_2} f\,dz = 0 \qquad \to \qquad \int_{C_1} f\,dz = \int_{C_2} f\,dz$$

(**b**) By a similar argument to that of part (a), the contour integral $\int_C f\,dz$ (within the above specifications) is path-independent if f has no singularity in R_1. This is because in this case the contour integral of f over any loop passing through z_1 and z_2 and confined inside R_1 is zero and hence the contour integrals over the two curves that connect z_1 to z_2 and form the loop must be equal, i.e. the value of the contour integral over any C (within the above specifications) is the same which means this

[183] The functions P and Q are required (according to Green's theorem) to have continuous first order partial derivatives. We should also note that we indicated the sense of orientation of C (corresponding to the above formulation) by the integral symbol. Also, Green's theorem can be extended to multiply-connected region.

contour integral is independent of the specific path inside R_1.[184] So we can say in brief, the integral of f is path-independent in a given region in the complex plane if f is analytic in the entire region.
Note: the restriction "non-intersecting" in the question may be relaxed by noting that if C_1 and C_2 intersect or touch each other then we can consider each individual loop separately (and this should apply even if the two curves share a common segment or segments).

3. Verify Cauchy's theorem for the following contour integrals using a parameterization approach (see § 3.2):
   (**a**) $\oint_C 3z\,dz$ where C is the circle $|z-1+i|=4$.
   (**b**) $\oint_C iz\,dz$ where C is the ellipse $\frac{x^2}{4}+y^2=1$.
   (**c**) $\oint_C \cos z\,dz$ where C is the triangle with vertices at $z_1=0$, $z_2=1$ and $z_3=i2$.
   Answer:
   (**a**) Here, we have $w=3z$ which is a linear polynomial and hence it is analytic over C and the region enclosed by C. So, according to Cauchy's theorem we expect this contour integral to be zero. Now, in part (a) of Problem 1 of § 3.2 we found that the value of this contour integral over the upper half C_u of this circle is $-24+i24$. So, all we need to do here is to evaluate this integral over the lower half of this circle and add the two values to find out if they add up to zero (as required by Cauchy's theorem) or not. Now, if we follow a similar method to that of Problem 1 of § 3.2 then the value of this contour integral over the lower half C_l of this circle is:

$$\int_{C_l} 3z\,dz = 12\Big[e^{i\theta} - ie^{i\theta} + 2e^{i2\theta}\Big]_{\pi}^{2\pi} = 12\Big[1-i+2\Big] - 12\Big[-1+i+2\Big] = 24 - i24$$

   Hence:

$$\oint_C 3z\,dz = \int_{C_u} 3z\,dz + \int_{C_l} 3z\,dz = (-24+i24) + (24-i24) = 0$$

   as required by Cauchy's theorem. Hence, Cauchy's theorem is verified for this contour integral.
   Note: a simpler approach for solving this part of the Problem is to evaluate the contour integral in one go over the entire curve as a single piece, i.e.

$$\oint_C 3z\,dz = 12\Big[e^{i\theta} - ie^{i\theta} + 2e^{i2\theta}\Big]_{0}^{2\pi} = 12\Big[1-i+2\Big] - 12\Big[1-i+2\Big] = 0$$

   However, for the sake of demonstration (to show how the values of the integrals cancel each other over different parts of the curve), we split the curve into two parts and added the values over the two parts.
   (**b**) Here, we have $w=iz$ which is a linear polynomial and hence it is analytic over C and the region enclosed by C. So, according to Cauchy's theorem we expect this contour integral to be zero. Now, in part (b) of Problem 1 of § 3.2 we found that the value of this contour integral over the quarter of this ellipse C_1 in the first quadrant is $-i\frac{5}{2}$. So, all we need to do here is to evaluate this integral over the other three quarters of the ellipse and add the two values to find out if they add up to zero (as required by Cauchy's theorem) or not. Now, if we follow a similar method to that of Problem 1 of § 3.2 then the value of this contour integral over the other three quarters $C_2 \cup C_3 \cup C_4$ of this ellipse is:

$$\int_{C_2\cup C_3\cup C_4} iz\,dz = \Big[-\sin 2\theta + i\frac{5}{2}\cos^2\theta\Big]_{\frac{\pi}{2}}^{2\pi} = \Big[-0+i\frac{5}{2}\Big] - \big[-0+i0\big] = i\frac{5}{2}$$

   Hence:

$$\oint_C iz\,dz = \int_{C_1} iz\,dz + \int_{C_2\cup C_3\cup C_4} iz\,dz = -i\frac{5}{2} + i\frac{5}{2} = 0$$

[184] In fact, the last bit of this argument may be improved by choosing a specific (preferably boundary) curve (say C_1 with f being analytic over the entire C_1) as one part of any loop considered in this argument.

as required by Cauchy's theorem. Hence, Cauchy's theorem is verified for this contour integral.
Note: as in the note of part (a) of this Problem, we can evaluate the contour integral in one go over the entire curve as a single piece, i.e.

$$\oint_C iz\,dz = \left[-\sin 2\theta + i\frac{5}{2}\cos^2\theta\right]_0^{2\pi} = \left[-0 + i\frac{5}{2}\right] - \left[-0 + i\frac{5}{2}\right] = 0$$

(**c**) Here, we have $w = \cos z$ which is a trigonometric cosine and hence it is analytic over C and the region enclosed by C. So, according to Cauchy's theorem we expect this contour integral to be zero. Now, the triangle is made of 3 straight line segments that can be parameterized as follows:
C_1 from z_1 to z_2: $z = t$ $(0 \le t \le 1)$ and hence $dz = dt$.
C_2 from z_2 to z_3: $z = 1 - t + i2t$ $(0 \le t \le 1)$ and hence $dz = -dt + i2dt$.
C_3 from z_3 to z_1: $z = i2 - i2t$ $(0 \le t \le 1)$ and hence $dz = -i2dt$.
Therefore, we have:

$$\begin{aligned}
\oint_C \cos z\,dz &= \int_{C_1} \cos z\,dz + \int_{C_2} \cos z\,dz + \int_{C_3} \cos z\,dz \\
&= \int_0^1 \cos t\,dt + \int_0^1 \cos(1 - t + i2t)\,(-dt + i2dt) + \int_0^1 \cos(i2 - i2t)\,(-i2dt) \\
&= \int_0^1 \Big[\cos t + (-1 + i2)\cos(1 - t + i2t) - i2\cos(i2 - i2t)\Big]\,dt \\
&= \Big[\sin t + \sin(1 - t + i2t) + \sin(i2 - i2t)\Big]_0^1 \\
&= \Big[\sin 1 + \sin(i2) + \sin 0\Big] - \Big[\sin 0 + \sin 1 + \sin(i2)\Big] = 0
\end{aligned}$$

as required by Cauchy's theorem. Hence, Cauchy's theorem is verified for this contour integral.

4. Re-solve Problem 3 using this time a real and imaginary variables approach (see § 3.2).[185]
Answer:
(**a**) We have $w = 3z$ which is analytic over C and the region enclosed by C and hence we expect the integral to be zero. Now, the circle is made of 2 semi-circles: the upper half C_u from $z_1 = 5 - i$ to $z_2 = -3 - i$ and the lower half C_l from z_2 to z_1. In part (a) of Problem 2 of § 3.2 we found that the value of this contour integral over C_u is $-24 + i24$. So, all we need to do here is to evaluate this integral over C_l and add the two values to find out if they add up to zero (as required by Cauchy's theorem) or not. Now, if we follow a similar method to that of part (a) of Problem 2 of § 3.2 then the curve C_l can be represented as $y = -\sqrt{16 - (x-1)^2} - 1$ (where $-3 \le x \le 5$) with $dy = \frac{(x-1)}{\sqrt{16-(x-1)^2}}dx$. We also have:

$$w = u + iv = 3z = 3x + i3y = 3x + i\left(-3\sqrt{16 - (x-1)^2} - 3\right)$$

So, from Eq. 80 we have:

$$\begin{aligned}
\int_{C_l} 3z\,dz &= \int_{C_l} (u\,dx - v\,dy) + i\int_{C_l} (u\,dy + v\,dx) \\
&= \int_{-3}^{5} \left[3x\,dx - \left(-3\sqrt{16 - (x-1)^2} - 3\right)\frac{(x-1)}{\sqrt{16 - (x-1)^2}}dx\right] + \\
&\qquad i\int_{-3}^{5} \left[3x\frac{(x-1)}{\sqrt{16 - (x-1)^2}}dx + \left(-3\sqrt{16 - (x-1)^2} - 3\right)dx\right] \\
&= \int_{-3}^{5} \left(6x - 3 + \frac{3(x-1)}{\sqrt{16 - (x-1)^2}}\right)dx +
\end{aligned}$$

[185] In this Problem, we use this approach (which is rather messy) for the purpose of demonstration and practice.

$$\begin{aligned}
& i\int_{-3}^{5}\left(\frac{3x^2-3x}{\sqrt{16-(x-1)^2}}-3\sqrt{16-(x-1)^2}-3\right)dx \\
= & \left[\left(3x^2-3x-3\sqrt{16-(x-1)^2}\right)+i\left(-3x\sqrt{16-(x-1)^2}-3x\right)\right]_{-3}^{5} \\
= & \left[60-i15\right]-\left[36+i9\right]=24-i24
\end{aligned}$$

Hence:

$$\oint_C 3z\,dz=\int_{C_u}3z\,dz+\int_{C_l}3z\,dz=(-24+i24)+(24-i24)=0$$

as required by Cauchy's theorem. Therefore, Cauchy's theorem is verified for this contour integral.
(**b**) We have $w=iz$ which is analytic over C and the region enclosed by C and hence we expect the integral to be zero. Now, the ellipse is made of 2 semi-ellipses: the upper half C_u from $z_1=2$ to $z_2=-2$ and the lower half C_l from z_2 to z_1. Now, if we follow a similar method to that of part (b) of Problem 2 of § 3.2 then C_u can be represented as $y=\sqrt{1-(x/2)^2}$ (where $2\ge x\ge -2$) with $dy=\frac{-x}{4\sqrt{1-(x/2)^2}}dx$ while C_l can be represented as $y=-\sqrt{1-(x/2)^2}$ (where $-2\le x\le 2$) with $dy=\frac{x}{4\sqrt{1-(x/2)^2}}dx$. We also have:

$$\begin{aligned}
w &= u+iv=iz=-y+ix=-\sqrt{1-(x/2)^2}+ix && \text{(for } C_u\text{)}\\
w &= u+iv=iz=-y+ix=+\sqrt{1-(x/2)^2}+ix && \text{(for } C_l\text{)}
\end{aligned}$$

So, from Eq. 80 we have:

$$\begin{aligned}
\oint_C iz\,dz &= \int_{C_u}iz\,dz+\int_{C_l}iz\,dz\\
&= \int_{C_u}(u\,dx-v\,dy)+i\int_{C_u}(u\,dy+v\,dx)+\int_{C_l}(u\,dx-v\,dy)+i\int_{C_l}(u\,dy+v\,dx)\\
&= \int_2^{-2}\left[-\sqrt{1-(x/2)^2}dx-x\,\frac{-x}{4\sqrt{1-(x/2)^2}}dx\right]+\\
&\quad i\int_2^{-2}\left[-\sqrt{1-(x/2)^2}\frac{-x}{4\sqrt{1-(x/2)^2}}dx+x\,dx\right]+\\
&\quad \int_{-2}^{2}\left[\sqrt{1-(x/2)^2}dx-x\,\frac{x}{4\sqrt{1-(x/2)^2}}dx\right]+\\
&\quad i\int_{-2}^{2}\left[\sqrt{1-(x/2)^2}\frac{x}{4\sqrt{1-(x/2)^2}}dx+x\,dx\right]\\
&= \int_2^{-2}\left[-\sqrt{1-(x/2)^2}+\frac{x^2}{4\sqrt{1-(x/2)^2}}\right]dx+i\int_2^{-2}\frac{5x}{4}\,dx+\\
&\quad \int_{-2}^{2}\left[\sqrt{1-(x/2)^2}-\frac{x^2}{4\sqrt{1-(x/2)^2}}\right]dx+i\int_{-2}^{2}\frac{5x}{4}\,dx\\
&= \left[\left(-x\sqrt{1-(x/2)^2}\right)+i\left(\frac{5x^2}{8}\right)\right]_2^{-2}+\left[\left(x\sqrt{1-(x/2)^2}\right)+i\left(\frac{5x^2}{8}\right)\right]_{-2}^{2}\\
&= \left[0+i0\right]+\left[0+i0\right]=0
\end{aligned}$$

as required by Cauchy's theorem. Hence, Cauchy's theorem is verified for this contour integral.
(**c**) We have $w=\cos z$ which is analytic over C and the region enclosed by C and hence we expect the

integral to be zero. Now, from Eq. 137 we have:

$$w = u + iv = \cos z = \cos x \,\cosh y - i \sin x \,\sinh y$$

The triangle is made of 3 straight line segments that can be represented as follows:
C_1 from z_1 to z_2: $y = 0$ and $0 \le x \le 1$ and hence $dy = 0$.
C_2 from z_2 to z_3: $y = -2x + 2$ and $1 \ge x \ge 0$ and hence $dy = -2dx$.
C_3 from z_3 to z_1: $x = 0$ and $(2 \ge y \ge 0)$ and hence $dx = 0$.
Therefore, from Eq. 80 we have:

$$\begin{aligned}
\oint_C \cos z \, dz &= \oint_C (u\,dx - v\,dy) + i \oint_C (u\,dy + v\,dx) \\
&= \int_{C_1} (u\,dx - v\,dy) + \int_{C_2} (u\,dx - v\,dy) + \int_{C_3} (u\,dx - v\,dy) + \\
&\quad\; i \int_{C_1} (u\,dy + v\,dx) + i \int_{C_2} (u\,dy + v\,dx) + i \int_{C_3} (u\,dy + v\,dx) \\
&= \int_{x=0}^{x=1} \Big[\cos x \,\cosh 0 \, dx + (\sin x \,\sinh 0) \times 0 \Big] + \\
&\quad\; \int_{x=1}^{x=0} \Big[\cos x \,\cosh(-2x+2)\, dx + \sin x \,\sinh(-2x+2)\,(-2dx) \Big] + \\
&\quad\; \int_{y=2}^{y=0} \Big[(\cos 0 \,\cosh y) \times 0 + \sin 0 \,\sinh y \, dy \Big] + \\
&\quad\; i \int_{x=0}^{x=1} \Big[(\cos x \,\cosh 0) \times 0 - \sin x \,\sinh 0 \, dx \Big] + \\
&\quad\; i \int_{x=1}^{x=0} \Big[\cos x \,\cosh(-2x+2)\,(-2dx) - \sin x \,\sinh(-2x+2)\, dx \Big] + \\
&\quad\; i \int_{y=2}^{y=0} \Big[\cos 0 \,\cosh y \, dy - (\sin 0 \,\sinh y) \times 0 \Big] \\
&= \int_0^1 \cos x \, dx + \int_1^0 \Big[\cos x \,\cosh(-2x+2) - 2 \sin x \,\sinh(-2x+2) \Big] dx + 0 + \\
&\quad\; i0 + i \int_1^0 \Big[-2 \cos x \,\cosh(-2x+2) - \sin x \,\sinh(-2x+2) \Big] dx + i \int_2^0 \cosh y \, dy \\
&= \Big[\sin x \Big]_0^1 + \Big[\sin x \,\cosh(-2x+2) \Big]_1^0 + 0 + i0 + i \Big[\cos x \,\sinh(-2x+2) \Big]_1^0 + i \Big[\sinh y \Big]_2^0 \\
&= \Big[\sin 1 - 0 \Big] + \Big[0 - \sin 1 \Big] + 0 + i0 + i \Big[\sinh 2 - 0 \Big] + i \Big[0 - \sinh 2 \Big] = 0
\end{aligned}$$

as required by Cauchy's theorem. Hence, Cauchy's theorem is verified for this contour integral.

5. Discuss some of the obvious implications of Cauchy's theorem.
Answer: For example:
• The vanishing of the contour integrals over closed curves (within the given region and the stated conditions) implies path independence of the line integrals along open curves connecting two given points in that region.
• Path independence (over a given region which could be the entire complex plane) implies path alternativeness in evaluating line integrals (i.e. as line integrals and not as ordinary integrals) over different paths connecting two given points in the region of analyticity.
• Path independence implies that the line integrals can be evaluated as ordinary integrals since the value of the line integrals do not depend on the open curves that connect the two points and hence

the value of the line integrals should depend only on the two end points of these curves (as in ordinary integrals).

• From a practical point of view, the theorem can facilitate the calculations of difficult line integrals either through path alternativeness over the region of analyticity (which makes the replacement of difficult paths by easy paths possible; see Problem 6) or through path independence (which makes the use of ordinary integration techniques possible; see Problem 6).

6. Calculate the following line integrals over the given curves C:

(**a**) $\int_C 3z\,dz$ where C is the upper half of the circle $|z-1+i|=4$ from $z_1=5-i$ to $z_2=-3-i$.

(**b**) $\int_C iz\,dz$ where C is the quarter of the ellipse $\frac{x^2}{4}+y^2=1$ in the first quadrant from $z_1=2$ to $z_2=i$.

(**c**) $\int_C (2z^2-z+5)\,dz$ where C is the parabola $y=x^2-2x$ from $z_1=0$ to $z_2=3+i3$.

(**d**) $\int_C e^z\,dz$ where C is the hyperbola $y=1/x$ from $z_1=1+i$ to $z_2=2+i0.5$.

Answer:[186]

(**a**) As we see, $w=3z$ is analytic over the entire circle and the region surrounded by this circle and hence Cauchy's theorem applies. So, according to Problem 5 the line integral over C is independent of the path (i.e. within the region of analyticity) and hence we can use path alternativeness which allows us to evaluate this as a line integral but over a simpler path connecting the two end points. For example, let replace the semi-circle path C by the straight line path C_s that connects z_1 to z_2. In this case, the path can be parameterized as $z=x+iy=(5-8t)-i$ $(0\le t\le 1)$ with $dz=dx=-8dt$ and accordingly:

$$\begin{aligned}\int_C 3z\,dz &= \int_{C_s} 3z\,dz = \int_0^1 3\big[(5-8t)-i\big]\,(-8dt) = -24\int_0^1 \big[(5-8t)-i\big]\,dt \\ &= -24\left[\left(5t-4t^2\right)-it\right]_0^1 = -24\big[1-i\big]+24\big[0-i0\big] = -24+i24\end{aligned}$$

This result is identical to the result that we obtained in part (a) of Problems 1 and 2 of § 3.2 from evaluating this as a line integral over C (not over C_s).

Alternatively, we can evaluate it as an ordinary integral that depends only on the two end points of C. In fact, in part (a) of Problem 3 of § 3.2 we evaluated this as an ordinary integral and also obtained $-24+i24$.

(**b**) As we see, $w=iz$ is analytic over the entire ellipse and the region surrounded by this ellipse and hence Cauchy's theorem applies. So, according to Problem 5 the line integral over C is independent of the path (i.e. within the region of analyticity) and hence we can use path alternativeness which allows us to evaluate this as a line integral but over a simpler path connecting the two end points. For example, let replace the quarter-ellipse path C by the straight line path C_s that connects z_1 to z_2. In this case, the path can be parameterized as $z=x+iy=(2-2t)+it$ $(0\le t\le 1)$ with $dz=dx+idy=(-2+i)dt$ and accordingly:

$$\begin{aligned}\int_C iz\,dz &= \int_{C_s} iz\,dz = \int_0^1 i\big[(2-2t)+it\big]\,(-2+i)dt = i(-2+i)\int_0^1 \big[(2-2t)+it\big]\,dt \\ &= i(-2+i)\left[\left(2t-t^2\right)+i\frac{t^2}{2}\right]_0^1 = i(-2+i)\left[1+i\frac{1}{2}\right] - i(-2+i)\big[0+i0\big] = -i\frac{5}{2}\end{aligned}$$

This result is identical to the result that we obtained in part (b) of Problems 1 and 2 of § 3.2 from evaluating this as a line integral over C (not over C_s).

Alternatively, we can evaluate it as an ordinary integral that depends only on the two end points of C.

[186] The integrands in all parts of this Problem are analytic over the entire z plane. However, we consider a bounded region of analyticity (which encloses the closed curve and its interior region) to indicate the importance of considering the region of analyticity in applying Cauchy's theorem since the application of this theorem is restricted to the regions of analyticity (which are bounded in general).

In fact, in part (b) of Problem 3 of § 3.2 we evaluated this as an ordinary integral and also obtained $-i\frac{5}{2}$.
(**c**) Let add a straight line C_s that connects z_2 to z_1 and hence we have a closed curve $C \cup C_s$ made of the union of C and C_s. Accordingly, $w = 2z^2 - z + 5$ is analytic over the entire curve $C \cup C_s$ and the region surrounded by this curve and hence Cauchy's theorem applies. So, according to Problem 5 the line integral over C is independent of the path (i.e. within the region of analyticity) and hence we can use path alternativeness which allows us to evaluate this as a line integral but over a simpler path connecting the two end points in this region. For example, let replace the parabola path C by the straight line path C_s that connects z_1 to z_2. As we saw in part (c) of Problem 2 of § 3.2, the value of the line integral over C_s is $-21 + i42$ which is identical to the result that we obtained in part (d) of Problem 2 of § 3.2 from evaluating this as a line integral over C (not over C_s).
Alternatively, we can evaluate it as an ordinary integral that depends only on the two end points of C. In fact, in part (c) of Problem 3 of § 3.2 we evaluated this as an ordinary integral and also obtained $-21 + i42$.
(**d**) Let add a straight line C_s that connects z_2 to z_1 and hence we have a closed curve $C \cup C_s$ made of the union of C and C_s. Accordingly, $w = e^z$ is analytic over the entire curve $C \cup C_s$ and the region enclosed by this curve and hence Cauchy's theorem applies. So, according to Problem 5 the line integral over C is independent of the path (i.e. within the region of analyticity) and hence we can use path alternativeness which allows us to evaluate this as a line integral but over a simpler path connecting the two end points in this region. For example, let replace the hyperbola path C by the straight line path C_s that connects z_1 to z_2. Now, the curve C_s can be parameterized as $z = t + i(1.5 - 0.5t)$ (where $1 \le t \le 2$) with $dz = (1 - i0.5)dt$ and hence we have:

$$\begin{aligned}\int_{C_s} e^z\,dz &= \int_1^2 e^{t+i(1.5-0.5t)}(1 - i0.5)dt = \Big[e^{t+i(1.5-0.5t)}\Big]_1^2 = \big[e^{2+i0.5}\big] - \big[e^{1+i}\big] \\ &\simeq \big[6.4845 + i3.5425\big] - \big[1.4687 + i2.2874\big] \simeq 5.0158 + i1.2551\end{aligned}$$

Alternatively, we can evaluate it as an ordinary integral that depends only on the two end points of C, that is:

$$\int_C e^z\,dz = \int_{1+i}^{2+i0.5} e^z\,dz = \big[e^z\big]_{1+i}^{2+i0.5} \simeq \big[6.4845 + i3.5425\big] - \big[1.4687 + i2.2874\big] \simeq 5.0158 + i1.2551$$

which is identical to the value obtained by contour integration over C_s.
Now, let verify the result that we obtained already (i.e. by line integration over C_s as well as by ordinary integration) by line integration over C itself. The curve C can be parameterized as $z = t + (i/t)$ (where $1 \le t \le 2$) with $dz = dt - (i/t^2)dt$ and hence we have:

$$\begin{aligned}\int_C e^z\,dz &= \int_1^2 e^{t+(i/t)}\left(1 - \frac{i}{t^2}\right)dt = \Big[e^{t+(i/t)}\Big]_1^2 = \big[e^{2+i0.5}\big] - \big[e^{1+i}\big] \\ &\simeq \big[6.4845 + i3.5425\big] - \big[1.4687 + i2.2874\big] \simeq 5.0158 + i1.2551\end{aligned}$$

as before.

7. Investigate Cauchy's theorem considering the following contour integrals over the given curves C:
   (**a**) $\oint_C \frac{1}{z}\,dz$ where C is the origin-centered unit circle, i.e. $|z| = 1$.
   (**b**) $\oint_C \frac{1}{z}\,dz$ where C is the unit circle centered on $z = i2$, i.e. $|z - i2| = 1$.
   (**c**) $\oint_C z^*\,dz$ where C is the origin-centered circle with radius 5, i.e. $|z| = 5$.
   Answer:
   (**a**) Here, C can be parameterized as $z = e^{i\theta}$ (where $0 \le \theta < 2\pi$). Therefore $dz = ie^{i\theta}d\theta$ and we have:

$$\oint_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{e^{i\theta}}\,ie^{i\theta}d\theta = \int_0^{2\pi} i\,d\theta = i\Big[\theta\Big]_0^{2\pi} = i2\pi \ne 0$$

This seeming failure of Cauchy's theorem is due to the existence of a singularity (i.e. at $z = 0$) inside the circle and hence the conditions of Cauchy's theorem are not fulfilled.
(**b**) Here, C can be parameterized as $z = i2 + e^{i\theta}$ (where $0 \le \theta < 2\pi$). Therefore $dz = ie^{i\theta}d\theta$ and we have:

$$\begin{aligned}\oint_C \frac{1}{z}\,dz &= \int_0^{2\pi} \frac{1}{i2+e^{i\theta}}\,ie^{i\theta}d\theta = \int_0^{2\pi} \frac{ie^{i\theta}}{i2+e^{i\theta}}d\theta = \Big[\ln\left(i2+e^{i\theta}\right)\Big]_0^{2\pi} \\ &= \Big[\ln\left(i2+1\right)\Big] - \Big[\ln\left(i2+1\right)\Big] = 0\end{aligned}$$

So, despite the existence of a singularity (i.e. at $z = 0$) in the neighborhood, it is not inside C and hence the conditions of Cauchy's theorem are fulfilled and the contour integral is zero.
(**c**) Here, C can be parameterized as $z = 5e^{i\theta}$ (where $0 \le \theta < 2\pi$). Therefore $dz = i5e^{i\theta}d\theta$ and we have:

$$\oint_C z^*\,dz = \int_0^{2\pi} (5e^{i\theta})^*\,i5e^{i\theta}d\theta = \int_0^{2\pi} 5e^{-i\theta}\,i5e^{i\theta}d\theta = i25\int_0^{2\pi} d\theta = i25\Big[\theta\Big]_0^{2\pi} = i50\pi \ne 0$$

This seeming failure of Cauchy's theorem is due to the fact that z^* is not analytic (see part d of Problem 7 of § 3.1) and hence the conditions of Cauchy's theorem are not fulfilled.

4.2.1 Extension of Cauchy's Theorem

As stated in Cauchy's theorem (see § 4.2), the integrated function $w = f(z)$ must be analytic over the closed curve C and the entire (simply-connected) region surrounded by it if the theorem should apply. Accordingly, if there is a singularity (or singularities) inside the enclosed region then the contour integral over C is not zero in general (as demonstrated vividly in Problem 7 of § 4.2). In such cases, we need a remedy (or extension) to make the theorem usable even in these cases. It may be suggested that a potential remedy is to isolate the singularity (or singularities) by surrounding it by another contour and hence the theorem should apply within the original region excluding the region(s) of singularity. However, this exclusion will obviously violate the statement of the theorem which requires the analyticity of the function over the closed curve and the entire surrounded (simply-connected) region. Nevertheless, this suggested remedy can be improved and developed further by excluding the singularity but with the extension of C in such a way that ensures there is no singularity inside the (simply-connected) region enclosed by the extended C and hence the overall analyticity is restored. This extension is achieved by connecting C to the introduced contour(s) C_s that surround the singularity(s) by contour(s) C_c that to be tracked twice (in opposite directions) so that the union of C with C_s and C_c becomes a closed curve (which we label as C_E to indicate the extension, i.e. $C_E = C \cup C_s \cup C_c$) that surrounds a (simply-connected) region which the function f is analytic over its entirety.

The idea of this remedy and extension is demonstrated in Figure 26 where in the left frame we illustrated the case of having one singularity while in the right frame we illustrated the case of having two singularities (which can be easily extended to the cases of having more than two singularities). Now, let formalize this remedy so that it can be used in practice as a tool for evaluating contour integrals over closed and open curves in regions that contain singularities. Referring to Figure 26 (left frame), if we track C_E anticlockwise then we can see that we are actually tracking C anticlockwise and C_s clockwise while we are tracking C_c in both directions. As we know, the value of a contour integral over a curve in a given direction is equal to minus the value of that integral over the curve in the opposite direction (see Problem 5 of § 1.10) and hence the overall value of the integrals over C_c is zero. Now, if we apply Cauchy's theorem over C_E (which f is analytic on it and on the entire region surrounded by it) then we have (according to the left frame of Figure 26 noting the sense and direction of arrows):

$$\begin{aligned}\oint_{C_E} w\,dz &= \oint_C w\,dz + \int_{C_c^{\uparrow}} w\,dz + \oint_{C_s} w\,dz + \int_{C_c^{\downarrow}} w\,dz \\ &= \oint_C w\,dz + \int_{C_c^{\uparrow}} w\,dz + \oint_{C_s} w\,dz - \int_{C_c^{\uparrow}} w\,dz\end{aligned}$$

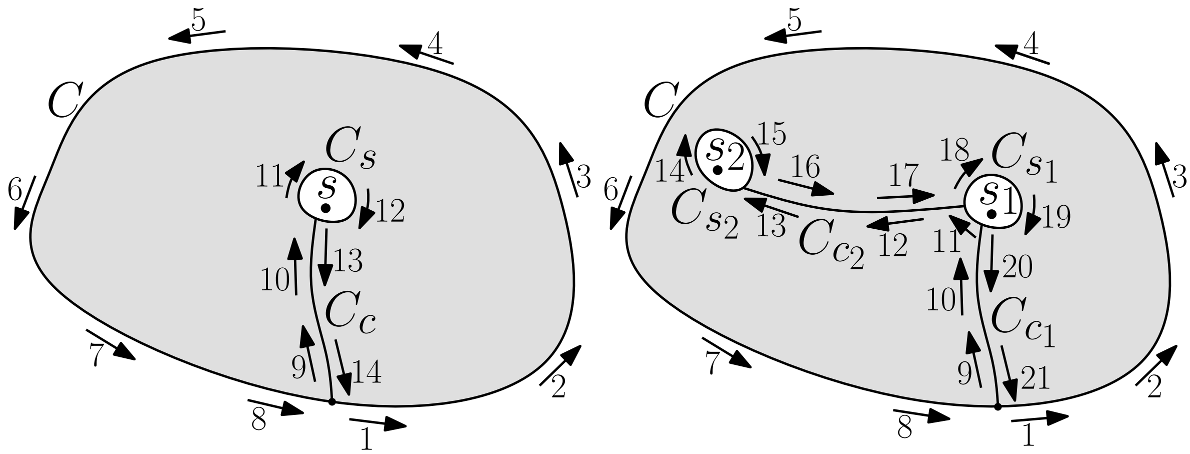

Figure 26: Graphic representation of the remedy and extension of Cauchy's theorem where in the left frame the curve C surrounds only one singularity s which is surrounded by the introduced curve C_s while in the right frame the curve C surrounds two singularities s_1, s_2 which are surrounded by the introduced curves C_{s_1}, C_{s_2}. In both frames, the sense and sequence of tracking the curves are indicated by the numbered arrows where the direction of the arrows indicates the sense of tracking while the numbering of the arrows indicates the sequence of tracking (or progression). As we see, in both frames the original curve C is tracked anticlockwise while the introduced curves C_s, C_{s_1}, C_{s_2} are tracked clockwise (and if the sense of tracking C is reversed to become clockwise the sense of tracking C_s, C_{s_1}, C_{s_2} is reversed to become anticlockwise). Also, in both frames the introduced curves C_c, C_{c_1}, C_{c_2} are tracked twice but in opposite directions and hence the integrals over these curves cancel each other. As we see, in the left frame the closed curve C_E (which consists of the union $C \cup C_c^{\uparrow} \cup C_s \cup C_c^{\downarrow}$) surrounds a region (shaded gray) that contains no singularity (or hole). This similarly applies to the closed curve C_E in the right frame which consists of a similar union surrounding a similar singularity-free (and hole-free) region (also shaded gray).

$$= \ointctrclockwise_C w\,dz + \ointclockwise_{C_s} w\,dz = 0 \tag{172}$$

where in the second line we reversed the direction of tracking C_c and the sign of the integral in the last term. Accordingly, we have:

$$\ointctrclockwise_C w\,dz = -\ointclockwise_{C_s} w\,dz \tag{173}$$

$$\ointctrclockwise_C w\,dz = +\ointctrclockwise_{C_s} w\,dz \tag{174}$$

where in the last line we reversed the sense of tracking C_s and the sign of the integral.[187] This means that the value of the contour integral around C in a given sense (say anticlockwise) is equal to the value of that integral around any curve C_s (within the region of analyticity) that surrounds the enclosed singularity in the same sense.[188] This formalism provides a handy method for evaluating difficult integrals over complicated closed curves C by evaluating these integrals over simpler curves that surround singularities inside C (as will be explained further in Problem 3).[189] The generalization of this formalism to the cases

[187] As indicated briefly earlier, the value of contour integral along a curve in one sense (e.g. clockwise) or in one direction (e.g. from z_1 to z_2) is equal to minus the value of that integral along that curve in the opposite sense (e.g. anticlockwise) or opposite direction (e.g. from z_2 to z_1). See Problem 5 of § 1.10.

[188] We note that as long as we observe the above-stated conditions (i.e. analyticity over the region and enclosure of no other singularity) the curve C_s could be inside or outside the curve C (because this is just a matter of labeling). The two curves may also be intersecting (and hence their surrounded regions partially overlap).

[189] In fact, this method is not limited to closed curves but it can be extended to open curves (as will also be explained in Problem 3).

of having more than one surrounded singularity (say n singularities) inside C is straightforward, that is:

$$\begin{aligned}\oint_C w\,dz &= -\oint_{C_{s_1}} w\,dz - \oint_{C_{s_2}} w\,dz - \cdots - \oint_{C_{s_n}} w\,dz \\ &= +\oint_{C_{s_1}} w\,dz + \oint_{C_{s_2}} w\,dz + \cdots + \oint_{C_{s_n}} w\,dz \end{aligned} \tag{175}$$

where $C_{s_1}, C_{s_2}, \cdots, C_{s_n}$ are closed curves surrounding the $1^{st}, 2^{nd}, \cdots, n^{th}$ singularities individually (i.e. each one of these n singularities is surrounded by one and only one of these n closed curves). This can be easily inferred from the right frame of Figure 26 for $n = 2$ (which can be easily generalized to any $n > 2$).

We should finally note that the (original) Cauchy's theorem has two major limitations: one on the integrated function (i.e. being analytic) and one on the enclosed region (i.e. being simply-connected). In the above extension we dealt with the first of these limitations (by extending the applicability of the theorem to cases in which the integrated function is not entirely analytic because it has singularities in the region enclosed by the contour). So, how to extend the (original) Cauchy's theorem to deal with the second limitation (i.e. by lifting the "simply-connected" condition to include multiply-connected regions)? In fact, the same technique that we used to extend the theorem with regard to the first limitation can be used to extend the theorem with regard to the second limitation and hence the theorem can be applied even to multiply-connected regions. More explicitly, if we surround the hole(s) of a multiply-connected region by closed interior contour(s) and connect the interior contour(s) to the exterior contour by oppositely-tracked curve(s) then we can deal with the situation as if the region is simply-connected (whether it is singularity-free, and hence the original Cauchy's theorem should apply, or not and hence the aforementioned extended Cauchy's theorem should apply). So in brief, to deal with both limitations we need to introduce interior contour(s) that surround the problematic spots (whether because these spots have singularities or because they have holes) to exclude them and apply the above technique of connection, tracking and canceling. Accordingly, the value of a contour integral over a closed curve that surrounds singularities or/and holes is equal to the sum of contour integrals over closed curves that surround these singularities or/and holes individually (where the sense of tracking all curves is the same).[190]

Problems

1. Outline the essence of the extended form of Cauchy's theorem.
   Answer: If C_1 and C_2 are two closed curves with C_2 being inside C_1 and f is analytic on the two curves and in the (simply-connected) region between them then $\oint_{C_1} f\,dz = \oint_{C_2} f\,dz$.
2. Outline the main implications of the extended form of Cauchy's theorem.
   Answer: There are two main (and obvious) implications:
   • Path deformability which means that we can deform the closed curve (or contour) of a contour integral (within the stated conditions of the original Cauchy's theorem) without affecting the value of the integral as long as we do not cross a singularity of the integrand or a hole in the enclosed region during the deformation process. This gives us the freedom to choose the shape and location of the curve (within the given conditions) that facilitate the evaluation of the integral.
   • Path replacement which means that we can replace the contour of our contour integral by the sum of the contours surrounding the singularities or/and holes inside the enclosed region, that is:

$$\oint_C f\,dz = \sum_{k=1}^{n} \oint_{C_k} f\,dz$$

   where C_k $(k = 1, \cdots, n)$ are contours surrounding all the singularities or/and holes inside the region enclosed by C.[191] This also provides us with more freedom about how to evaluate the integral.

[190] As we will see (refer to Problems 9 and 10), we have a third extension to Cauchy's theorem related to the type of singularity.

[191] We note that any C_k can enclose more than one singularity or/and hole although no singularity or hole can be enclosed by more than one C_k contour.

3. Explain how can we make use of Cauchy's theorem for evaluating contour integrals over closed and open curves (assuming that the conditions for the application of the theorem are fulfilled).
Answer: Regarding closed curves we have two main cases:
• There is no singularity or hole inside the enclosed region. In this case the contour integral is evaluated immediately without effort by applying Cauchy's theorem trivially because the integral is zero regardless of the form of the integrated function or the shape of the curve.
• There is a singularity or/and hole in the enclosed region. In this case if the contour integral can be easily evaluated over the actual curve in the problem then the integral can be evaluated over this curve (without use of Cauchy's theorem) by using the familiar techniques of contour integration (see § 3.2). However, if the contour integral cannot be easily evaluated over the actual curve in the problem (because the shape of the actual curve is complicated for instance) then the integral can be evaluated by using the extended form of Cauchy's theorem where the integral over the actual curve is evaluated indirectly by evaluating it over a simpler curve (e.g. unit circle) surrounding the singularity/hole with the employment of the familiar techniques of contour integration.
Regarding open curves, use can be made of Cauchy's theorem or its extension by adding a curve C_a that closes the region (i.e. with the exclusion of any singularity/hole in the neighborhood if Cauchy's theorem to be used and with the inclusion if its extension to be used) and hence the problem is converted to a closed curve problem. In the former case, the given integral can be evaluated (through exploiting path alternativeness) by choosing an alternative closing curve C_a that makes the evaluation of the integral simpler and hence the integral is evaluated over C_a instead of over the actual curve. In the latter case, another closed curve (or curves) C_b (over which the integral can be easily evaluated) is also introduced and hence the integral is evaluated over C_a and C_b and their difference (which is equal to the value of the integral over the actual curve) is taken.[192] It should be obvious that if ordinary integration is employable and more feasible in the given problem (as it is normally the case considering the relative ease of ordinary integration compared to contour integration but noting the demand for analyticity and simple connectivity) then the integral can be evaluated (noting path independence) by the techniques of ordinary integration (instead of the techniques of contour integration) without use of Cauchy's theorem or its extension.
4. Explain how to evaluate the contour integral $\oint_C w\,dz$ where C is a simple closed contour that surrounds a singularity of w and a hole.
Answer: If we surround the singularity s and the hole h by closed interior contours C_s and C_h and connect these interior contours to C through oppositely-tracked curves C_{cs} and C_{ch} (as shown in Figure 27) then from the (original) Cauchy's theorem we have (noting the sense of tracking as indicated by the arrows):

$$\begin{aligned} \oint_C w\,dz + \oint_{C_s} w\,dz + \oint_{C_h} w\,dz + \left(\int_{C_{cs}^{\uparrow}} w\,dz + \int_{C_{cs}^{\downarrow}} w\,dz\right) + \left(\int_{C_{ch}^{\uparrow}} w\,dz + \int_{C_{ch}^{\downarrow}} w\,dz\right) &= 0 \\ \oint_C w\,dz + \oint_{C_s} w\,dz + \oint_{C_h} w\,dz + 0 + 0 &= 0 \\ \oint_C w\,dz - \oint_{C_s} w\,dz - \oint_{C_h} w\,dz &= 0 \end{aligned}$$

i.e. $\oint_C w\,dz = \oint_{C_s} w\,dz + \oint_{C_h} w\,dz$.
Note: it is obvious that there are other choices for the connecting curves C_{cs} and C_{ch}. However, all these types of connection produce the same result.
5. Verify the extended form of Cauchy's theorem for the contour integral $\oint_C \frac{1}{z}\,dz$ which we evaluated in part (a) of Problem 7 of § 4.2 (with C being the circle $|z| = 1$) by evaluating this integral again but around different curves C that surround the singularity at $z = 0$ and hence confirming that the same value will be obtained.

[192] As we will see, other complex analysis techniques (such as the calculus of residues) may be used instead of introducing C_b.

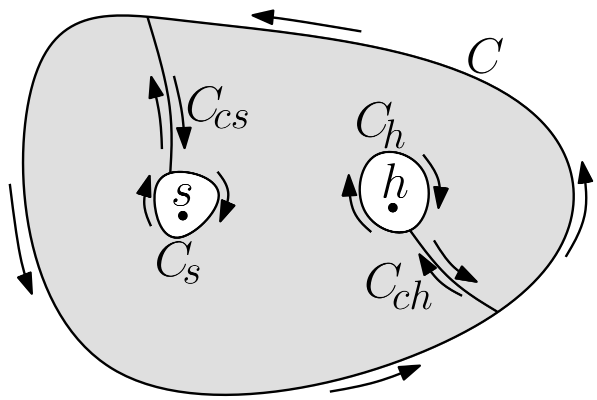

Figure 27: The setting of Problem 4 of § 4.2.1 where Cauchy's integral theorem is extended to include a region with a hole h and a singularity s.

Answer: We found in part (a) of Problem 7 of § 4.2 that the value of $\oint_C \frac{1}{z}\,dz$ (where C is the circle $|z| = 1$) is $i2\pi$. In the following parts we evaluate this integral around other curves[193] that surround the singularity at $z = 0$. As we will see, the value of this integral around every one of the investigated curves is also $i2\pi$ (as it should be according to the extended form of Cauchy's theorem) and hence this theorem is verified for the investigated cases.

(**a**) Let C be the circle $|z| = \rho$ $(0 < \rho < \infty)$ and hence it can be parameterized as $z = \rho e^{i\theta}$ (where $0 \le \theta < 2\pi$). Therefore, $dz = i\rho e^{i\theta} d\theta$ and we have:

$$\oint_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{\rho e^{i\theta}}\, i\rho e^{i\theta} d\theta = \int_0^{2\pi} i d\theta = i\Big[\theta\Big]_0^{2\pi} = i2\pi$$

(**b**) Let C be the square with vertices at $z_1 = -1 - i$, $z_2 = 1 - i$, $z_3 = 1 + i$ and $z_4 = -1 + i$. Now, if we parameterize this square as we did in part (c) of Problem 1 of § 3.2 then we have:

$$\begin{aligned}
\oint_C \frac{1}{z}\,dz &= \int_{C_1} \frac{1}{z}\,dz + \int_{C_2} \frac{1}{z}\,dz + \int_{C_3} \frac{1}{z}\,dz + \int_{C_4} \frac{1}{z}\,dz \\
&= \int_{-1}^{+1} \frac{1}{t-i}\,dt + \int_{-1}^{+1} \frac{1}{1+it}\,idt + \int_{-1}^{+1} \frac{1}{-t+i}\,(-dt) + \int_{-1}^{+1} \frac{1}{-1-it}\,(-idt) \\
&= \int_{-1}^{+1} \left[\frac{1}{t-i} + \frac{i}{1+it} + \frac{1}{t-i} + \frac{i}{1+it}\right] dt = \int_{-1}^{+1} \left[\frac{4}{t-i}\right] dt \\
&= 4\Big[\ln(t-i)\Big]_{-1}^{+1} = 4\Big[\ln(1-i) - \ln(-1-i)\Big] \\
&= 4\left[\log_e \sqrt{2} + i\left(-\frac{\pi}{4} + 2n\pi\right) - \log_e \sqrt{2} - i\left(-\frac{3\pi}{4} + 2n\pi\right)\right] = 4\Big[i\frac{\pi}{2}\Big] = i2\pi
\end{aligned}$$

(**c**) Let C be the rhombus with vertices at $z_1 = 2$, $z_2 = i$, $z_3 = -2$ and $z_4 = -i$. Now, the rhombus is made of 4 straight line segments that can be parameterized as follows:
C_1 from z_1 to z_2: $z = (2 - 2t) + it$ $(0 \le t \le 1)$ and hence $dz = (-2 + i)dt$.
C_2 from z_2 to z_3: $z = -2t + i(1 - t)$ $(0 \le t \le 1)$ and hence $dz = (-2 - i)dt$.
C_3 from z_3 to z_4: $z = (2t - 2) - it$ $(0 \le t \le 1)$ and hence $dz = (2 - i)dt$.

[193] It should be obvious that these other curves which we selected in this Problem (and in any similar Problem) meet the requirements of the extended form of Cauchy's theorem (i.e. analyticity of w over these curves and the surrounded simply-connected region excluding the enclosed singularity) and they share the original curve (which is the circle $|z| = 1$ in our case) the property of enclosing the same singularity (with no other singularity).

C_4 from z_4 to z_1: $z = 2t + i(t-1)$ $(0 \le t \le 1)$ and hence $dz = (2+i)dt$.
Therefore, we have:

$$\begin{aligned}
\oint_C \frac{1}{z}\,dz &= \int_{C_1} \frac{1}{z}\,dz + \int_{C_2} \frac{1}{z}\,dz + \int_{C_3} \frac{1}{z}\,dz + \int_{C_4} \frac{1}{z}\,dz \\
&= \int_0^1 \frac{-2+i}{(2-2t)+it}dt + \int_0^1 \frac{-2-i}{-2t+i(1-t)}dt + \int_0^1 \frac{2-i}{(2t-2)-it}dt + \int_0^1 \frac{2+i}{2t+i(t-1)}dt \\
&= \int_0^1 \left[\frac{-4+i2}{(2-2t)+it} + \frac{4+i2}{2t+i(t-1)}\right] dt \\
&= \int_0^1 \left[\frac{-8+10t}{4+5t^2-8t} + i\frac{4}{4+5t^2-8t} + \frac{-2+10t}{1-2t+5t^2} + i\frac{4}{1-2t+5t^2}\right] dt \\
&= \int_0^1 \left(\frac{10t-8}{5t^2-8t+4} + \frac{10t-2}{5t^2-2t+1}\right) dt + i\int_0^1 \left(\frac{4}{5t^2-8t+4} + \frac{4}{5t^2-2t+1}\right) dt \\
&= \Big[\ln\left(5t^2-8t+4\right) + \ln\left(5t^2-2t+1\right)\Big]_0^1 + i\left[2\arctan\left(\frac{10t-8}{4}\right) + 2\arctan\left(\frac{10t-2}{4}\right)\right]_0^1 \\
&= \Big[(\ln 1 + \ln 4) - (\ln 4 + \ln 1)\Big] + i\Big[\pi + \pi\Big] = 0 + i2\pi = i2\pi
\end{aligned}$$

6. Show that the contour integral $\oint_C \frac{dz}{(z-z_0)^n}$ (where z_0 is a given complex number, n is an integer and C is a closed curve) is zero in all cases except when $n = 1$ and C encloses the point z_0 where in this case the integral is $i2\pi$.
Answer:[194] If C does not enclose the singularity of the integrand at z_0 then by the (original) Cauchy's theorem (which we investigated and proved in § 4.2 considering that in this case the surrounded region is singularity-free) the integral should be zero for any integer n.
If C does enclose the singularity at $z = z_0$ then by the extended Cauchy's theorem (which we investigated and established in the present subsection) the integral is equal to an integral $\oint_{C_s} \frac{dz}{(z-z_0)^n}$ where C_s is a closed curve (say inside the surrounded region) that encloses the singularity. Now, let C_s be a circle centered on the point $z = z_0$ with radius ρ and hence C_s is represented as $z = z_0 + \rho e^{i\theta}$ $(0 \le \theta < 2\pi)$ with $dz = i\rho e^{i\theta}d\theta$. Accordingly, we have:

$$\oint_{C_s} \frac{1}{(z-z_0)^n}\,dz = \int_0^{2\pi} \frac{1}{(z_0+\rho e^{i\theta}-z_0)^n}\,i\rho e^{i\theta}d\theta = \int_0^{2\pi} \frac{1}{\rho^n e^{in\theta}}\,i\rho e^{i\theta}d\theta = \int_0^{2\pi} \frac{i}{\rho^{n-1}e^{i(n-1)\theta}}d\theta$$

Now, if $n \ne 1$ then we have:

$$\begin{aligned}
\oint_{C_s} \frac{1}{(z-z_0)^n}\,dz &= \int_0^{2\pi} \frac{i}{\rho^{n-1}e^{i(n-1)\theta}}d\theta = \frac{i}{\rho^{n-1}}\int_0^{2\pi} e^{i(1-n)\theta}d\theta = \frac{i}{\rho^{n-1}}\left[\frac{e^{i(1-n)\theta}}{i(1-n)}\right]_0^{2\pi} \\
&= \frac{1}{\rho^{n-1}(1-n)}\Big[1-1\Big] = 0
\end{aligned}$$

On the other hand, if $n = 1$ then we have:

$$\oint_{C_s} \frac{1}{z-z_0}\,dz = \int_0^{2\pi} \frac{i}{\rho^{1-1}e^{i(1-1)\theta}}d\theta = i\int_0^{2\pi} d\theta = i\Big[\theta\Big]_0^{2\pi} = i\Big[2\pi - 0\Big] = i2\pi$$

Hence, the integral is zero in all cases except when $n = 1$ and C encloses the singularity at z_0 (where in this case the integral is $i2\pi$).

[194] For any contour integral to be defined and convergent the contour should not pass through any singularity of the integrand that causes the divergence of the integrand (such as pole). Hence, when we discuss and investigate contour integrals that are supposed to be defined and convergent (as in the present Problem) this condition should be understood implicitly if it is not stated explicitly. Accordingly, when we investigate contour integrals (of this kind) over closed contours it should be understood that this type of singularities must be either inside the contour or outside the contour.

7. Investigate (non-thoroughly) the value of the contour integral $\oint_C w\,dz$ where w is a rational function and C is a closed curve assuming that this integral meets the requirements and conditions of the extended Cauchy's theorem.
Answer: To avoid unwanted complications and confusion we restrict our attention to the rather simple case where w can be split (by partial fractions) into a sum of m rational functions with constant (complex) numerators and (complex) *powered* linear polynomial denominators, that is:

$$\begin{aligned}\oint_C w\,dz &= \oint_C \left[\frac{a_1}{(z-z_1)^{n_1}} + \frac{a_2}{(z-z_2)^{n_2}} + \cdots + \frac{a_m}{(z-z_m)^{n_m}}\right] dz \\ &= \left[\oint_C \frac{a_1}{(z-z_1)^{n_1}}\,dz\right] + \left[\oint_C \frac{a_2}{(z-z_2)^{n_2}}\,dz\right] + \cdots + \left[\oint_C \frac{a_m}{(z-z_m)^{n_m}}\,dz\right]\end{aligned}$$

where $a_1, a_2, \cdots, a_m$ and $z_1, z_2, \cdots, z_m$ (which are not necessarily different) are complex constants and $n_1, n_2, \cdots, n_m$ are positive integers. Now, we can use the results of Problem 6 to determine the value of each sub-integral (i.e. each of the bracketed integrals in the last line) and hence determine the overall value of the main integral, i.e. $\oint_C w\,dz$. In brief, the value of each sub-integral $\oint_C \frac{a_k}{(z-z_k)^{n_k}}dz$ (where $k = 1, \cdots, m$) is zero except when $n_k = 1$ and C encloses the singularity at z_k where in this case the value of the sub-integral is $i2\pi a_k$ and hence the value of the main integral is the sum of the values of the non-zero sub-integrals, i.e. $\oint_C w\,dz = i2\pi \sum_l a_l$ (where l ranges over the non-zero sub-integrals).[195]

8. Evaluate the contour integrals of the following rational functions w around the given closed curves C:
(**a**) $w = \frac{1}{z^2-1}$ and C is the (anticlockwise) origin-centered circle with radius $\rho > 1$.
(**b**) $w = \frac{7z-23+i2z}{(z-3)(z+i)}$ and C is the (anticlockwise) ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.
(**c**) $w = \frac{-z^4+4z^3-3z^2+z+i\left(4z^2-4z+2\right)}{z^2(z-1)^2(z+i2)}$ and C is the (anticlockwise) triangle with vertices at $z_a = -i5$, $z_b = -2+i$ and $z_c = 2 - i\frac{3}{4}$.
Answer: We note first that all these integrals meet the requirements and conditions of the extended Cauchy's theorem and hence we can use the results of Problems 6 and 7.
(**a**) The denominator of w vanishes at $z_1 = 1$ and at $z_2 = -1$ and hence we have two singularities (i.e. one singularity at each one of these points). On decomposing w (by partial fractions), the integral can be written as:

$$\oint_C w\,dz = \frac{1}{2}\oint_C \left[\frac{1}{z-1} - \frac{1}{z+1}\right] dz = \frac{1}{2}\left[\oint_C \frac{1}{z-1}\,dz\right] - \frac{1}{2}\left[\oint_C \frac{1}{z+1}\,dz\right]$$

Since C encloses both singularities the value of the main integral is equal to the sum of the values of the two sub-integrals (multiplied by their coefficients).[196] Now, from the result of Problem 6 the value of each one of these sub-integrals is $i2\pi$. Hence, the value of the main integral is zero.
(**b**) The denominator of w vanishes at $z_1 = 3$ and at $z_2 = -i$ and hence we have two singularities (i.e. one singularity at each one of these points). On decomposing w (by partial fractions), the integral can be written as:

$$\oint_C w\,dz = \oint_C \left[\frac{i2}{z-3} + \frac{7}{z+i}\right] dz = i2\left[\oint_C \frac{1}{z-3}\,dz\right] + 7\left[\oint_C \frac{1}{z+i}\,dz\right]$$

Since C encloses only the singularity at $z_2 = -i$ the first sub-integral is zero and hence the value of the main integral is equal to the value of the second sub-integral times 7. Now, from the result of Problem 6 the value of the second sub-integral is $i2\pi$. Hence, the value of the main integral is $7 \times i2\pi = i14\pi$.
(**c**) The denominator of w vanishes at $z_1 = 0$, at $z_2 = 1$ and at $z_3 = -i2$ and hence we have three

[195] In fact, we are considering the sub-integrals which have non-trivial contribution to the value of the main integral; otherwise the value of the main integral is the sum of the values of all sub-integrals.

[196] We mean that both sub-integrals have non-trivial contribution to the main integral; otherwise the main integral is always equal to the sum of the values of its sub-integrals regardless of the inclusion or non-inclusion of singularities.

singularities (i.e. one singularity at each one of these points). On decomposing w (by partial fractions), the integral can be written as:

$$\oint_C w\,dz = \oint_C \left[\frac{1}{z^2} + \frac{1}{(z-1)^2} - \frac{1}{z+i2}\right] dz = \left[\oint_C \frac{1}{z^2}\,dz\right] + \left[\oint_C \frac{1}{(z-1)^2}\,dz\right] - \left[\oint_C \frac{1}{z+i2}\,dz\right]$$

Since the singularity at $z_2 = 1$ is not enclosed inside C the second sub-integral is zero[197] and hence the value of the main integral is equal to the sum (or rather the difference) of the values of the first and third sub-integrals (whose singularities are enclosed inside C). However, the value of the first sub-integral is also zero because $n_1 = 2 \neq 1$, and hence the value of the main integral is equal to the value of the third sub-integral times -1. Now, from the result of Problem 6 the value of the third sub-integral is $i2\pi$ and hence the value of the main integral is $-i2\pi$.

9. Let R be a simply-connected and bounded region and z_0 is a given point in R. Moreover, $g(z)$ is a function which is continuous on the entire R and analytic on R excluding z_0. Show that for any (piecewise-smooth and oriented) closed curve C inside R that encloses z_0 we have $\oint_C g\,dz = 0$.
Answer: If C_c is a z_0-centered circle of radius ρ inside R then by the extension of Cauchy's theorem we should have:
$$\oint_C g\,dz = \oint_{C_c} g\,dz \tag{176}$$
Now, since g is continuous then $|g|$ should be bounded over R, i.e. $|g| \leq M$ with M being an upper bound (see part b of Problem 7 of § 1.9; also see Problem 9 of § 1.11). Hence, we should have (see Eq. 45):[198]
$$\left|\oint_{C_c} g\,dz\right| \leq \oint_{C_c} |g|\,|dz| \leq 2\pi\rho M$$
Now, since the size of C_c is arbitrary (i.e. within the given conditions) ρ can be as small as we wish which in the limit can go to zero leading to $\left|\oint_{C_c} g\,dz\right| \leq \oint_{C_c} |g|\,|dz| \leq 0$ and thus $\left|\oint_{C_c} g\,dz\right| = 0$ (noting that $\left|\oint_{C_c} g\,dz\right|$ is non-negative) and hence $\oint_{C_c} g\,dz = 0$. Accordingly, from Eq. 176 we get $\oint_C g\,dz = 0$, as required.
Note 1: although the statement of the Problem and the proof are about curves enclosing z_0 it should be obvious that the result applies to any closed curve in R because if the curve does not enclose z_0 then by Cauchy's theorem (rather than its extension) the integral should also be zero noting that g in this case is analytic on and inside such a curve (since z_0 is outside the curve). We should also note that the simplicity of the curve is not required. This also applies to the particular orientation of the curve (although we use the integral symbol of positively oriented curve).
Note 2: this theorem actually extends the application of Cauchy's theorem by treating "continuous singularities" like analytic points (i.e. by restricting the offending singularities to those that blow up the integrand and make it unbound). Accordingly, the contour integral around a curve that encloses a "continuous singularity" (i.e. a singularity at which the integrand remains continuous and hence bounded) is zero as if the "continuous singularity" is an "analytic point".[199]

10. List the main extensions of the (original) Cauchy's theorem.
Answer: As we noted earlier, we have three main extensions:
• Extension related to the type of the integrated function, i.e. by extending "analytic" to "analytic with some singularities" through introducing inner loops to exclude the singularities.

[197] In fact, the value of the second sub-integral is zero even if the singularity is enclosed inside C because $n_2 = 2 \neq 1$.
[198] In fact, M in the following equation is only required to be an upper bound of g over C_c (rather than over R).
[199] It may be shown (using for instance the fact that a continuous function that has a primitive is analytic) that this type of "continuous singularity" does not destroy the analyticity of g at z_0 and hence g is effectively "analytic" at z_0 despite this "casual singularity" (or "apparent singularity"). Accordingly, a function can be "analytic" at this type of "singularity". For example, $\frac{\sin z}{z}$ has a "continuous singularity" at $z = 0$ but it has a Taylor series (or rather Maclaurin series) there and hence it is "analytic" there (see § 5.1) despite this "singularity". For more details see Problem 6 of § 5.2 (also refer to Problem 17 of § 1.5).

• Extension related to the type of region, i.e. by extending "simply-connected" to "multiply-connected" (i.e. region with holes) through introducing inner loops to exclude the holes.
• Extension related to the type of singularity, i.e. by extending "analytic point" to include "continuous singularity" through treating "continuous singularity" like "analytic point".

4.2.2 The Residue Theorem

This theorem (which may also be called Cauchy's residue theorem) is based on the extension of Cauchy's theorem (see Problem 1 of § 5.4) and hence it is appropriate to be investigated here. However, because of its dependency on the subject of Laurent series (which will be investigated later on in § 5.2) we defer the detailed investigation of this theorem (or rather the technique that is based on it) to § 5.4. Nevertheless, we can provide at this point some useful remarks that do not depend on our pending investigations (apart from the definition of "residue" which will be given in § 5.4):
• Because the residue theorem is based in essence on the extension of Cauchy's theorem with regard to singularities, it can be regarded as a generalization of Cauchy's theorem (and hence Cauchy's theorem is a special case of the residue theorem). In fact, if the residue theorem is stated properly then it can be regarded as an extension to Cauchy's theorem even with regard to the nature of the region (i.e. simply-connected or multiply-connected; see § 4.2.1) as will be clarified in the upcoming remarks.
• The residue theorem essentially deals with the evaluation of contour integrals of complex analytic functions in regions that potentially contain some singularities of these functions (and possibly lack of simple connectivity of the region by having holes for instance).
• The residue theorem can be stated in simple terms as: if $f(z)$ is a function analytic over a piecewise-smooth, simple and closed curve C and the entire simply-connected region surrounded by C in the complex plane except (possibly) at a finite number n of singularities inside C then:

$$\oint_C f\,dz = \sum_{k=1}^{n} \oint_{C_{s_k}} f\,dz = i2\pi \sum_{k=1}^{n} {}_k a_{-1} \tag{177}$$

where C_{s_k} $(k = 1, \cdots, n)$ are (closed anticlockwise) contours surrounding the n singularities (individually) and ${}_k a_{-1}$ $(k = 1, \cdots, n)$ are the n residues of f corresponding to its Laurent series expansions around these singularities (see § 5.4 for the definition of residue).
• If we have to extend the residue theorem to include the second extension of Cauchy's theorem (i.e. the extension from simply-connected region to multiply-connected region by having holes) then we need to add to the right side of Eq. 177 the (closed anticlockwise) contour integrals around the holes (i.e. to isolate the holes exclusively) as explained in § 4.2.1 (and in Problem 4 of § 4.2.1 in particular). In brief, if we have n singularities and m holes inside C then we have (according to the residue theorem that includes both extensions):

$$\oint_C f\,dz = \left(\sum_{k=1}^{n} \oint_{C_{s_k}} f\,dz\right) + \left(\sum_{l=1}^{m} \oint_{C_{h_l}} f\,dz\right) = \left(i2\pi \sum_{k=1}^{n} {}_k a_{-1}\right) + \left(\sum_{l=1}^{m} \oint_{C_{h_l}} f\,dz\right) \tag{178}$$

where C_{h_l} $(l = 1, \cdots, m)$ are (closed anticlockwise) contours surrounding the m holes individually (i.e. without including any other hole) and exclusively (i.e. without including any singularity).

4.3 The Integral Formula Theorem

One of the main pillars of complex analysis is the integral formula theorem (which may also be called Cauchy's integral theorem among other names and labels that exacerbate the confusion). The essence of this theorem (which leads to the formula) is that: if $f(z)$ is a function analytic over a (simply-connected) region R containing a given point z_0 and C is a piecewise-smooth, simple, closed and positively oriented curve in R with z_0 being inside C then:

$$f(z_0) = \frac{1}{i2\pi} \oint_C \frac{f(z)}{z - z_0}\,dz \tag{179}$$

This equation is commonly known as Cauchy's integral formula. If we inspect Eq. 179 carefully we can appreciate its significance and exceptional power because according to this formula we can know anything about the region inside C with regard to f (since z_0 is arbitrary and hence it can represent the entire region inside C) by just knowing what is on the border of this region. On the other hand, we can know the value of the contour integral over C if we know the value of f at a given point z_0 inside C.[200]

Now, if we manipulate the symbols in Eq. 179 slightly by replacing z_0 by z and replacing z by w then Eq. 179 becomes:[201]

$$f(z) = \frac{1}{i2\pi} \oint_C \frac{f(w)}{w - z} \, dw \tag{180}$$

which is another form of this formula that is commonly used in the texts of complex analysis (noting that these two forms are the same in essence although each form may have some marginal advantages in certain contexts such as being more aesthetic or suggestive or indicative or easier to mix with other expressions and formulations in that context). In fact, this formula may also be given in the literature by other forms such as $f(w) = \frac{1}{i2\pi} \oint_C \frac{f(z)}{z-w} \, dz$ (and hence the reader should be careful when reading for different authors to avoid confusion).

An important corollary of the integral formula theorem is the following differentiation formula (which may be called Cauchy's integral formula for derivatives or Cauchy's differentiation or derivative formula):

$$\left. \frac{d^n f}{dz^n} \right|_{z=z_0} \equiv f^{(n)}(z_0) = \frac{n!}{i2\pi} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \tag{181}$$

where $f^{(n)}(z_0)$ is the n^{th} derivative of f at z_0 and $n!$ is the factorial of n and where the assumptions and conditions of the integral formula theorem are inherited here (i.e. f, C, R and z_0 satisfy the assumptions and conditions of Cauchy's integral formula as stated above).[202] Again, if we manipulate the symbols in Eq. 181 slightly by replacing z_0 by z and replacing z by w then Eq. 181 becomes:

$$f^{(n)}(z) = \frac{n!}{i2\pi} \oint_C \frac{f(w)}{(w - z)^{n+1}} \, dw \tag{182}$$

which is another form of the differentiation formula that is commonly used in the texts of complex analysis (noting again that the two forms are the same in essence).[203]

Now, if we mean by derivative its common meaning then n in the above formulae (i.e. Eqs. 181 and 182) should be a positive integer (i.e. $n = 1, 2, 3, \cdots$). However, if we want to include f itself (as the zeroth derivative of itself which is sensible) then n is a non-negative integer (i.e. $n = 0, 1, 2, \cdots$). Accordingly, what we actually have is a single formula (i.e. Eq. 181 or Eq. 182) with Eq. 179 (or Eq. 180) being an instance of Eq. 181 corresponding to $n = 0$. In fact, this will ease the derivations and manipulations.

We should now draw the attention to the following useful remarks:

• From Cauchy's integral formula for derivatives we can conclude that if $f(z)$ is an analytic function on a region R then f should have derivatives of all orders over R. To be more formal, let z_0 be a point inside R. Now, if we take a sufficiently small disk centered on z_0 and contained inside R then we can apply the integral formula theorem on the boundary of this disk (noting that all the conditions of this theorem are satisfied) to obtain Cauchy's integral formula, and from this formula we can obtain the derivatives of f at z_0 of all orders using Cauchy's integral formula for derivatives which is based on and derived from Cauchy's integral formula (as will be seen in Problem 2). Now, since z_0 is an arbitrary point in R it can represent all the interior points of R (see Problem 6).

---

[200] In fact, theorems and features like these are behind the exceptional power and beauty of complex analysis.

[201] This form of the integral formula is based on the fact that z_0 in Eq. 179 represents all the points inside C, and hence if we represent these points (or the region inside C) with z and replace z in Eq. 179 (where z there represents the points on the curve C) by w then we get this form of the integral formula.

[202] Noting that $f^{(n)}$ represents derivative, n should be a positive integer. However, as we will see this could be generalized to include $n = 0$ and hence n can represent a non-negative integer.

[203] We should repeat what we said about the significance of the Cauchy's integral formula since the differentiation formula also links what is inside a region to what is on its boundary (i.e. with regard to the derivatives of all orders).

• It is important to keep in mind that z_0 in Eqs. 179 and 181 (and z in Eqs. 180 and 182) represents points inside C because if it is outside C then the integrand in these equations is analytic on and inside C and hence the integral vanishes (by the original Cauchy's theorem) while if it is on C then the conditions of Cauchy's theorem (which the integral formula theorem is ultimately based on) are violated and hence it cannot be used (in fact some of the conditions required by the proof of the integral formula theorem which is given in Problem 1 will be violated; also see footnote [194] on page 191).

Problems

1. Prove the integral formula theorem.
   Answer: Let first define a new function $g(z)$ as:

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0} \tag{183}$$

Since f is analytic over R and $\frac{1}{z-z_0}$ is analytic over R excluding z_0 then g is analytic (and hence continuous; see part a of Problem 7 of § 1.9) over R excluding z_0. Also, because f is analytic at z_0 it has a derivative at z_0, and hence (see Eq. 17):

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} g(z)$$

This means that $g(z)$ has a limit at z_0 which is equal to $f'(z_0)$. So, if we define $g(z)$ at z_0 as $g(z_0) = f'(z_0)$ then g becomes continuous over the entire R. Now, if we use the result of Problem 9 of § 4.2.1 (noting that g is continuous on the entire R and analytic on R excluding z_0) then we should have (noting that C is the curve defined in the statement of the theorem above which satisfies the conditions of C in Problem 9 of § 4.2.1):

$$\begin{aligned}
\oint_C g\,dz &= 0 && \text{(Problem 9 of § 4.2.1)} \\
\oint_C \frac{f(z) - f(z_0)}{z - z_0}\,dz &= 0 && \text{(Eq. 183)} \\
\oint_C \frac{f(z)}{z - z_0}\,dz - \oint_C \frac{f(z_0)}{z - z_0}\,dz &= 0 \\
\oint_C \frac{f(z_0)}{z - z_0}\,dz &= \oint_C \frac{f(z)}{z - z_0}\,dz \\
f(z_0) \oint_C \frac{1}{z - z_0}\,dz &= \oint_C \frac{f(z)}{z - z_0}\,dz \\
i2\pi\, f(z_0) &= \oint_C \frac{f(z)}{z - z_0}\,dz && \text{(Problem 6 of § 4.2.1)} \\
f(z_0) &= \frac{1}{i2\pi} \oint_C \frac{f(z)}{z - z_0}\,dz
\end{aligned}$$

which is Eq. 179.

2. Verify Cauchy's differentiation formula (Eq. 182).
   Answer: Because all the assumptions and conditions of the integral formula theorem are presumably satisfied (as stated in the text) we can start from Cauchy's integral formula (using the form of Eq. 180). Now, if we take the first order derivative of f (using the definition of derivative as a limit) then we have:

$$\begin{aligned}
f^{(1)}(z) &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[\frac{1}{i2\pi} \oint_C \frac{f(w)}{w - (z + \Delta z)}\,dw - \frac{1}{i2\pi} \oint_C \frac{f(w)}{w - z}\,dw \right] && \text{(Eq. 180)}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\Delta z\to 0} \frac{1}{i2\pi\Delta z} \oint_C \left[\frac{f(w)}{w-z-\Delta z} - \frac{f(w)}{w-z}\right] dw \\
&= \lim_{\Delta z\to 0} \frac{1}{i2\pi\Delta z} \oint_C f(w) \left[\frac{(w-z)-(w-z-\Delta z)}{(w-z-\Delta z)(w-z)}\right] dw \\
&= \lim_{\Delta z\to 0} \frac{1}{i2\pi\Delta z} \oint_C f(w) \frac{\Delta z}{(w-z)^2-\Delta z(w-z)} dw \\
&= \lim_{\Delta z\to 0} \frac{1}{i2\pi} \oint_C \frac{f(w)}{(w-z)^2-\Delta z(w-z)} dw \\
&= \frac{1}{i2\pi} \oint_C \lim_{\Delta z\to 0} \left[\frac{f(w)}{(w-z)^2-\Delta z(w-z)}\right] dw \qquad \text{(continuity of integrand)} \\
&= \frac{1}{i2\pi} \oint_C \frac{f(w)}{(w-z)^2} dw
\end{aligned}
$$

which is Eq. 182 for $n = 1$. This means that Eq. 182 is valid for $n = 1$. Now, if we repeat this process of differentiation we can obtain the formulae for higher derivatives. These formulae can then be generalized to all orders of derivative (i.e. for all n) using mathematical induction (as will be explained in the following). So, let assume that Eq. 182 is valid for $n = m$ and hence we will show that it is also valid for $n = m+1$, that is:

$$
\begin{aligned}
f^{(m+1)}(z) &= \lim_{\Delta z\to 0} \frac{f^{(m)}(z+\Delta z) - f^{(m)}(z)}{\Delta z} \\
&= \lim_{\Delta z\to 0} \frac{1}{\Delta z}\left[\frac{m!}{i2\pi} \oint_C \frac{f(w)}{(w-z-\Delta z)^{m+1}} dw - \frac{m!}{i2\pi} \oint_C \frac{f(w)}{(w-z)^{m+1}} dw\right] \\
&= \lim_{\Delta z\to 0} \frac{m!}{i2\pi\Delta z} \oint_C \left[\frac{f(w)}{(w-z-\Delta z)^{m+1}} - \frac{f(w)}{(w-z)^{m+1}}\right] dw \\
&= \lim_{\Delta z\to 0} \frac{m!}{i2\pi\Delta z} \oint_C f(w) \left[\frac{1}{(w-z-\Delta z)^{m+1}} - \frac{1}{(w-z)^{m+1}}\right] dw \\
&= \lim_{\Delta z\to 0} \frac{m!}{i2\pi} \oint_C f(w) \left[\frac{\frac{1}{(w-z-\Delta z)^{m+1}} - \frac{1}{(w-z)^{m+1}}}{\Delta z}\right] dw \\
&= \frac{m!}{i2\pi} \oint_C \lim_{\Delta z\to 0} \left\{f(w) \left[\frac{\frac{1}{(w-z-\Delta z)^{m+1}} - \frac{1}{(w-z)^{m+1}}}{\Delta z}\right]\right\} dw \\
&= \frac{m!}{i2\pi} \oint_C f(w) \left\{\lim_{\Delta z\to 0} \left[\frac{\frac{1}{(w-z-\Delta z)^{m+1}} - \frac{1}{(w-z)^{m+1}}}{\Delta z}\right]\right\} dw \\
&= \frac{m!}{i2\pi} \oint_C f(w) \left\{\frac{d}{dz}\left[\frac{1}{(w-z)^{m+1}}\right]\right\} dw \\
&= \frac{m!}{i2\pi} \oint_C f(w) \left\{\frac{m+1}{(w-z)^{m+2}}\right\} dw \\
&= \frac{(m+1)!}{i2\pi} \oint_C \frac{f(w)}{(w-z)^{m+2}} dw
\end{aligned}
$$

where line 2 is justified by the presumed validity of Eq. 182 for $n = m$, line 6 is justified by the continuity of the integrand over C, and line 7 is justified by the fact that the limit (and hence the derivative) is with respect to z. As we see, the last equation is of the same form as Eq. 182 for $n = m+1$ and hence Eq. 182 is valid for $n = m+1$. So, we have shown that Eq. 182 is valid for $n = 1$ and if it is valid for $n = m$ then it is also valid for $n = m+1$ and hence by mathematical induction Eq. 182 should be valid for all n, as required.

Note: Cauchy's differentiation formula (in the form of Eq. 181) may also be obtained (more easily)

by differentiating Eq. 179 (using the method of differentiation under the integral sign) with respect to z_0, that is:

$$f^{(n)}(z_0) = \frac{d^n}{dz_0^n}\left[\frac{1}{i2\pi}\oint_C \frac{f(z)}{z-z_0}\,dz\right] = \frac{1}{i2\pi}\oint_C \frac{\partial^n}{\partial z_0^n}\left[\frac{f(z)}{z-z_0}\right]dz = \frac{n!}{i2\pi}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Similarly, Eq. 182 can be obtained by differentiating Eq. 180 with respect to z. However, the theoretical justification of this (i.e. the use of differentiation under the integral sign in this case) may be more difficult (and possibly problematic).

3. Evaluate the following contour integrals around the contours shown in Figure 28:

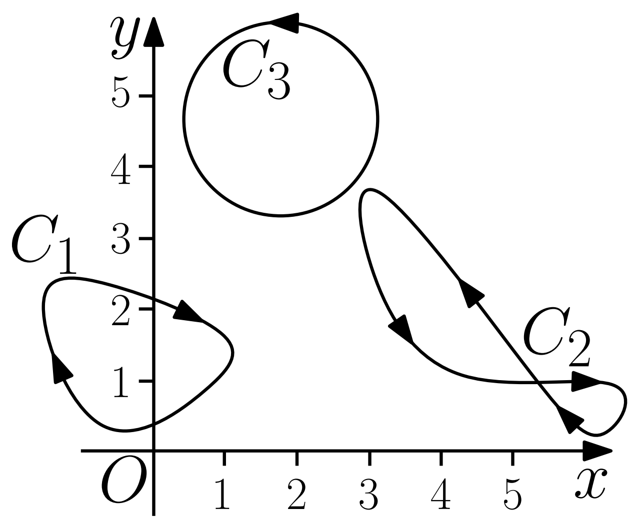

Figure 28: Graphic representation of the contours of Problem 3 of § 4.3.

(**a**) $\oint_{C_1} \frac{2z^4}{z-i}\,dz$. (**b**) $\oint_{C_1} \frac{2z^4}{z+i}\,dz$. (**c**) $\oint_{C_2} \frac{e^z}{z-4-i2}\,dz$. (**d**) $\oint_{C_3} \frac{\cos z}{z-3-i}\,dz$.

Answer:

(**a**) Let $f(z) = 2z^4$ and $z_0 = i$. Now, f is analytic on and inside C_1 and z_0 is inside C_1 and therefore we can use the integral formula (where f and z_0 in the formula are as defined here). Hence, from Eq. 179 (noting that C_1 is clockwise and thus a minus sign is added) we have:

$$\begin{aligned}\frac{1}{i2\pi}\oint_{C_1} \frac{2z^4}{z-i}\,dz &= -f(i)\\ \oint_{C_1} \frac{2z^4}{z-i}\,dz &= -2i^4 \times i2\pi = -i4\pi\end{aligned}$$

(**b**) Here, the integrand $\frac{2z^4}{z+i}$ is analytic on and inside C_1 (noting that $z = -i$ which is a singularity of the integrand is not inside C_1) and hence by Cauchy's theorem (whose conditions are obviously satisfied) the integral should be zero.

(**c**) C_2 is not a simple curve but it can be split into two simple curves, i.e. the small (clockwise) loop which we label C_{2a} and the large (anticlockwise) loop which we label C_{2b} (where these loops share the point of intersection). Hence:

$$I = \oint_{C_2} \frac{e^z}{z-4-i2}\,dz = \oint_{C_{2a}} \frac{e^z}{z-4-i2}\,dz + \oint_{C_{2b}} \frac{e^z}{z-4-i2}\,dz = I_a + I_b$$

Now, the integrand of I_a is analytic on and inside C_{2a} (noting that $z = 4+i2$ which is a singularity of the integrand is not inside C_{2a}) and hence by Cauchy's theorem (whose conditions are obviously

satisfied) I_a should be zero. Regrading I_b, let $f(z) = e^z$ and $z_0 = 4 + i2$. Now, f is analytic on and inside C_{2b} and z_0 is inside C_{2b} and therefore we can use the integral formula (where f and z_0 in the formula are as defined here). Hence, from Eq. 179 we have:

$$\begin{aligned} \frac{1}{i2\pi}\oint_{C_{2b}} \frac{e^z}{z-4-i2}\,dz &= f(4+i2) \\ \oint_{C_{2b}} \frac{e^z}{z-4-i2}\,dz &= e^{4+i2}\times i2\pi = i2\pi e^{4+i2} \end{aligned}$$

Hence, $I = I_a + I_b = 0 + i2\pi e^{4+i2} = i2\pi e^{4+i2} \simeq -311.9347 - i142.7593$.
(**d**) Here, the integrand $\frac{\cos z}{z-3-i}$ is analytic on and inside C_3 (noting that $z = 3+i$ which is a singularity of the integrand is not inside C_3) and hence by Cauchy's theorem (whose conditions are obviously satisfied) the integral should be zero.

4. Evaluate the following contour integrals around the contours shown in Figure 29:

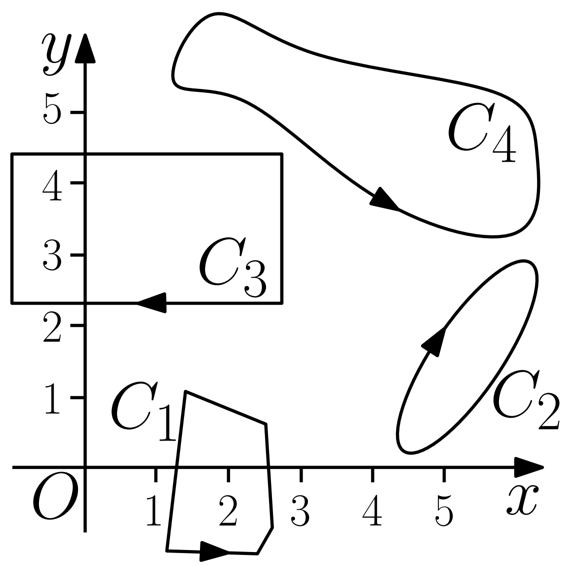

Figure 29: Graphic representation of the contours of Problem 4 of § 4.3.

(**a**) $\oint_{C_1} \frac{\sin z}{(z-2)^6}\,dz$. (**b**) $\oint_{C_2} \frac{e^{3z^2}}{(z-5-i)^3}\,dz$. (**c**) $\oint_{C_3} \frac{\cosh^5 z}{z^9}\,dz$. (**d**) $\oint_{C_4} \frac{3z^3-iz^2+6}{(z-4-i5)^2}\,dz$.

Answer:
(**a**) If $f(z) = \sin z$ and $z_0 = 2$ then from Eq. 181 (noting that the conditions for applying Cauchy's derivative formula are satisfied, i.e. f is analytic on and inside C_1 and z_0 is inside C_1, etc.) we have:

$$\begin{aligned} \left.\frac{d^5 \sin z}{dz^5}\right|_{z=2} &= \frac{5!}{i2\pi}\oint_{C_1} \frac{\sin z}{(z-2)^6}\,dz \\ \cos 2 &= \frac{5!}{i2\pi}\oint_{C_1} \frac{\sin z}{(z-2)^6}\,dz \\ \oint_{C_1} \frac{\sin z}{(z-2)^6}\,dz &= \frac{i2\pi\cos 2}{5!} = \frac{i\pi\cos 2}{60} \simeq -i0.02179 \end{aligned}$$

(**b**) If $f(z) = e^{3z^2}$ and $z_0 = 5 + i$ then from Eq. 181 (noting that the conditions for applying Cauchy's derivative formula are satisfied, i.e. f is analytic on and inside C_2 and z_0 is inside C_2, etc.) we have:

$$\left.\frac{d^2 e^{3z^2}}{dz^2}\right|_{z=5+i} = -\frac{2!}{i2\pi}\oint_{C_2} \frac{e^{3z^2}}{(z-5-i)^3}\,dz$$

$$\begin{aligned} 6e^{3z^2} + 36z^2e^{3z^2}\Big|_{z=5+i} &= -\frac{1}{i\pi}\oint_{C_2} \frac{e^{3z^2}}{(z-5-i)^3}\,dz \\ \oint_{C_2} \frac{e^{3z^2}}{(z-5-i)^3}\,dz &= -i\pi\left[6e^{3(5+i)^2} + 36(5+i)^2e^{3(5+i)^2}\right] = \pi e^{72+i30}(360-i870) \end{aligned}$$

where a minus sign is added because C_2 is clockwise.
(**c**) Here, the integrand $\frac{\cosh^5 z}{z^9}$ is analytic on and inside C_3 (noting that $z = 0$ which is a singularity of the integrand is not inside C_3) and hence by Cauchy's theorem (whose conditions are obviously satisfied) the integral should be zero.
(**d**) If $f(z) = 3z^3 - iz^2 + 6$ and $z_0 = 4 + i5$ then from Eq. 181 (noting that the conditions for applying Cauchy's derivative formula are satisfied, i.e. f is analytic on and inside C_4 and z_0 is inside C_4, etc.) we have:

$$\begin{aligned} \frac{d(3z^3 - iz^2 + 6)}{dz}\bigg|_{z=4+i5} &= \frac{1!}{i2\pi}\oint_{C_4} \frac{3z^3 - iz^2 + 6}{(z-4-i5)^2}\,dz \\ 9z^2 - i2z\Big|_{z=4+i5} &= \frac{1}{i2\pi}\oint_{C_4} \frac{3z^3 - iz^2 + 6}{(z-4-i5)^2}\,dz \\ \oint_{C_4} \frac{3z^3 - iz^2 + 6}{(z-4-i5)^2}\,dz &= i2\pi\left[9(4+i5)^2 - i2(4+i5)\right] = -(704 + i142)\pi \end{aligned}$$

5. Explain and prove the mean value theorem which states that if $f(z)$ is an analytic function on the closed disk $|z - z_0| \le \rho$ then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})\,d\theta$$

Answer: This theorem means that the value of f at the center of the disk z_0 is equal to the average value of f over its boundary, i.e. the circle C_c that defines this disk. To prove this theorem let parameterize C_c as $z = z_0 + \rho e^{i\theta}$ $(0 \le \theta < 2\pi)$ and hence $dz = i\rho e^{i\theta}d\theta$. Now, from Cauchy's integral formula (see Eq. 179) we have:

$$f(z_0) = \frac{1}{i2\pi}\oint_{C_c} \frac{f(z)}{z - z_0}\,dz = \frac{1}{i2\pi}\int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}}\,i\rho e^{i\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})\,d\theta$$

as required.

6. Verify the following statement: if $f(z)$ is analytic over a region R in the z plane, then f is infinitely differentiable and its derivatives of all orders are analytic over R.
Answer: Let z_0 be a point inside R.[204] Now, since f is analytic over R then f is analytic in a neighborhood N of z_0. So, if C is a contour (say circle) inside N that encloses z_0 then f is analytic on C and inside C and hence by Eq. 181 we have:

$$f^{(n)}(z_0) = \frac{n!}{i2\pi}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (n = 1, 2, \cdots) \tag{184}$$

Now, on inspecting and considering the motivation and rationale of the integral formula theorem (which led to Eq. 181), we can see that Eq. 184 holds for any z inside C, and this means that for any $n > 1$ the function $f^{(n-1)}$ is differentiable on a neighborhood of z_0 inside C and hence $f^{(n-1)}$ is analytic at z_0.[205] Noting that $n = 1, 2, \cdots$ and z_0 is arbitrary (i.e. it can represent the entire region R) we can conclude that f is infinitely differentiable and its derivatives of all orders are analytic over R.

[204] In fact, we are considering R as an open region (or domain).
[205] In this sentence we use $n > 1$ because the analyticity of f (i.e. $f^{(0)} = f^{(1-1)}$ which corresponds to $n = 1$) is given in the statement and hence it does not need to be established by this argument. However, if n in Eq. 184 includes 0 then we can use $n \ge 1$ which may make our argument more straightforward. Anyway, this is a trivial matter and hence it should not affect our argument at all.

Note: as indicated earlier and will be investigated further later on, analytic function can be represented by a Taylor series. Hence, we may obtain the result of the present Problem from the fact that Taylor series is infinitely differentiable (assuming no circularity or inter-dependency and this depends on the sequence of results and proofs). We should also not that a rather simpler argument (similar to the above argument) for establishing the above statement was presented in the text of the present section.

4.3.1 Cauchy's Inequality

As we will see in the Problems, Cauchy's inequality is a direct result of Cauchy's differentiation formula which we investigated in § 4.3 (and this should explain why we put our investigation of Cauchy's inequality as a subsection of § 4.3). This inequality is widely used in complex analysis to give a bound on (or an estimate of) the magnitude of derivatives of a given complex function at a given point in the complex plane. It is also used in the proofs of some theorems of complex analysis.

Problems

1. Apply Cauchy's differentiation formula (i.e. Eq. 181) to a circle C of radius ρ and center z_0 to derive the following relation (called Cauchy's inequality or Cauchy's estimate) which puts an upper limit on the magnitude of the derivatives of f at z_0:

$$\left|f^{(n)}(z_0)\right| \le \frac{n!M_C}{\rho^n} \tag{185}$$

where M_C is an upper bound of $|f(z)|$ over C.
Answer: As stated in the question, the curve C in Cauchy's differentiation formula here is a circle of radius ρ and center z_0, i.e. $|z - z_0| = \rho$. On taking the modulus of both sides of Cauchy's differentiation formula and applying the rules of inequalities, we get:

$$\begin{aligned}\left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| = \left|\frac{n!}{2\pi i}\right|\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \\ &\le \frac{n!}{2\pi}\oint_C \frac{|f(z)|}{|z-z_0|^{n+1}}|dz| \le \frac{n!}{2\pi}\frac{M_C}{\rho^{n+1}}\oint_C |dz| = \frac{n!}{2\pi}\frac{M_C}{\rho^{n+1}}2\pi\rho = \frac{n!M_C}{\rho^n}\end{aligned}$$

which is Eq. 185. We note that because this formula is based on Cauchy's differentiation formula it inherits the definitions and conditions of that formula (which inherits the definitions and conditions of Cauchy's integral formula).
Note: although we were talking above about the derivatives of f, Cauchy's inequality should hold even for f itself (i.e. the zeroth derivative as indicated earlier in § 4.3) where the inequality of Eq. 185 becomes $|f(z_0)| \le M_C$. In fact, this result can be obtained not only from Eq. 185 (by extending Eq. 181 to include $n = 0$ as indicated in § 4.3) but can also be obtained directly from Cauchy's integral formula (i.e. Eq. 179) following the same reasoning and derivation as above. It may also be obtained from the mean value theorem (see Problem 5 of § 4.3).

2. Verify Cauchy's inequality for the following n^{th} derivatives of the given functions f at the given points z_0:
(**a**) $n = 2$ and $f = e^z$ at $z_0 = 0$. (**b**) $n = 2$ and $f = z^2$ at $z_0 = 1 + i$.
Answer:
(**a**) We have $f^{(1)} = f^{(2)} = e^z$ and hence:

$$\left|f^{(2)}(z_0)\right| = \left|f^{(2)}(0)\right| = |e^{z_0}| = \left|e^0\right| = 1$$

Now, let C be the origin-centered unit circle and hence $|e^z|$ on C is given by:

$$|e^z| = \left|e^{x+iy}\right| = |e^x|\left|e^{iy}\right| = |e^x|\sqrt{\cos^2 y + \sin^2 y} = |e^x| = e^x \qquad (-1 \le x \le 1)$$

which takes its maximum on C at $x = 1$, i.e. $M_C = e$. Hence, from Cauchy's inequality we should have (noting that $\rho = 1$):

$$\left|f^{(2)}(0)\right| \le \frac{2! \times e}{1^2} = 2e$$

which is true since $\left|f^{(2)}(0)\right| = 1$ (as we found above).
(**b**) We have $f^{(1)} = 2z$ and $f^{(2)} = 2$ and hence:

$$\left|f^{(2)}(z_0)\right| = \left|f^{(2)}(1+i)\right| = 2$$

Now, let C be the unit circle centered on z_0 and hence $\left|z^2\right|$ on C is given by:

$$\left|z^2\right| = |z \times z| = |z| \times |z| = |z|^2 = x^2 + y^2 \qquad (0 \le x \le 2,\ 0 \le y \le 2)$$

which takes its maximum on C at $x = y = 1 + \frac{1}{\sqrt{2}}$, i.e. $M_C = 2\left(1 + \frac{1}{\sqrt{2}}\right)^2 \simeq 5.8284$. Hence, from Cauchy's inequality we should have (noting that $\rho = 1$):

$$\left|f^{(2)}(1+i)\right| \le \frac{2! \times 2\left(1 + \frac{1}{\sqrt{2}}\right)^2}{1^2} \simeq 11.6569$$

which is true since $\left|f^{(2)}(1+i)\right| = 2$ (as we found above).

4.4 Morera's Theorem

According to Morera's theorem, if a complex function $f(z)$ is continuous in an open region R and $\oint_C f\,dz = 0$ for every closed contour C within R then f is analytic throughout R. Morera's theorem and Cauchy's theorem are generally regarded as converses of each other, which is seemingly correct although rigorously it is not. The reality is that according to Cauchy's theorem if f is analytic in such a region (within the other stated conditions) then $\oint_C f\,dz = 0$ for every closed contour C, while according to Morera's theorem if $\oint_C f\,dz = 0$ for every closed contour C (within the other stated conditions) then f is analytic. So, what we have according to Cauchy's theorem is the conditional statement $P \to Q$ (with P being "f is analytic" and Q being "$\oint_C f\,dz = 0$"), while what we have according to Morera's theorem is the conditional statement $Q \to P$. In other words, one theorem seems to be an "if" statement while the other theorem seems to be an "only if" statement and hence if they are combined together they seem to form an *iff* statement. So, apparently they are converses of each other according to this sense of "converse".

However, this claim ignores the detailed conditions and attributes of each theorem (e.g. with regard to the domain of definition). Moreover, it is based (at least with regard to Cauchy's theorem) on taking their implications rather than exact form and statement. Accordingly, they are not converses in a rigorous sense (although they are converses in a rough sense) and hence they do not form (i.e. when combined) an *iff* statement. Anyway, the discussion of this issue (i.e. being converses or not) is provided here for the sake of clarification and information to highlight the relation between these theorems and improve the understanding of their contents. Otherwise, as long as each theorem is stated rigorously within its conditions and attributes and can be applied correctly the entire matter of being converses or not is irrelevant and useless (at least from a practical point of view).

An important issue about Morera's theorem is regarding its practicability because it seems very difficult (if not impossible) to show that $\oint_C f\,dz = 0$ for *every* closed contour[206] C (within the stated conditions) noting that we have no access to Cauchy's theorem since the analyticity of f is what is required to establish rather than being established (as required by Cauchy's theorem). However, Morera's theorem has many useful applications in theoretical arguments and proofs although it may not be usable in solving familiar

[206] We note that there is a potential relaxation on this condition (which may ease the situation a little bit), but we will not go through this.

practical problems. An example of the use of Morera's theorem is outlined in the Problems and this should give an idea about how Morera's theorem is exploited and employed in theoretical arguments.

Problems

1. Outline a proof of Morera's theorem.

   Answer: We should note first that there are many technicalities and delicate details about a rigorous proof of this theorem. However, in this answer we present a simple version (or outline) of a proof that highlights the basic idea behind this theorem and why it should work. To ease the situation, let the region R be simply-connected (although we did not impose this condition in the statement).[207] Also, let accept (without proof) the intuitive claim that if a (continuous) function f has a primitive F then f is analytic. Now, let have two random points in R (which we label as z_1 and z_2) and let C_1 be an arbitrary curve (within R) that goes from z_1 to z_2 and C_2 be another arbitrary curve (within R) that goes from z_2 to z_1 (and hence the union $C_1 \cup C_2$ forms an arbitrary closed contour C within R).[208] Now, according to the statement of Morera's theorem we have $\oint_C f\,dz = 0$ for every closed contour C within R, and hence we can write:

$$\begin{aligned}
\oint_C f\,dz &= 0 \\
\int_{C_1} f\,dz + \int_{C_2} f\,dz &= 0 \\
\int_{C_1} f\,dz &= -\int_{C_2} f\,dz \\
\int_{C_1} f\,dz &= \int_{C_3} f\,dz
\end{aligned}$$

   where C_3 is the opposite of C_2 (i.e. it goes from z_1 to z_2 along the same path). The last line means (noting that C_1 and C_3 are arbitrary within R) that the integral on either side is independent of the path and hence it should solely depend on the end points z_1 and z_2. This is equivalent to having a primitive F of f throughout R (noting that z_1 and z_2 are arbitrary within R), that is:

$$\int_{C_1} f\,dz = \int_{C_3} f\,dz = F(z_2) - F(z_1)$$

   and hence f should be analytic throughout R.

   Note: if C_1 and C_2 intersect or touch each other then we can consider each individual loop separately (and hence the above proof is still valid). This similarly applies if the two curves share a common segment (or segments).

2. The function $f(z) = 1/z^2$ is obviously not analytic at the origin since it has a pole there. However, $\oint_C f\,dz = 0$ for every closed contour in a region R that contains the origin (see Problem 6 of § 4.2.1). So, despite the apparent fact that $\oint_C f\,dz = 0$ for every closed contour in R, f is not analytic throughout R, and hence this may be seen as a counterexample to Morera's theorem. Discuss this issue.

   Answer: First, $\oint_C f\,dz = 0$ does not apply to closed contours that pass through the origin and hence the condition "for every closed contour" is not satisfied. Second, this function is discontinuous at the origin and hence the condition "continuous" is not satisfied. Therefore, this is not a counterexample to Morera's theorem.

3. Give an example of the use of Morera's theorem.

   Answer: For instance, let $F(z) = \int_a^b f(t, z)\,dt$ (with a and b being constants) and we want to prove

[207] Imposing the condition "simply-connected" will have an impact on the relation between Morera's theorem and Cauchy's theorem. In fact, imposing this condition is required if we should have an analytic antiderivative of f. The details should be sought in the literature (noting that many impose this condition in the statement of Morera's theorem).

[208] At this stage, we are seemingly suggesting that the two curves do not share any point (other than z_1 and z_2), i.e. they do not intersect or touch. However, this seeming restriction on the type of curves (which affects their arbitrariness) will be dealt with in the upcoming note.

the analyticity of F over a given region R (which satisfies the requirements of Cauchy's and Morera's theorems) given that F is continuous and f is analytic over the entire R. Now, if we integrate F over an arbitrary closed contour C within R and we assume that the given problem allows the change of the order of integration then we have:

$$\oint_C F(z)\,dz = \oint_C \left[\int_a^b f(t,z)\,dt\right] dz = \int_a^b \left[\oint_C f(t,z)\,dz\right] dt$$

At this point we can use Cauchy's theorem because since f is analytic over R (as assumed above) then by Cauchy's theorem the contour integral $\oint_C f(t,z)\,dz$ is zero and hence we have:

$$\oint_C F(z)\,dz = 0$$

Now, because C is arbitrary (i.e. it can represent every closed contour within R) then by Morera's theorem F is analytic throughout R. So, we used Morera's theorem to prove the analyticity of F over R.

4.5 Liouville's Theorem

According to Liouville's theorem, if $f(z)$ is an entire function (i.e. analytic over the entire *finite* complex plane) and is bounded (i.e. over the *extended* complex plane) then f is constant over the entire (finite) complex plane (and even the extended complex plane).[209] To put it in more simple terms, the constant functions are the only entire bounded functions of complex variables, so all entire non-constant functions are unbounded. Accordingly, the entire functions can be split into two mutually exclusive (or disjoint) categories: constant bounded and non-constant unbounded (with no existence of non-constant bounded, noting that constant unbounded is meaningless). This theorem (which may not seem intuitive) has a surprisingly simple proof, as we will see in the Problems (refer to Problem 3).

It is important to note that being "entire" (i.e. analytic over the entire complex plane) is crucial for avoiding potential confusion (as well as for the validity of the theorem itself). For example, the trigonometric cosine and sine functions of *real variables* are "analytic"[210] over the "entire real line" and they are bounded (since they do not exceed 1 in magnitude) and hence we seemingly expect them (according to Liouville's theorem) to be constant. Nevertheless, they are not constants because they are not entire (i.e. analytic over the "entire complex plane") since they are restricted (by definition) to the real line.

Problems

1. State Liouville's theorem in simple words.
   Answer: Liouville's theorem simply states that all entire bounded functions are constant.
2. For a function to be unbounded it should blow up somewhere in the complex plane and this means having a singularity there and hence it cannot be analytic there. So, it seems that having "entire unbounded" functions is contradictory because "entire" implies having no singularity while "unbounded" implies having singularity. Discuss this issue.
   Answer: As indicated above and explained earlier (see note 2 of Problem 3 of § 1.5), entirety is restricted to the finite complex plane and hence "entire unbounded" functions blow up at infinity and thus they are entire (because they have no singularity in the finite complex plane) and unbounded (because they blow up at infinity) at the same time with no contradiction.
   Note: this seeming contradiction may also be avoided by imposing the condition "bounded" (i.e. in the statement of Liouville's theorem) on the entire complex plane including infinity (as indicated in

[209] We remind the reader that "extended complex plane" means the finite complex plane plus (the point at) infinity (see § 1.5).

[210] As indicated earlier, "analytic" here (and in similar contexts) should mean in the sense of real analysis and not in the sense of complex analysis.

the text by "extended") and hence an entire function that is bounded (even at infinity) is constant. In fact, Liouville's theorem may be stated as: a function that is analytic in the *extended* complex plane is constant (noting that being analytic in the extended complex plane implies boundedness everywhere because if it is not bounded at a point such as infinity then it is not analytic there; see part e of Problem 7 of § 1.9).

3. Prove Liouville's theorem.
   Answer: If M_C is an upper bound on $|f(z)|$ then by Cauchy's inequality (see Eq. 185) with $n = 1$ we have:
   $$|f'(z_0)| \equiv \left|f^{(1)}(z_0)\right| \le \frac{1!M_C}{\rho^1} = \frac{M_C}{\rho}$$
   Now, since $f(z)$ is an entire function then this should be true for any z_0 and any ρ, and hence if we let $\rho \to \infty$ we get $|f'(z_0)| = 0$ over the entire complex plane which means that f is a constant (i.e. throughout the complex plane).
4. Verify Liouville's theorem for the following entire functions:
   (**a**) Non-constant polynomials.
   (**b**) Exponentials.
   (**c**) Trigonometric and hyperbolic cosines and sines.
   Answer: All these are entire non-constant functions and hence Liouville's theorem (or rather its contrapositive) implies that they should be unbounded.
   (**a**) Liouville's theorem is obviously true in this case because non-constant polynomials are unbounded (i.e. blow up at infinity). Also, see Figure 18.
   (**b**) Liouville's theorem is obviously true in this case because complex exponentials split into a product of an exponential of a real number times trigonometric functions (i.e. $e^z = e^x \cos y + ie^x \sin y$) and hence the exponentials are unbounded (i.e. blow up at infinity) due to the presence of the exponential of real number (i.e. e^x). Also, see Figure 20.
   (**c**) Liouville's theorem is obviously true in this case because these functions are synthesized of exponentials (see Eqs. 131 and 133) and hence from the result of part (b) they should be unbounded (i.e. blow up at infinity). Also, see Figures 22 and 23.
5. Prove the following propositions using Liouville's theorem:
   (**a**) If $f(z)$ is an entire function with a constant modulus $|f(z)|$ then f is constant.
   (**b**) If $f(z)$ is an entire function with $|f(z)| \ge 1$ everywhere in the complex plane then f is constant.
   (**c**) If $w = f(z) = u + iv$ is entire and $u \le x$ for all $z = x + iy$ then w is a linear polynomial.
   (**d**) If $w = f(z) = u + iv$ is entire and $x \le v$ for all $z = x + iy$ then w is a linear polynomial.
   Answer:
   (**a**) This is obvious because $f(z)$ is an entire function and is bounded (since its modulus is constant) and hence f should be constant according to Liouville's theorem.
   (**b**) Because $f(z)$ is an entire function that does not vanish $\big[$since $|f(z)| \ge 1\big]$, its reciprocal (i.e. $1/f$) should be entire. Moreover, $|1/f| \le 1$ because $|f(z)| \ge 1$. So, $1/f$ is entire and bounded and hence by Liouville's theorem $1/f$ is constant over the entire complex plane. Therefore, f must also be constant over the entire complex plane (because the reciprocal of a non-zero constant is a constant).
   (**c**) We note first that e^{w-z} is entire since it is a composition of an exponential function (which is entire) and a difference of two entire functions (which is also entire). Moreover:
   $$\left|e^{w-z}\right| = \left|e^{(u-x)+i(v-y)}\right| = \left|e^{(u-x)}\right|\left|e^{i(v-y)}\right| = \left|e^{(u-x)}\right| = e^{u-x} \le 1$$
   where the last step (i.e. ≤ 1) is justified by the presumption that $u \le x$. So, e^{w-z} is entire and bounded and hence according to Liouville's theorem it is constant over the entire complex plane. Accordingly, the derivative of e^{w-z} with respect to z should vanish for all z, that is:
   $$\frac{d}{dz}e^{w-z} = \left(\frac{dw}{dz} - \frac{dz}{dz}\right)e^{w-z} = \left(\frac{dw}{dz} - 1\right)e^{w-z} = 0$$

Now, since $e^{w-z} \neq 0$ then $\frac{dw}{dz} - 1 = 0$, i.e. $\frac{dw}{dz} = 1$ and hence by integration we get $w = z + C$ (with C being a complex constant). This means that $w = f(z)$ is a linear polynomial, as required.
(**d**) As in part (c), e^{z+iw} is entire because it is a composition of two entire functions. Moreover:

$$\left|e^{z+iw}\right| = \left|e^{(x-v)+i(y+u)}\right| = \left|e^{(x-v)}\right| \left|e^{i(y+u)}\right| = \left|e^{(x-v)}\right| = e^{x-v} \leq 1$$

where the last step (i.e. ≤ 1) is justified by the presumption that $x \leq v$. So, e^{z+iw} is entire and bounded and hence according to Liouville's theorem it is constant over the entire complex plane. Accordingly, the derivative of e^{z+iw} with respect to z should vanish for all z, that is:

$$\frac{d}{dz}e^{z+iw} = \left(\frac{dz}{dz} + i\frac{dw}{dz}\right)e^{z+iw} = \left(1 + i\frac{dw}{dz}\right)e^{z+iw} = 0$$

Now, since $e^{z+iw} \neq 0$ then $1 + i\frac{dw}{dz} = 0$, i.e. $\frac{dw}{dz} = i$ and hence by integration we get $w = iz + C$ (with C being a complex constant). This means that $w = f(z)$ is a linear polynomial, as required.

6. Use Liouville's theorem to show that if f is a non-constant function in the complex plane then it should have at least one singularity.
   Answer: This is because either f has a singularity in the finite complex plane or not. In the former case f obviously has at least one singularity, while in the latter case f should be (according to the implication of Liouville's theorem[211] noting that in this case f is entire) unbounded and hence it has a singularity (i.e. at infinity).

4.6 The Maximum Modulus Theorem

According to this theorem (or principle) if $f(z)$ is an analytic function over a connected and bounded region R and it is not constant in R then its modulus $|f|$ has no relative maximum inside R and the absolute maximum of $|f|$ is on the boundary of R.[212] It should be obvious that this theorem can also be stated for the minimum modulus (by replacing "maximum" by "minimum" in the above statement with some adjustments) and hence we get the minimum modulus theorem which will be investigated later (see Problem 2).

Problems

1. Prove the maximum modulus theorem.
   Answer: We note first that this theorem can be split into two parts:
   (**a**) If z_0 is a point inside R and $|f|$ has a relative maximum at z_0 then f should be constant over the interior of R (and hence over the entire R).
   (**b**) The absolute maximum of $|f|$ should be on the boundary of R.
   Therefore, we need to prove both these parts. However, before we start proving these parts we note that since f is analytic over a bounded region R then it must be continuous and bounded over R (see Problem 7 of § 1.9; also see Problem 9 of § 1.11).[213]
   Regarding part (a), it can be proved by showing that if $|f|$ has a relative maximum at z_0 then $f(z)$ should be constant in a neighborhood of z_0 and hence it should be constant over the interior of R by extending this neighborhood in all directions and subsequently to the entire R including its boundary (as will be explained in the following). So, let $|f|$ have a relative maximum at z_0 and hence for a sufficiently-small z_0-centered disk $|z - z_0| \leq \rho$ that identifies a neighborhood of z_0 inside R we

[211] According to Liouville's theorem: if a function (i.e. entire) is bounded then it is constant. Accordingly, if a function (i.e. entire) is non-constant then it is unbounded. So, we are actually using the contrapositive of Liouville's theorem.

[212] To be more specific, f is analytic over the interior of R (assuming to be closed) and continuous over R including its boundary. However, we do not go through these (rather messy) technicalities and their implications although we will (implicitly) observe these conditions in the proof of the maximum modulus theorem (which will be given in Problem 1).

[213] As indicated earlier, f should be continuous even on the boundary (see § 1.5 for the continuity on the boundary).

have:[214]

$$|f(z)| \le |f(z_0)| \tag{186}$$

Now, if we parameterize the circle C_c that defines the boundary of this disk as $z = z_0 + \rho e^{i\theta}$ $(0 \le \theta < 2\pi)$ with $dz = i\rho e^{i\theta} d\theta$ then we have:

$$\begin{aligned}
|f(z_0)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta \right| && \text{(Problem 5 of § 4.3)} \\
&\le \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + \rho e^{i\theta}) \right| d\theta && (187) \\
&\le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\, d\theta && \text{(Eq. 186)} \quad (188) \\
&= \frac{|f(z_0)|}{2\pi} \int_0^{2\pi} d\theta \\
&= |f(z_0)|
\end{aligned}$$

As we see, the above semi-inequalities are bounded on both sides by an equality to the same value, i.e. $|f(z_0)|$, and hence these semi-inequalities should be equalities.[215] Accordingly, from the second and third lines (i.e. Eqs. 187 and 188) we get:

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + \rho e^{i\theta}) \right| d\theta &= \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|\, d\theta \\
\int_0^{2\pi} \Big[|f(z_0)| - \left| f(z_0 + \rho e^{i\theta}) \right| \Big] d\theta &= 0
\end{aligned}$$

Noting that the integrand in the last equation is continuous and it is non-negative (according to Eq. 186), we can conclude from the last equation that the integrand is identically zero, that is:

$$\left| f(z_0 + \rho e^{i\theta}) \right| = |f(z_0)| \qquad (0 \le \theta < 2\pi)$$

As we see, the last equality means that the modulus of f on C_c is constant since $|f(z_0)|$ is constant. Now, since f is analytic and $\left| f(z_0 + \rho e^{i\theta}) \right|$ is constant then $f(z_0 + \rho e^{i\theta})$ is constant (see Problem 16 of § 3.1). So in brief, f is constant on C_c. By the mean value theorem (see Problem 5 of § 4.3) $f(z_0)$ is the average of f on C_c and hence we should have $f(z) = f(z_0)$ on C_c (noting that f is constant on C_c). Now,[216] since the disk is no more than a series of z_0-centered circles and because the above reasoning applies to each one of these circles $\big[$i.e. f is constant on the circle and it equals $f(z_0)\big]$ then we should have $f(z) = f(z_0)$ on the entire disk (and not only on its boundary). Now, if we note that this disk can be used as a platform to launch similar disks centered on points on the disk (where these disks define neighborhoods to their corresponding central points) then we can extend this argument to the entire interior of the region R (to conclude that f is constant over the entire interior of R). We can finally argue that f is continuous over R (including its boundary) and hence this constant value on the interior of R should extend even to the boundary of R and therefore we can conclude that f is constant over the entire R including its boundary.

Regarding part (b), f is bounded on R and hence its modulus $|f|$ should have an absolute maximum

[214] The reader should note that the disk "identifies" (rather than "defines") a neighborhood of z_0 inside R and hence the boundary may or may not be included in this neighborhood. So, this does not contradict the terminology and concepts given in § 1.5. Moreover, the upcoming application of the mean value theorem (of Problem 5 of § 4.3) which is based on using a closed disk is fully justified.

[215] This is because if $a \le b \le a$ then $a = b$ (noting that $a \le b \le a$ means $a \le b$ and $a \ge b$).

[216] The following argument to establish the constancy of f over the entire R is intuitive but non-rigorous. A rigorous argument can be made by using the fact that the zeros of (non-zero) analytic function are isolated (see part a of Problem 3 of § 7.1) where we exploit the fact that $f(z) - f(z_0) = 0$ over C_c and hence it should be zero over the entire R $\big[$considering the analyticity of $f(z) - f(z_0)\big]$ which should lead to the constancy of f over the entire R.

in R. Now, we have two cases: either f is constant over R and hence the absolute maximum of $|f|$ is on the boundary of R (and indeed everywhere in R) or f is not constant over R and hence from the result of part (a) $|f|$ cannot have a relative maximum (and by priority cannot have absolute maximum which is stronger) inside R and hence the absolute maximum should be on the boundary of R, as required.
Note: in the proof of part (b) we actually use the contrapositive of part (a) because the statement in part (a) is "if $|f|$ has a relative maximum inside R then f is constant over R" while in the proof of part (b) we use the contrapositive of this conditional statement, i.e. "if f is not constant over R then $|f|$ has no relative maximum inside R".

2. State and prove the minimum modulus theorem.
Answer: The minimum modulus theorem can be stated as: if $f(z)$ is an analytic function over a connected and bounded region R and it is not constant in R then the minimum of its modulus $|f|$ is either at a zero of f or on the boundary of R.
This theorem can be proved as follows: if f has a zero in R then the minimum of $|f|$ should be there (noting that $|f|$ is non-negative); otherwise we take the reciprocal of f and apply the maximum modulus theorem to $1/f$ to conclude that the maximum of $|f|^{-1}$ is on the boundary of R and hence the minimum of $|f|$ is on the boundary of R.

3. Find the value and location of the maximum modulus of the following functions on the identified regions R in the z plane:
(**a**) $f = z + i3$ with R being the square region defined by $-2 \le x \le 2$ and $-2 \le y \le 2$.
(**b**) $f = e^z$ with R being the square region defined by $-2 \le x \le 2$ and $-2 \le y \le 2$.
(**c**) $f = 2\cos z$ with R being the square region defined by $-1 \le x \le 1$ and $-1 \le y \le 1$.
(**d**) $f = e^{i\pi z}$ where R is the rhombus with vertices at $z_1 = 5$, $z_2 = i3$, $z_3 = -5$ and $z_4 = -i3$.
Answer: We note first that all the functions and regions in this Problem satisfy the conditions and requirements of the maximum modulus theorem (refer to the statement of the theorem) and hence we should expect to find the maximum moduli on the boundary of the regions.
(**a**) The modulus of f is:

$$|f| = |z + i3| = |x + i(y + 3)| = \sqrt{x^2 + (y + 3)^2}$$

So, the maximum modulus of f (noting that $-2 \le x \le 2$ and $-2 \le y \le 2$) occurs when x^2 is maximum (i.e. when $x = \pm 2$) and $(y+3)^2$ is maximum (i.e. when $y = 2$). Therefore, the maximum modulus of f over R is $|f| = \sqrt{29} \simeq 5.3852$ and it occurs at the two points $z_{1,2} = \pm 2 + i2$ which are on the boundary of R (as required by the maximum modulus theorem). This is graphically illustrated in Figure 30.
(**b**) The modulus of f is:

$$|f| = |e^z| = \left|e^{x+iy}\right| = |e^x|\left|e^{iy}\right| = |e^x| = e^x$$

So, the maximum modulus of f (noting that $-2 \le x \le 2$) is $e^2 \simeq 7.3891$ and it occurs on the line $x = 2$ (with $-2 \le y \le 2$) which is on the boundary of R (as required by the maximum modulus theorem). This is graphically illustrated in Figure 16 (in part a of Problem 6 of § 1.11).
(**c**) Noting that $\cos z = \cos x \cosh y - i \sin x \sinh y$ (see Eq. 137), the modulus of f is:

$$\begin{aligned} |f| &= |2\cos z| = 2\sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} = 2\sqrt{\cos^2 x \cosh^2 y + (1 - \cos^2 x)\sinh^2 y} \\ &= 2\sqrt{\cos^2 x \left(\cosh^2 y - \sinh^2 y\right) + \sinh^2 y} = 2\sqrt{\cos^2 x + \sinh^2 y} \end{aligned}$$

So, the maximum of $|f|$ occurs at the points where $\cos^2 x + \sinh^2 y$ is maximum which is (noting that $-1 \le x \le 1$ and $-1 \le y \le 1$) when $x = 0$ (and hence $\cos^2 x = 1$) and $y = \pm 1$ (and hence $\sinh^2 y = \sinh^2 1$). Therefore, the maximum modulus of f over R is $|f| = 2\sqrt{1 + \sinh^2 1} = 2\cosh 1$ and it occurs at the two points $z_{1,2} = \pm i$ which are on the boundary of R (as required by the maximum modulus theorem). This is graphically illustrated in Figure 31.
(**d**) The modulus of f is:

$$|f| = \left|e^{i\pi z}\right| = \left|e^{i\pi(x+iy)}\right| = \left|e^{\pi(-y+ix)}\right| = \left|e^{-\pi y}\right|\left|e^{i\pi x}\right| = \left|e^{-\pi y}\right| = e^{-\pi y}$$

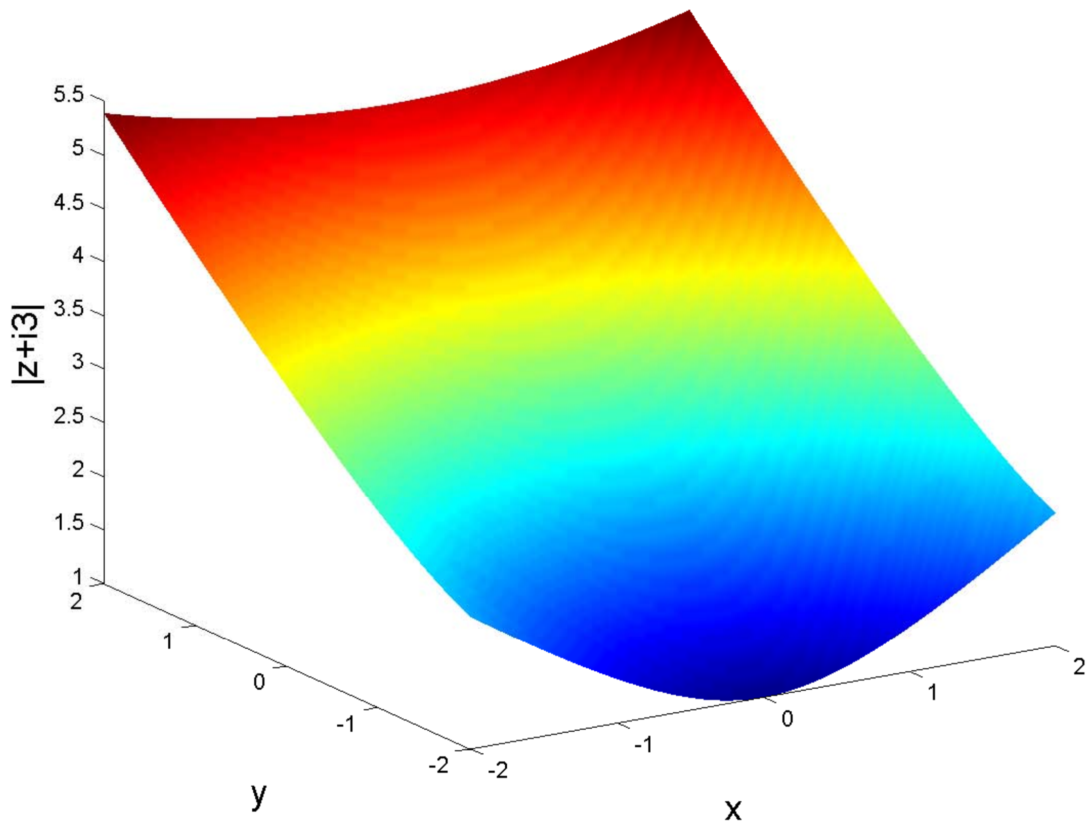

Figure 30: Graphic illustration of the modulus $|f|$ of the complex function $f = z + i3$ over the square region in the z plane defined by $-2 \le x \le 2$ and $-2 \le y \le 2$. As we see, the maximum of the modulus ($\simeq 5.3852$) occurs at $\pm 2 + i2$ (i.e. at $x = \pm 2$ and $y = 2$) and the minimum of the modulus ($= 1$) occurs at $-i2$ (i.e. at $x = 0$ and $y = -2$). See part (a) of Problem 3 of § 4.6 and part (a) of Problem 4 of § 4.6.

So, the maximum modulus of f (noting that $-3 \le y \le 3$) is $e^{3\pi}$ and it occurs at z_4 (where $x = 0$ and $y = -3$) which is on the boundary of R (as required by the maximum modulus theorem).

4. Re-solve Problem 3 but this time for the minimum modulus.
   Answer: We note first that all the functions and regions in this Problem satisfy the conditions and requirements of the minimum modulus theorem (refer to the statement of the theorem in Problem 2). Moreover, all these functions have no zero inside R. Hence, we should expect to find the minimum moduli on the boundary of the regions.
   (**a**) From part (a) of Problem 3 we have $|f| = \sqrt{x^2 + (y+3)^2}$ ($-2 \le x \le 2$ and $-2 \le y \le 2$) and hence the minimum modulus occurs when x^2 is minimum (i.e. when $x = 0$) and $(y+3)^2$ is minimum (i.e. when $y = -2$). Therefore, the minimum modulus of f over R is $|f| = 1$ and it occurs at the point $z = -i2$ which is on the boundary of R (as required by the minimum modulus theorem). This is graphically illustrated in Figure 30.
   (**b**) From part (b) of Problem 3 we have $|f| = e^x$ ($-2 \le x \le 2$) and hence the minimum modulus occurs when x is minimum (i.e. when $x = -2$ and $-2 \le y \le 2$). Therefore, the minimum modulus of f over R is $|f| = e^{-2}$ and it occurs at the line $x = -2$ ($-2 \le y \le 2$) which is on the boundary of R (as required by the minimum modulus theorem). This is graphically illustrated in Figure 16 (in part a of Problem 6 of § 1.11).
   (**c**) From part (c) of Problem 3 we have $|f| = 2\sqrt{\cos^2 x + \sinh^2 y}$ ($-1 \le x \le 1$ and $-1 \le y \le 1$) and hence the minimum modulus occurs when $\cos^2 x$ is minimum (i.e. when $x = \pm 1$) and $\sinh^2 y$ is

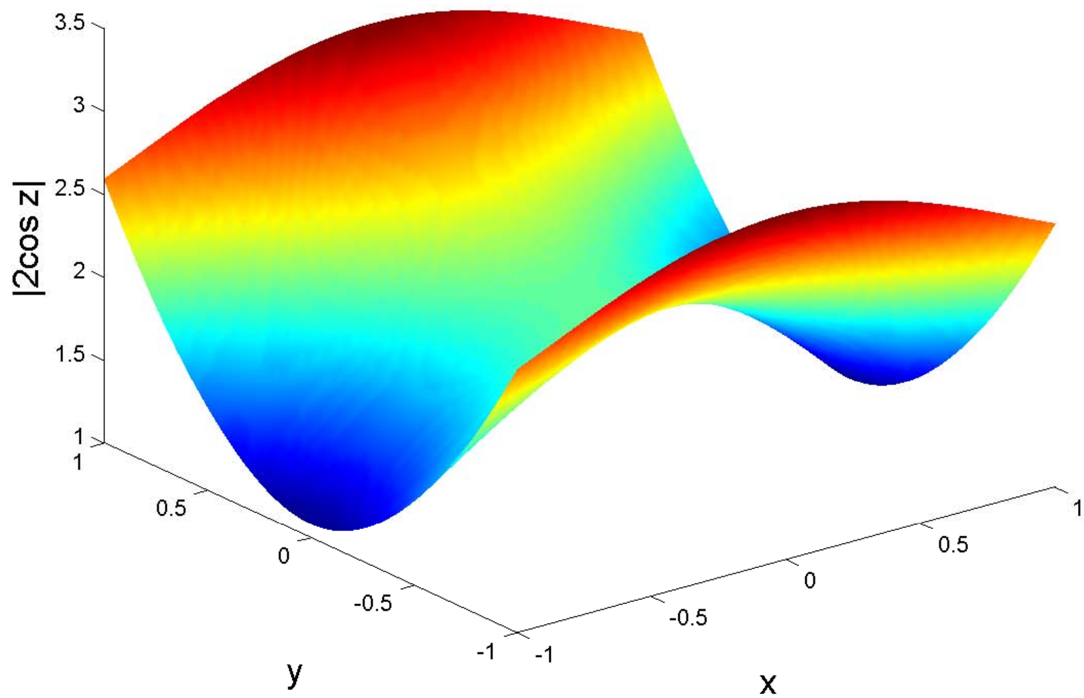

Figure 31: Graphic illustration of the modulus $|f|$ of the complex function $f = 2\cos z$ over the square region R in the z plane defined by $-1 \le x \le 1$ and $-1 \le y \le 1$. As we see, the maximum of the modulus ($\simeq 3.08616$) occurs at $\pm i$ (i.e. at $x = 0$ and $y = \pm 1$) and the minimum of the modulus ($\simeq 1.0806$) occurs at ± 1 (i.e. at $x = \pm 1$ and $y = 0$). See part (c) of Problem 3 of § 4.6 and part (c) of Problem 4 of § 4.6.

minimum (i.e. when $y = 0$). Therefore, the minimum modulus of f over R is $|f| = 2\cos 1 \simeq 1.0806$ and it occurs at the two points $z_{3,4} = \pm 1$ which are on the boundary of R (as required by the minimum modulus theorem). This is graphically illustrated in Figure 31.

(**d**) From part (d) of Problem 3 we have $|f| = e^{-\pi y}$. So, the minimum modulus of f (noting that $-3 \le y \le 3$) is $e^{-3\pi}$ and it occurs at z_2 (where $x = 0$ and $y = 3$) which is on the boundary of R (as required by the minimum modulus theorem).

5. Give some examples of the failure of the maximum modulus theorem due to the violation of some of its conditions.
   Answer: For example:
   • The function $f(z) = 1/z$ does not take its maximum modulus at the boundary of the origin-centered unit disk because it is not analytic over the entire disk (due to the singularity at $z = 0$).
   • The function $f(z) = 1+i$ does not take its maximum modulus at the boundary (specifically) because it is constant.
6. Give some examples of the failure of having the minimum modulus on the boundary according to the minimum modulus theorem.
   Answer: For example:
   • The function $f(z) = \sin z$ does not take its minimum modulus at the boundary of the origin-centered unit disk because it has a zero inside this disk (i.e. at $z = 0$) and hence it takes its minimum modulus there.
   • The function $f(z) = 2-i5$ does not take its minimum modulus at the boundary (specifically) because it is constant.
7. Show that if $f(z)$ is entire and $\lim_{z\to\infty} f(z) = 0$ then f is identically zero, i.e. $f(z) \equiv 0$.
   Answer: This statement can be seen as an application of the maximum modulus theorem because if

z_0 is a given point in the complex plane and D is an origin-centered disk of radius ρ that encloses z_0 then according to the maximum modulus theorem the maximum M of $|f|$ over D is on the boundary of D and cannot be at z_0 and hence we should have $|f(z_0)| \le M$ (where the equality holds if f is constant). Now, if $\rho \to \infty$ then $f(z) \to 0$ $\left[\text{since } \lim_{z\to\infty} f(z) = 0\right]$ and hence $|f| \to 0$ and $M \to 0$ and thus we get $|f(z_0)| \le 0$, i.e. $|f(z_0)| = 0$ since $|f(z_0)|$ is non-negative. Noting that z_0 can represent any point in the complex plane, we conclude that f is identically zero on the entire complex plane.

The statement may also be seen (rather more straightforwardly) as an application of Liouville's theorem (see § 4.5) because since f is entire then it is bounded on the (finite) complex plane (see part c of Problem 7 of § 1.9) and since its limit at infinity is zero then it is bounded there as well and hence it is bounded on the extended complex plane. So, f is entire (on the complex plane) and bounded (on the extended complex plane) and hence it should be constant (i.e. zero since its value at infinity is zero) according to Liouville's theorem.

Note: the second proof (i.e. the one based on Liouville's theorem) may seem conceptually problematic. However, it should become more acceptable according to some forms of the statement of Liouville's theorem, e.g. "a function that is analytic in the extended complex plane is constant" (see Problem 2 of § 4.5).

Chapter 5
Series Expansion of Complex Functions

In this chapter we investigate series expansion of complex functions which bears a strong resemblance to the series expansion of real functions in real analysis. In fact, complex series is a big subject and hence what we present in this chapter is just a glimpse (where we rely in many cases and aspects on the presumed familiarity with real series from calculus). Accordingly, there are many gaps and lack of details in the investigation of this chapter. The inquisitive readers should therefore look for more specialized texts on complex series.

In the following bullet points we outline some of the properties and issues of complex series:[217]

• An infinite complex series is convergent if the sequence of its partial sums has a limit as the number of terms tends to infinity.[218]

• The value of a convergent infinite series at any point in its domain of validity (i.e. where it converges) is the value of the limit which we mentioned in the previous bullet point.

• A complex series, like any complex variable or function, is made of a real part and an imaginary part where each one of these parts is a real series.

• A complex series is convergent *iff* its real and imaginary parts are convergent (where the real part series converges to the real part of the series and the imaginary part series converges to the imaginary part of the series).

• As indicated above, most of the properties and rules of real series apply to complex series. For example, the convergence tests of real series (which are investigated in calculus) generally apply to complex series. Also, the types of convergence of a series (e.g. absolute convergence, conditional convergence, uniform convergence, pointwise convergence, etc.) generally apply to complex series as to real series where they have similar definitions and criteria in both cases (with some minor adaptations). We also have a Taylor (and Maclaurin) type series in the complex domain as in the real domain (although a Laurent type series is generally restricted to the complex domain for reasons that will become clear in the future).

• As we will see, any convergent series represents a function whether this function has a known standard closed form (like e^z or $\cosh z$) or not. Accordingly, when we talk about series expansion of complex functions (which is the title of the present chapter and is used in the text) "function" should be understood in this extended sense although we are generally interested in (and will deal with) only familiar standard functions of known closed forms (as well as their combinations and compositions).

• Power series can be added and subtracted to produce a new series whose radius of convergence[219] is the smaller of the radii of convergence of the added/subtracted series.

• Two power series can be multiplied to produce a new series called the Cauchy product of the multiplied series (where the radius of convergence is also the smaller).

• Power series can be differentiated and integrated termwise to produce new series representing the derivative and integral of the function represented by the original series (where the radius of convergence is the same as that of the original series).[220]

• Power series generally converge absolutely and uniformly in their disk of convergence.

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[217] We note that some of these points are roughly-stated generalizations to give a basic idea about complex series and their properties and behavior and hence they may require some restrictions and polishing to be thorough and precise.

[218] It should be noted that the definition of "convergence of complex sequence" is similar to the definition of convergence of real sequence (as given in calculus). More specifically, a complex sequence $z_1, z_2, \cdots, z_n, \cdots$ is convergent to a given limit L if for any positive real number ε there exists a positive integer N such that $|z_n - L| < \varepsilon$ for all $n \geq N$.

[219] If a complex function $f(z)$ represented by a given complex power series around a given point z_0 in the complex plane has a singularity or singularities then the distance between z_0 and the nearest singularity of f is known as the radius of convergence of the series, i.e. the series converges to the function inside the z_0-centered disk identified by that radius (and this disk is called the disk of convergence). Also see § 5.3.

[220] To be more accurate, uniform convergence is required (which is generally satisfied by power series).

5.1 Taylor and Maclaurin Series of Complex Functions

In general, complex analysis follows the style and formalism of real analysis with regard to the Taylor and Maclaurin series of functions.[221] For example, in real analysis the cosine function has the Maclaurin series $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ ($\forall\ x \in \mathbb{R}$) while in complex analysis it has the Maclaurin series $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$ ($\forall\ z \in \mathbb{C}$) which is identical in form. Also, the convergence tests (such as the ratio test) of complex series are generally similar to those of real series (as we will see). However, certain characterizations and distinctions naturally arise between real and complex series due to the obvious difference between real and complex variables. For example, in real analysis the "radius of convergence" of a series identifies a 1D "interval of convergence" on a line (since the domain of real series is the real line) while in complex analysis the radius of convergence of a series identifies a 2D disk in the complex plane (since the domain of complex series is the complex plane).

Accordingly, the standard expansion of the Taylor series of a given analytic complex function $f(z)$ around a given point z_0 is:[222]

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \frac{f'''(z_0)}{3!}(z-z_0)^3 + \cdots \quad (189)$$

where $f^{(n)}(z_0)$ is the n^{th} derivative of f at z_0 (with $f^{(0)}$ being the function itself), the prime means derivative with respect to z (i.e. d/dz), and ! stands for factorial. Similarly, the Maclaurin series of that function is:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \cdots \quad (190)$$

which is no more than Eq. 189 with the replacement of z_0 by 0 since the Maclaurin series is just the Taylor series at $z_0 = 0$. Some examples for the application and use of these series expansions will be given in the upcoming Problems.

However, before we go through these Problems we should draw the attention of the reader to the following important remarks:

• Since the Maclaurin series is a special case of the Taylor series, any discussion of Taylor series in the future should include Maclaurin series (as a special case) even if it is not mentioned explicitly.

• As indicated above, for a function $f(z)$ to have a Taylor series expansion at a given point z_0, f should be analytic at z_0. This may be inferred from Eqs. 189 and 190 where derivatives of f at z_0 are required (also see the next point).

• Because $f(z)$ in the above formulation is assumed to be analytic at z_0 then (according to the result of Problem 6 of § 4.3) it is infinitely differentiable (with its derivatives of all orders being analytic) and this should guarantee the existence of its Taylor series as given by Eq. 189 (noting that other issues related to its Taylor series, such as the convergence of the series to f and the radius of this convergence, still require establishment and verification).

• The Taylor series of a given function $f(z)$ at a given point z_0 is unique and hence if it is found then that is it regardless of the method by which it is obtained. In other words, if we found "a" Taylor series (within the indicated specifications and restrictions) then we found "the" Taylor series (within these specifications and restrictions).

• The point z_0 (or 0) in the above expansions of Eqs. 189 and 190 is the center of the disk of convergence of the Taylor series.[223] This is graphically illustrated in Figure 32.

[221] This similarity, in fact, will ease our task in presenting the subject of complex series since we rely on the presumed familiarity of the reader with the mathematics of real series. So, we will not go through many details in this regard because they are assumed to be part of the background knowledge.

[222] We are assuming implicitly that f has a Taylor series (which is justified by assuming analyticity of f as will be explained next). This also applies to the upcoming Maclaurin series. More clarifications about these issues will follow.

[223] As indicated earlier, the disk of convergence (or disk of analyticity) is the z_0-centered disk D in the z plane where the Taylor series at any point in D converges to a definite value and this value is equal to the value of f at that point. This will be investigated and clarified further later on.

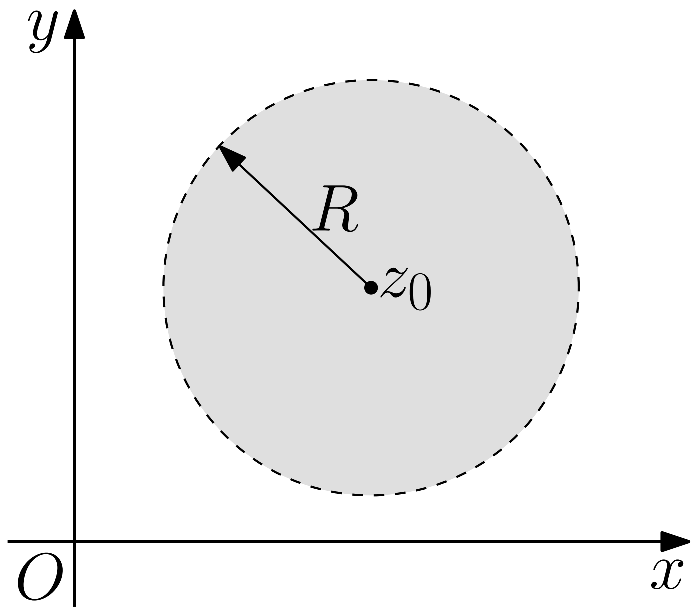

Figure 32: Graphic illustration of the disk of convergence (shaded gray) with radius of convergence R for a Taylor series expansion of a function $f(z)$ at $z = z_0$ (where for Maclaurin series expansion $z_0 = 0$, i.e. the origin of the complex plane). See § 5.1.

• As indicated earlier, the Taylor series as a sum that has a given value for a given z is an analytic function (in itself) inside its disk of convergence (i.e. where it converges to a definite finite value) regardless of being representing a given function (of standard closed form such as e^z or $\cos z$) or not. Accordingly, it can be treated like any formal analytic function and hence it can, for instance, be differentiated or integrated as such.
• The termwise derivative of the Taylor series of an analytic function (at a given point) is the Taylor series of the derivative of that function (at that point).
• The Taylor series of a linear combination of two analytic functions (at a given point) is the linear combination of the Taylor series of these functions (at that point).
• The Taylor series of the product of two analytic functions (at a given point) is the product of the Taylor series of these functions (at that point).

Problems

1. List the main elements that identify a Taylor series.
   Answer: We have:
   • The center of the series which is the point z_0 at which the series is generated (or expanded).
   • The disk of convergence which is the z_0-centered disk inside which the series converges and outside which the series diverges.[224]
   • The circle of convergence which is the perimeter of the disk of convergence (where the series may or may not converge at individual points on it).
   • The radius of convergence R which is the radius of the disk of convergence and it can take any non-negative real value (including 0 when the series converges only at its center and ∞ when the series converges on the entire complex plane).
2. Propose a simple and general reason to justify why the complex series should have the same form as their corresponding real series.

[224] Although the *series* diverges outside, the function represented by the series may still be analytic (and hence defined and convergent) there thanks, for instance, to analytic continuation (see § 1.5 and Problem 4 of § 7.1) by a series expansion around a neighboring point.

Answer: The justification of this is consistency because real numbers are a subset of complex numbers and hence if a complex series should apply to this subset (as to the set of complex numbers) then it should produce the same form when applied to this subset and this requires the corresponding complex and real series to have the same form.

3. Give some properties of Taylor series that can be imported from real analysis to complex analysis (i.e. they apply to complex as to real).
Answer:[225] For instance:
• The Taylor series of a given function at a given point is unique.
• Two (or more) Taylor series (at a given point) can be added or subtracted or multiplied where the resultant series is convergent inside the common disk of convergence of the original series. Also, two Taylor series can be divided where the convergence of the resulting series depends on the convergence of the original series and the zeros of the denominator series (as well as if these zeros are shared by the numerator series or not with some extra conditions and details).
• If $S_1(z)$ is a Taylor series convergent at z_1 and S_2 is a Taylor series convergent at the value of S_1 at z_1 $\big[$i.e. $S_1(z_1)\big]$ then the series S obtained by substituting S_1 into S_2 converges at z_1 to the value of S_2 at $S_1(z_1)$, i.e. $S(z_1) = S_2(S_1(z_1))$.
• The Taylor series S of a given function f can be differentiated termwise where S' (i.e. the termwise derivative) converges to f' (i.e. the derivative of f) at their corresponding points (noting that S' converges to f' inside the disk of convergence of S to f). Similarly, the Taylor series S of a given function f can be integrated termwise where $\int S$ (i.e. the termwise integral) converges to $\int f$ (i.e. the integral of f) at their corresponding points (noting that $\int S$ converges to $\int f$ inside the disk of convergence of S to f).

4. Give general criteria and conditions for expanding a complex function in a convergent Taylor series.
Answer: In brief, any complex function $f(z)$ that is analytic (although it may have some singularities) in a given region R within the complex plane can be expanded uniquely in a Taylor series around any (non-singular) point z_0 inside R where this series converges to f within a z_0-centered disk of convergence that extends to the nearest singularity of f to z_0.

5. Give the complex version of some examples of common Maclaurin series found in the texts of real analysis.
Answer: For example:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \qquad (\forall\ z \in \mathbb{C}) \tag{191}$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \qquad (\forall\ z \in \mathbb{C}) \tag{192}$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad (\forall\ z \in \mathbb{C}) \tag{193}$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots \qquad (|z| < 1) \tag{194}$$

$$\ln(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \qquad (|z| < 1) \tag{195}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \qquad (\forall\ z \in \mathbb{C}) \tag{196}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots \qquad (\forall\ z \in \mathbb{C}) \tag{197}$$

[225] The purpose of the question and answer is to have a taste of the common properties between real and complex Taylor series and hence some points lack sufficient details and restrictions.

6. Give some examples of the methods used to find the Taylor or Maclaurin series of a given function.
Answer:[226] For example:
• We may use the general standard expansion of that series. For instance, the standard expansion of the Taylor series or Maclaurin series (as given by Eqs. 189 and 190) may be used to find the series of a given complex function $f(z)$ at a given point z_0. This method is general and flexible but it is normally lengthy and demanding.
• We may use combination and composition relations in association with known standard series. For example, if we know the Maclaurin series of e^z then we can simply generate the Maclaurin series of e^{-z} or e^{iz} or e^{2z+1} (and any similar Maclaurin series of an exponential function) by substituting these arguments (i.e. $-z$ and iz and $2z+1$ which correspond to z) in the place of z in the known standard Maclaurin series.
• We may use the series of another function which is related to the function of interest. For instance, we can find the series of the hyperbolic/trigonometric functions from the series of the trigonometric/hyperbolic functions by exploiting the relations between these functions (as given and investigated for instance in Problem 5 of § 2.3). So, if we know the Maclaurin series of $\cos z$ and $\sinh z$ and we do not know the Maclaurin series of $\cosh z$ and $\sin z$ then we can generate these series as follows (see Eqs. 192 and 197 as well as Eqs. 196 and 193):

$$\begin{aligned}
\cosh z &= \cos(iz) = \sum_{n=0}^{\infty} \frac{(-1)^n (iz)^{2n}}{(2n)!} = 1 - \frac{i^2 z^2}{2!} + \frac{i^4 z^4}{4!} - \cdots = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \\
\sin z &= -i \sinh(iz) = -i \sum_{n=0}^{\infty} \frac{(iz)^{2n+1}}{(2n+1)!} = -i \left(iz + \frac{i^3 z^3}{3!} + \frac{i^5 z^5}{5!} + \cdots \right) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}
\end{aligned}$$

Similarly, if we know the Maclaurin series of e^z (and hence we know the Maclaurin series of e^{iz} and e^{-iz}) and we do not know the Maclaurin series of $\cos z$ and $\sin z$ then we can generate the unknown series from the known series by using the relations $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \frac{e^{iz}-e^{-iz}}{i2}$ (see Eq. 131). On the other hand, the (unknown) series of e^{iz} can be obtained from the (known) series of $\cos z$ and $\sin z$ by using the relation $e^{iz} = \cos z + i \sin z$.
• We may use termwise differentiation or integration of a series of a given function to obtain the series of the derivative or integral of that function.
• We may express the function as an algebraic sum of simpler functions (using for example partial fractions) and use the series of these simpler functions to generate the series of the function.
• We may use specialized methods that apply in the particular situation. For example, the series of $\frac{1}{1-z}$ can be generated by applying the binomial theorem expansion on $(1-z)^{-1}$ or by performing long division of 1 by $(1-z)$.

7. Find the Maclaurin series expansion of the polynomial functions.
Answer: The general form of an m^{th} order polynomial function is:

$$\sum_{k=0}^{m} a_k z^k = a_0 + a_1 z + a_2 z^2 + \cdots + a_{m-1} z^{m-1} + a_m z^m$$

where a_k's are constants (with $a_m \neq 0$) and m is a non-negative integer. Now, if we use the Maclaurin series expansion then we have:

$$f(0) = a_0, \qquad f'(0) = a_1, \qquad f''(0) = 2!\, a_2, \qquad \cdots \qquad f^{(m-1)}(0) = (m-1)!\, a_{m-1}, \qquad f^{(m)}(0) = m!\, a_m$$

[226] We note that (unknown) complex series can in general be obtained from their corresponding (known) real series (and vice versa). However, this is not included or investigated in this answer because this is not a method but it is a kind of a rule. See Problem 2.

while all higher derivatives are zero. Hence, from the Maclaurin series (see Eq. 190) we get:

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{m} \frac{f^{(n)}(0)}{n!} z^n + \sum_{n=m+1}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{m} \frac{n!\, a_n}{n!} z^n + \sum_{n=m+1}^{\infty} 0 \\ &= \sum_{n=0}^{m} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots + a_{m-1} z^{m-1} + a_m z^m \end{aligned}$$

This means that the Maclaurin series expansion of a polynomial function is itself.

8. Generate the Maclaurin series of the following complex functions:

(**a**) $f(z) = \frac{1}{1-z}$. (**b**) $f(z) = \cos z^2$. (**c**) $f(z) = \ln(1-z)$. (**d**) $f(z) = \sinh z$.
(**e**) $f(z) = \ln \frac{1-z}{1+z}$. (**f**) $f(z) = \frac{1}{(1-z)^2}$. (**g**) $f(z) = \arctan z$. (**h**) $f(z) = \frac{2}{1-z^2}$.

Answer: In this answer we just consider the form of the series without any other consideration such as its radius of convergence. Moreover, the main purpose of this Problem is to demonstrate the various methods used to generate power series and hence some of the methods used may not be the best for generating the given series.

(**a**) We can, for instance, use the standard Maclaurin series expansion, that is:

$$f(0) = \frac{1}{1-0} = 1, \qquad f'(0) = \frac{1}{(1-0)^2} = 1!, \qquad f''(0) = \frac{1 \times 2}{(1-0)^3} = 2!, \qquad f^{(n)}(0) = \frac{n!}{(1-0)^{n+1}} = n!$$

Hence, from the Maclaurin series expansion (see Eq. 190) we get:

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{n!}{n!} z^n = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots$$

which is as given by Eq. 194.

(**b**) The simplest way is to replace z in the Maclaurin series for $\cos z$ (see Eq. 192) by z^2, that is:

$$\cos z^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(z^2\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{4n} = 1 - \frac{z^4}{2!} + \frac{z^8}{4!} - \cdots$$

(**c**) We simply replace $+z$ in the series of $\ln(1+z)$ which we gave earlier (see Eq. 195) by $-z$, that is:

$$\ln(1-z) = \ln\left(1 + \left[-z\right]\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-z)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} z^n = \sum_{n=1}^{\infty} \frac{-z^n}{n} = -z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots$$

(**d**) We can, for instance, use the definition of $\sinh z$ (see Eq. 133) in conjunction with the series of e^z, that is:

$$\begin{aligned} \sinh z &= \frac{e^z - e^{-z}}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{(-z)^n}{n!}\right) = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z^n}{n!} + \frac{(-1)^{n+1} z^n}{n!}\right) \\ &= \frac{1}{2}\left(0 + 2\frac{z}{1!} + 0 + 2\frac{z^3}{3!} + 0 + 2\frac{z^5}{5!} + \cdots\right) = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \end{aligned}$$

which is as given by Eq. 197.

(**e**) From the rules of logarithm (see Problem 5 of § 2.2) we have:

$$\ln \frac{1-z}{1+z} = \ln(1-z) - \ln(1+z)$$

Now, the Maclaurin series of $\ln(1+z)$ is given by Eq. 195, while the Maclaurin series of $\ln(1-z)$ can be obtained from Eq. 195 by replacing $+z$ by $-z$ (as done in part c). Hence:

$$\ln \frac{1-z}{1+z} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-z)^n - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1} + (-1)^{n+2}}{n} z^n$$

$$= \sum_{n=1}^{\infty} \frac{-1+(-1)^n}{n} z^n = -2\sum_{n=1}^{\infty} \frac{z^{2n-1}}{2n-1} = -2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots\right)$$

(**f**) $\frac{1}{(1-z)^2}$ is the derivative of $\frac{1}{1-z}$ and hence the Maclaurin series of $\frac{1}{(1-z)^2}$ can be obtained by termwise differentiation of the Maclaurin series of $\frac{1}{1-z}$ (which is given by Eq. 194 and obtained in part a), that is:

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n\right) = \sum_{n=1}^{\infty} nz^{n-1} = 1 + 2z + 3z^2 + 4z^3 + \cdots$$

(**g**) $\arctan z$ is the integral of $\frac{1}{1+z^2}$ (see Problem 6 of § 2.4) and hence the Maclaurin series of $\arctan z$ can be obtained by termwise integration of the Maclaurin series of $\frac{1}{1+z^2}$. Now, the series of $\frac{1}{1+z^2}$ can be obtained from the series of $\frac{1}{1-z}$ (which is given by Eq. 194 and obtained in part a), that is:

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + \cdots$$

Hence:

$$\begin{aligned}\arctan z &= \int \frac{dz}{1+z^2} = \int \left(\sum_{n=0}^{\infty}(-1)^n z^{2n}\right) dz = \sum_{n=0}^{\infty}(-1)^n \left[\int z^{2n} dz\right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1} = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots\end{aligned}$$

(**h**) On splitting f by partial fractions and using the series of part (a) we get:

$$\begin{aligned}\frac{2}{1-z^2} &= \frac{1}{1-z} + \frac{1}{1+z} = \frac{1}{1-z} + \frac{1}{1-(-z)} = \left(\sum_{n=0}^{\infty} z^n\right) + \left(\sum_{n=0}^{\infty}(-z)^n\right) \\ &= \sum_{n=0}^{\infty}\left[1+(-1)^n\right] z^n = 2\sum_{n=0}^{\infty} z^{2n} = 2\left(1 + z^2 + z^4 + z^6 + \cdots\right)\end{aligned}$$

9. Generate the Taylor series of the following complex functions around the given points z_0:
(**a**) $f(z) = \frac{1}{z}$ at $z_0 = -1$. (**b**) $f(z) = \cosh z$ at $z_0 = i$. (**c**) $f(z) = \frac{1}{z+2}$ at $z_0 = 1$.
Answer: Again, we consider only the form of the series without any other consideration.
(**a**) We can, for instance, use the Maclaurin series for $\frac{1}{1-z}$ which we gave and generated earlier (see Eq. 194 and part a of Problem 8) with some algebraic manipulation, that is:

$$f(z) = \frac{1}{z} = \frac{-1}{-z} = \frac{-1}{1-z-1} = \frac{-1}{1-(z+1)} = \sum_{n=0}^{\infty} -(z+1)^n = -1 - (z+1) - (z+1)^2 - \cdots \quad (198)$$

We can also use the standard Taylor series expansion at $z_0 = -1$, that is:

$$f(-1) = \frac{1}{-1} = -1, \quad f'(-1) = \frac{-1}{(-1)^2} = -1!, \quad f''(-1) = \frac{1\times 2}{(-1)^3} = -2!, \quad f^{(n)}(-1) = \frac{(-1)^n\, n!}{(-1)^{n+1}} = -n!$$

Hence, from the Taylor series expansion (see Eq. 189) we get:

$$f(z) = \frac{1}{z} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(z+1)^n = \sum_{n=0}^{\infty} \frac{-n!}{n!}(z+1)^n = \sum_{n=0}^{\infty} -(z+1)^n$$

(**b**) We can, for instance, use the standard Taylor series expansion at $z_0 = i$, that is:[227]

$$\begin{aligned}
f(i) &= \cosh i = \cos 1 \\
f'(i) &= \sinh i = i \sin 1 \\
f''(i) &= \cosh i = \cos 1 \\
f'''(i) &= \sinh i = i \sin 1 \\
&\vdots \\
f^{(n)}(i) &= \frac{(-1)^n + 1}{2} \cosh i + \frac{(-1)^{n+1} + 1}{2} \sinh i = \frac{(-1)^n + 1}{2} \cos 1 + i \frac{(-1)^{n+1} + 1}{2} \sin 1
\end{aligned}$$

Hence, from the Taylor series expansion (see Eq. 189) we get:

$$\begin{aligned}
f(z) &= \cosh z = \sum_{n=0}^{\infty} \frac{f^{(n)}(i)}{n!} (z-i)^n = \sum_{n=0}^{\infty} \left[\frac{(-1)^n + 1}{2} \cos 1 + i \frac{(-1)^{n+1} + 1}{2} \sin 1 \right] \frac{(z-i)^n}{n!} \\
&= \cos 1 + i \sin 1\, (z-i) + \frac{\cos 1}{2} (z-i)^2 + i \frac{\sin 1}{6} (z-i)^3 + \frac{\cos 1}{24} (z-i)^4 + \cdots
\end{aligned}$$

(**c**) We can, for instance, use the binomial theorem with some algebraic manipulation, that is:

$$\begin{aligned}
f(z) &= \frac{1}{z+2} = \frac{1}{3 + (z-1)} = \frac{1}{3} \left[\frac{1}{1 + \left(\frac{z-1}{3}\right)} \right] = \frac{1}{3} \left[1 + \left(\frac{z-1}{3} \right) \right]^{-1} \\
&= \frac{1}{3} \left[1 + \frac{(-1)}{1!} \left(\frac{z-1}{3} \right) + \frac{(-1)(-2)}{2!} \left(\frac{z-1}{3} \right)^2 + \frac{(-1)(-2)(-3)}{3!} \left(\frac{z-1}{3} \right)^3 + \cdots \right] \\
&= \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{3} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (z-1)^n = \frac{1}{3} - \frac{(z-1)}{9} + \frac{(z-1)^2}{27} - \frac{(z-1)^3}{81} + \cdots
\end{aligned}$$

We may also use the Maclaurin series for $\frac{1}{1-z}$ which we gave and verified earlier (see Eq. 194 and part a of Problem 8) with some algebraic manipulation, that is:

$$\begin{aligned}
f(z) &= \frac{1}{z+2} = \frac{1}{3 + (z-1)} = \frac{1}{3} \left[\frac{1}{1 - \left(\frac{1-z}{3}\right)} \right] = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{1-z}{3} \right)^n \\
&= \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{3} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (z-1)^n = \frac{1}{3} - \frac{(z-1)}{9} + \frac{(z-1)^2}{27} - \frac{(z-1)^3}{81} + \cdots
\end{aligned}$$

10. Explain briefly the ratio test.
Answer: The ratio test for examining the convergence of a series $\sum a_n$ (whether real or complex) states that: if

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

then the series converges if $L < 1$ and diverges if $L > 1$, and the test is inconclusive if $L = 1$ or the limit does not exist. We note that a_n and a_{n+1} in the limit of the ratio test represent the general expression of the n^{th} and $(n+1)^{th}$ term of the series. We also note that when we talk about the convergence of a series (in the context of the ratio test and other similar convergence tests) we mean the convergence of the series to the function that it represents, i.e. the series will yield the same value as the function that it represents (e.g. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ will yield the same value as the function e^z for a given z within its radius of convergence).
Note: there is a more general version of the ratio test, but we do not need it in this book.

[227] See Problem 5 of § 2.3 for the relations between the trigonometric and hyperbolic functions. We should also note that symbols like $f'(i)$ means $f'(z)$ at $z = i$ which may also be symbolized as $f'(z)|_{z=i}$ or $f'(z = i)$.

11. Which of the following series is convergent:
(**a**) $\sum_{n=0}^{\infty}(2-i)^n$. (**b**) $\sum_{n=0}^{\infty}\frac{i}{(1-i)^n}$. (**c**) $\sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^n$. (**d**) $\sum_{n=0}^{\infty}\frac{(1+i2)^n}{n!}$.
Answer: From the ratio test we can see that all these series are convergent except (a), that is:

$$(\mathbf{a})\quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2-i)^{n+1}}{(2-i)^n}\right| = \lim_{n\to\infty}|2-i| = \sqrt{5} > 1$$

$$(\mathbf{b})\quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{i/(1-i)^{n+1}}{i/(1-i)^n}\right| = \lim_{n\to\infty}\left|\frac{1}{1-i}\right| = \frac{1}{\sqrt{2}} < 1$$

$$(\mathbf{c})\quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(i/2)^{n+1}}{(i/2)^n}\right| = \lim_{n\to\infty}\left|\frac{i}{2}\right| = \frac{1}{2} < 1$$

$$(\mathbf{d})\quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(1+i2)^{n+1}}{(n+1)!}\,\frac{n!}{(1+i2)^n}\right| = \lim_{n\to\infty}\left|\frac{1+i2}{n+1}\right| = 0 < 1$$

12. Use the ratio test (see Problem 10) to determine for which values of z the following complex series converge:
(**a**) $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$. (**b**) $\cos z = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}$. (**c**) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$.
(**d**) $\sinh z = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}$. (**e**) $\ln(1+z) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}z^n$.
Answer:
(**a**)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{z^{n+1}}{(n+1)!}\,\frac{n!}{z^n}\right| = \lim_{n\to\infty}\left|\frac{z}{n+1}\right| = 0 < 1$$

As we see, this limit (which is < 1) is independent of z (i.e. it applies to any z). Hence, the complex series of e^z converges for all values of z (i.e. on the entire complex plane $\mathbb{C}$).
(**b**)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\,z^{2n+2}}{(2n+2)!}\,\frac{(2n)!}{(-1)^n\,z^{2n}}\right| = \lim_{n\to\infty}\left|\frac{z^2}{(2n+2)\,(2n+1)}\right| = 0 < 1$$

Again, this limit (which is < 1) is independent of z. Hence, the complex series of $\cos z$ converges for all values of z (i.e. on the entire complex plane $\mathbb{C}$).
(**c**)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{z^{n+1}}{z^n}\right| = \lim_{n\to\infty}|z| = |z|$$

Hence, the complex series of $\frac{1}{1-z}$ converges for all values of z with $|z| < 1$ (i.e. on the interior of the origin-centered unit disk).
(**d**)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{z^{2n+3}}{(2n+3)!}\,\frac{(2n+1)!}{z^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{z^2}{(2n+3)\,(2n+2)}\right| = 0 < 1$$

This limit is independent of z and hence the complex series of $\sinh z$ converges for all values of z (i.e. on the entire complex plane $\mathbb{C}$).
(**e**)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\,z^{n+1}}{n+1}\,\frac{n}{(-1)^{n+1}\,z^n}\right| = \lim_{n\to\infty}\left|\frac{nz}{n+1}\right| = |z|$$

Hence, the complex series of $\ln(1+z)$ converges for all values of z with $|z| < 1$ (i.e. on the interior of the origin-centered unit disk).

13. There seems to be a consensus in the literature that analytic functions and Taylor series are equivalent, i.e. each analytic function can be represented by a Taylor series and each Taylor series represents an analytic function (although it may not be of standard form). Discuss this issue.

Answer: We have some reservations on such generalizations. For example:
• A Taylor series that converges only on its center (e.g. $\sum_{n=1}^{\infty} n!\, z^n$) cannot represent an analytic function since it is defined only on a single point. So, to avoid this we should restrict the Taylor series (that represents an analytic function) to be of positive radius of convergence.
• Taylor series (or at least Taylor-like series) can be generated at removable singularities (e.g. the series $1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$ which represents the function $\frac{\sin z}{z}$ which has a removable singularity at $z = 0$) and hence we can claim that we have a "Taylor series" at a singular point.[228] So, to avoid this we should consider removable singularities as "analytic points" (i.e. points at which the function is analytic).

5.2 Laurent Series of Complex Functions

As indicated earlier (see § 5.1), for a function $f(z)$ to have a Taylor (or Maclaurin) series expansion at a given point z_0 f should be analytic at z_0. Moreover, this series is convergent only inside its disk of convergence (which is confined by the singularity nearest to the center of the series). So, if we need a series expansion of a complex function at a singular point or/and we need a series expansion beyond the nearest singularity then we should look for another type of series expansion (because we have no Taylor series in these cases). This other type of series expansion is the Laurent series which is a power series expansion that applies within an annular region (in the complex plane) centered on a given point z_0 (where z_0 is possibly singular).[229] To be more formal, let have a complex function $f(z)$ and let C_1 and C_2 be two circles centered on a given point z_0 (which is possibly a singularity of f) where these circles define a z_0-centered annular region. Now, if $f(z)$ is analytic over the annulus and C is a closed curve surrounding z_0 and it is inside the annulus (i.e. C is between C_1 and C_2) then the Laurent series expansion of $f(z)$ at z_0 is given by:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \cdots + a_{-2}(z - z_0)^{-2} + a_{-1}(z - z_0)^{-1} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \quad (199)$$

where

$$a_n = \frac{1}{i2\pi} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad (n = 0, \pm 1, \pm 2, \cdots) \quad (200)$$

Referring to Eq. 181, we can see that $a_n = f^{(n)}/n!$ and hence Laurent series is identical in form to Taylor series (see Eq. 189) provided that we can extend n to negative values (which requires a non-trivial generalization of $f^{(n)}$ and $n!$).

It is important to be aware of the following remarks about the Laurent series expansion (and related issues):
• The above expansion of $f(z)$ (as given by Eqs. 199 and 200) is valid (i.e. it converges to f) inside the annulus of analyticity (or the annulus of convergence). In fact, if the center of the annulus (i.e. z_0) is the only singularity of f (assuming z_0 to be a singularity of f) inside the annulus then the annulus of analyticity becomes a punctured disk. Moreover, if z_0 is the only singularity of f in the complex plane then the annulus of analyticity becomes the entire complex plane (excluding z_0).
• If a function $f(z)$ has a number of singularities at different distances from the center of the series z_0 then these singularities will divide the domain of f to concentric annuli where the Laurent series in each one of these annuli of analyticity is generally different from the series in the other annuli.
• The Laurent series (whenever it exists) for a given function around a given point and in a given annulus of analyticity is unique, and hence if it is found by any method then that is it, regardless of the method used to obtain the series (i.e. when we find "a" Laurent series we actually find "the" Laurent series).
• The Laurent series expansion is a (non-trivial) generalization of Taylor series expansion (as indicated above in the reference to Eq. 181 next to Eq. 200), and hence Taylor expansion can be seen as a special

[228] We should also note in this context what will be indicated later that is removable singularities do not define the radius of convergence of a Taylor series (see footnote [241] on page 233).

[229] We should note that we use "power series" here in a rather non-technical (or non-conventional) sense. However, the meaning should be clear.

case of Laurent expansion (noting that Maclaurin is a special case of Taylor and hence it is also a special case of Laurent).

• Although Laurent series is introduced (in part) as a demand for a series expansion around singular points (as well as a demand for a series expansion beyond the nearest singularity), Laurent series can also be obtained around non-singular points (as it should be obvious from the above introduction). In fact, this provides the link between Taylor (including Maclaurin) and Laurent and illuminates the relation between them because at a singular point we have a Laurent series but we do not have a Taylor series while at a non-singular point we have both Laurent and Taylor series (which are the same within the immediate neighborhood of the point), i.e. the Laurent series at a non-singular point is the same as the Taylor series at that point (and this should provide more clarification to the previous remark).[230] However, we should remember that beyond the nearest singularity we have only Laurent series (regardless of z_0 being singular or not). In fact, Laurent series (in its strict sense as opposite to Taylor series) is conditioned by the existence of a singular point inside the annulus (i.e. in the "hole" of the annulus not in the annular region itself) whether the series is centered at that point or not. So, for Taylor series to exist (i.e. as a special case of Laurent series in its general sense that includes Taylor series) there should be no singularity within the region of convergence (or the "annulus" of analyticity which becomes a disk in this case). This is illustrated graphically in Figure 33.

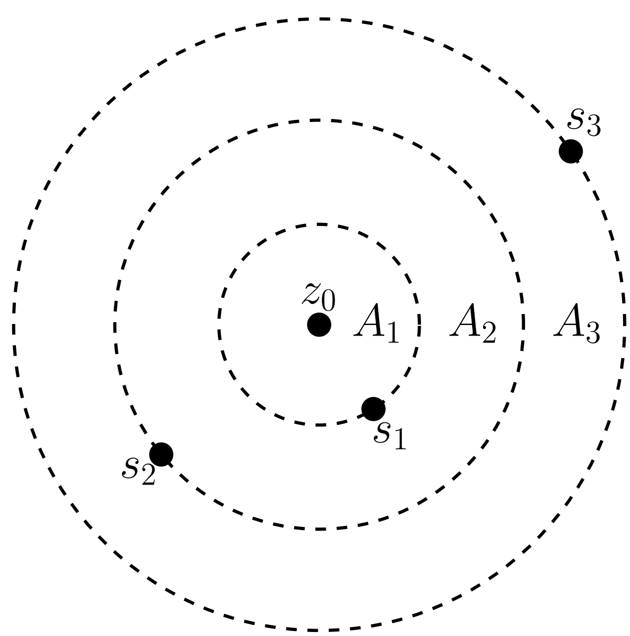

Figure 33: Graphic illustration of the annuli of analyticity (i.e. A_1, A_2, A_3) of an analytic function f (with singularities at s_1, s_2, s_3) which is expanded around the point z_0 where if z_0 is a singular point of f then f has Laurent series in all the annuli A_1, A_2, A_3 while if z_0 is an analytic point of f then f has Laurent series in the annuli A_2, A_3 and a Taylor series (which is the same as the Laurent series) in A_1 (with A_1 being a disk, rather than annulus, of analyticity in this case). See § 5.2.

• The above formulation of Laurent series (as given by Eqs. 199 and 200) is mainly of theoretical value since the generation of the Laurent series by Eqs. 199 and 200 is messy and not very practical (if it is viable at all). Hence, in reality the Laurent series is usually generated by more practical methods; mainly by combining and mixing known Taylor series with other functions and series.[231] This is justified by the fact that Taylor series (which is unique for a given function at a given point) is a special case of Laurent

[230] We are assuming that the conditions for these series are met at the indicated points.

[231] Yes, the above formulation is needed when other methods are not available or they are not less complex than the above formulation.

series and hence if a function can be expressed as a Taylor series or as a combination of Taylor series with other functions and series (as will be clarified in the Problems) then the obtained series should be the Laurent series for that case due to the uniqueness of the Laurent series expansion (as well as the fact that Taylor series is a special case of Laurent series as stated above).

• The part of the Laurent series with negative powers of $(z - z_0)$ is called the **principal part** while the other part is called the **analytic part**. It should be noted that the number of terms in the principal part can be infinite or finite or zero (where in the latter case the Laurent series becomes a Taylor series representing an analytic function, or at least a function with a removable singularity, and hence it may not be called Laurent series). Similarly, the number of terms in the analytic part can be infinite or finite or zero. So in brief, the number of terms in both parts can be infinite or finite or zero (individually or together with some details that can be worked out easily).

• Based on the above explanation about the convergence of Laurent series within z_0-centered annuli of analyticity (which are disk or punctured disk in the immediate neighborhood of z_0), the concept of "radius of convergence" does not apply to the Laurent series (i.e. in the sense of Taylor series) unless the Laurent series is actually a Taylor series (i.e. in the immediate neighborhood of z_0). Yes, the "radius of convergence of Laurent series" may be defined within the annuli of analyticity. Anyway, this "radius of convergence of Laurent series" can be infinite or finite or zero (like the radius of convergence of Taylor series).

• If the lowest negative power in the Laurent series is $-n$ (with n being a positive integer) then f has a pole of order n while if the principal part is made of an infinite number of terms then f has an essential singularity (which may also be called essential pole or pole of infinite order; see § 3.3). As indicated above, if the principal part is missing then the function is analytic (or has a removable singularity).

• The coefficient a_{-1} in the Laurent series of Eq. 199 is called the **residue** of $f(z)$ for reasons that will be explained later (see § 5.4; also see 4.2.2). The residue of analytic function (or with removable singularity) is zero (noting that it may also be zero otherwise).

• For a function that has singularities at separate points in the complex plane, the Laurent series of the function around a given point should consider the different annuli over which the function is analytic where these annuli are defined by the singularities (see Figure 33 and Problem 9). In other words, the Laurent series should be generated separately for each annulus of analyticity and hence in general we have a distinct Laurent series for each annulus.

Problems

1. Let a given function $f(z)$ be analytic in a region R except at z_0 (inside R) which is a pole of order m. Derive the Laurent series of f around z_0 (for this case) and hence justify some of our observations and remarks in the text.

   Answer: Since z_0 is a pole of order m then $g(z) = (z - z_0)^m f(z)$ is *analytic* in a z_0-centered disk in R (including z_0) and hence g should have a *Taylor* series (valid in the disk) around z_0, that is:

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^n \qquad \text{(Eq. 189)}$$

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^n \qquad \left[g(z) = (z - z_0)^m f(z)\right]$$

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} \left[\frac{1}{i2\pi}\oint_C \frac{g(z)}{(z - z_0)^{n+1}}\,dz\right](z - z_0)^n \qquad \text{(Eq. 181)}$$

$$(z - z_0)^m f(z) = \sum_{n=-m}^{\infty} \left[\frac{1}{i2\pi}\oint_C \frac{g(z)}{(z - z_0)^{n+1+m}}\,dz\right](z - z_0)^{n+m}$$

$$f(z) = \sum_{n=-m}^{\infty} \left[\frac{1}{i2\pi}\oint_C \frac{g(z)}{(z - z_0)^{n+1+m}}\,dz\right](z - z_0)^n$$

$$f(z) = \sum_{n=-m}^{\infty} \left[\frac{1}{i2\pi}\oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz\right](z - z_0)^n \qquad \left[g(z) = (z - z_0)^m f(z)\right]$$

From this result we note the following:
• This series has the form of the Laurent series of Eq. 199 with

$$a_n = \frac{1}{i2\pi} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad (n = -m, -m+1, \cdots) \tag{201}$$

noting that for finite m (as in our case) all the terms in Eq. 199 with indices lower than $-m$ are zero.
• As a generalization of Taylor series expansion, Eq. 201 may give sense to the derivatives of negative orders divided by the factorials of negative integers (since a_n in Eq. 201 correspond to $f^{(n)}/n!$ in the Taylor series noting the difference in the range of indices).

2. Let z_0 be a point in the z plane and $f(z)$ be a complex function having a Laurent series around z_0. Use the Laurent series (in its extended sense that includes Taylor series) to investigate the nature of f at z_0.
Answer: We have the following main cases (considering the terms of the principal and analytic parts of the Laurent series):
• If the principal part has an infinite number of terms then f is singular at z_0 with z_0 being an essential singularity (or essential pole).
• If the principal part has a finite (non-zero) number of terms then f is singular at z_0 with z_0 being a pole of order m (where m is a positive integer with $-m$ being the lowest index in the principal part).
• If the principal part is missing then f is analytic at z_0 (or z_0 is a removable singularity of f). Now, if a_0 (in the analytic part) is not zero then f has no zero at z_0; otherwise f has a zero of order $k > 0$ at z_0 where k is the index of the first non-zero term in the series.

3. Find the Laurent series of the following functions around the given points z_0 using known Taylor series:
(**a**) $f(z) = \frac{e^z}{z^3}$ at $z_0 = 0$. (**b**) $f(z) = \frac{\cos z}{z-1}$ at $z_0 = 1$. (**c**) $f(z) = \frac{\sinh z}{(z-i4)^2}$ at $z_0 = i4$.
Answer: As we said, the legitimacy of this method is based on the fact that if Laurent series is found then it is found regardless of the method used.
(**a**) As we see, $f(z)$ is analytic over the entire complex plane except at z_0 where it has a triple pole there and hence it can be represented by a Laurent series around z_0. On using the Taylor series for e^z (which is analytic everywhere even at z_0) we get:

$$f(z) = \frac{e^z}{z^3} = \frac{1}{z^3}\sum_{n=0}^{\infty}\frac{z^n}{n!} = \sum_{n=0}^{\infty}\frac{z^{n-3}}{n!} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!z} + \frac{1}{3!} + \frac{z}{4!} + \frac{z^2}{5!} + \frac{z^3}{6!} + \cdots$$

which is the Laurent series of f at $z_0 = 0$ with a principal part made of only three terms (i.e. $\frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!z}$) and an analytic part made of infinite number of terms (i.e. the rest).
(**b**) As we see, $f(z)$ is analytic over the entire complex plane except at z_0 where it has a simple pole there and hence it can be represented by a Laurent series around z_0. Now, if we transform the coordinates by moving the origin to z_0 then f can be expressed as $f(Z) = \frac{\cos(Z+1)}{Z}$ (where $Z = z - 1$ represents the new coordinates) and hence we can use the Taylor (i.e. Maclaurin) series for $\cos(Z+1)$ at $Z = 0$ $\big[$i.e. where the singularity of $f(Z)\big]$, that is:

$$\begin{aligned}
f(Z) &= \frac{\cos(Z+1)}{Z} = \frac{1}{Z}\sum_{n=0}^{\infty}\left[\frac{(-1)^n+1}{2}(-1)^{n/2}\cos 1 + \frac{(-1)^{n+1}+1}{2}(-1)^{(n+1)/2}\sin 1\right]\frac{Z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left[\frac{(-1)^n+1}{2}(-1)^{n/2}\cos 1 + \frac{(-1)^{n+1}+1}{2}(-1)^{(n+1)/2}\sin 1\right]\frac{Z^{n-1}}{n!} \\
&= \frac{\cos 1}{Z} - \sin 1 - \frac{\cos 1}{2!}Z + \frac{\sin 1}{3!}Z^2 + \frac{\cos 1}{4!}Z^3 - \frac{\sin 1}{5!}Z^4 - \frac{\cos 1}{6!}Z^5 + \frac{\sin 1}{7!}Z^6 + \frac{\cos 1}{8!}Z^7 - \cdots
\end{aligned}$$

Now, if we go back to the original coordinates (by replacing Z by $z - 1$) then we get:

$$f(z) = \frac{\cos 1}{z-1} - \sin 1 - \frac{\cos 1}{2!}(z-1) + \frac{\sin 1}{3!}(z-1)^2 + \frac{\cos 1}{4!}(z-1)^3$$

$$-\frac{\sin 1}{5!}(z-1)^4 - \frac{\cos 1}{6!}(z-1)^5 + \frac{\sin 1}{7!}(z-1)^6 + \frac{\cos 1}{8!}(z-1)^7 - \cdots$$

which is the Laurent series of $f(z)$ at $z_0 = 1$ with a principal part made of a single term (i.e. $\frac{\cos 1}{z-1}$) and an analytic part made of infinite number of terms (i.e. the rest).

(**c**) As we see, $f(z)$ is analytic over the entire complex plane except at z_0 where it has a double pole there and hence it can be represented by a Laurent series around z_0. Now, if we transform the coordinates by moving the origin to z_0 then f can be expressed as $f(Z) = \frac{\sinh(Z+i4)}{Z^2}$ (where $Z = z - i4$ represents the new coordinates) and hence we can use the Taylor (i.e. Maclaurin) series for $\sinh(Z+i4)$ at $Z = 0$ $\big[$i.e. where the singularity of $f(Z)\big]$, that is:

$$\begin{aligned}
f(Z) &= \frac{\sinh(Z+i4)}{Z^2} = \frac{1}{Z^2}\sum_{n=0}^{\infty}\left[\frac{(-1)^n+1}{2}\sinh(i4) + \frac{(-1)^{n+1}+1}{2}\cosh(i4)\right]\frac{Z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left[\frac{(-1)^n+1}{2} i\sin 4 + \frac{(-1)^{n+1}+1}{2}\cos 4\right]\frac{Z^{n-2}}{n!} \\
&= i\frac{\sin 4}{Z^2} + \frac{\cos 4}{Z} + i\frac{\sin 4}{2!} + \frac{\cos 4}{3!}Z + i\frac{\sin 4}{4!}Z^2 + \frac{\cos 4}{5!}Z^3 + i\frac{\sin 4}{6!}Z^4 + \frac{\cos 4}{7!}Z^5 + \cdots
\end{aligned}$$

Now, if we go back to the original coordinates (by replacing Z by $z - i4$) then we get:

$$\begin{aligned}
f(z) &= i\frac{\sin 4}{(z-i4)^2} + \frac{\cos 4}{(z-i4)} + i\frac{\sin 4}{2!} + \frac{\cos 4}{3!}(z-i4) + i\frac{\sin 4}{4!}(z-i4)^2 + \frac{\cos 4}{5!}(z-i4)^3 + \\
&\quad i\frac{\sin 4}{6!}(z-i4)^4 + \frac{\cos 4}{7!}(z-i4)^5 + \cdots
\end{aligned}$$

which is the Laurent series of $f(z)$ at $z_0 = i4$ with a principal part made of two terms (i.e. $i\frac{\sin 4}{(z-i4)^2} + \frac{\cos 4}{(z-i4)}$) and an analytic part made of infinite number of terms (i.e. the rest).

4. Show that the Laurent series of the following functions around $z_0 = 0$ are the same as the Maclaurin series of these functions:

(**a**) $f(z) = \ln(1+z)$. (**b**) $f(z) = \frac{1}{1-z}$.

Answer: In this answer we just consider the form of the series without any other consideration such as its radius of convergence. However, because of "Maclaurin series" in the question, it should be obvious that the question is about the series in the first "annulus of convergence" (i.e. the disk of convergence) which is the closest to the origin.

(**a**) The Maclaurin series of f was given earlier (see Eq. 195). So, all we need to do is to generate the Laurent series around $z_0 = 0$ and compare the two series to see if they are identical or not. Referring to Eq. 200 we have:

$$a_n = \frac{1}{i2\pi}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{1}{i2\pi}\oint_C \frac{\ln(1+z)}{z^{n+1}}\,dz$$

Now, if C is an origin-centered circle of radius $\rho < 1$ and $n < 0$ then the integrand of the last integral is an analytic function over C and the entire region surrounded by C and hence by Cauchy's theorem (see § 4.2) the integral is zero.[232] This means that the principal part of the Laurent series of $\ln(1+z)$ is zero and hence all we need to do is to find the analytic part (which is represented by the terms with $n \ge 0$). Now, if we compare Eq. 200 to Eq. 181 we can see that $a_n = \frac{f^{(n)}(z_0)}{n!}$ which means that the coefficients a_n for the analytic part (corresponding to $n = 0, 1, 2, \cdots$) are the same as the coefficients of the Maclaurin series (whose standard form is given by Eq. 190). Accordingly, the Laurent series of $\ln(1+z)$ around $z_0 = 0$ is the same as the Maclaurin series of this function (noting that Eq. 199 will reduce to Eq. 190 considering that the terms with $n < 0$ are zero, $z_0 = 0$ and $a_n = \frac{f^{(n)}(z_0)}{n!}$).

[232] We note that the condition $\rho < 1$ (which is justified by the existence of a singularity of f at $z = -1$) is inline with the radius of convergence of the series of $\ln(1+z)$.

(**b**) The Maclaurin series of f was given and generated earlier (see Eq. 194 and part a of Problem 8 of § 5.1). So, all we need to do is to generate the Laurent series around $z_0 = 0$ and compare the two series to see if they are identical or not. Referring to Eq. 200 we have:

$$a_n = \frac{1}{i2\pi} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz = \frac{1}{i2\pi} \oint_C \frac{1/(1-z)}{z^{n+1}}\, dz$$

Now, if C is an origin-centered circle of radius $\rho < 1$ and $n < 0$ then the integrand of the last integral is an analytic function over C and the entire region surrounded by C and hence by Cauchy's theorem (see § 4.2) the integral is zero.[233] This means that the principal part of the Laurent series of $1/(1-z)$ is zero and hence all we need to do is to find the analytic part. Now, if we compare Eq. 200 to Eq. 181 we can see that $a_n = \frac{f^{(n)}(z_0)}{n!}$ which means that the coefficients a_n for the analytic part are the same as the coefficients of the Maclaurin series. Accordingly, the Laurent series of $1/(1-z)$ around $z_0 = 0$ is the same as the Maclaurin series of this function.

5. Find the Laurent series of the following functions around the given points z_0 using the standard Laurent series expansion and verify the results by using known Taylor series:

(**a**) $f(z) = \frac{e^z}{z^3}$ at $z_0 = 0$. (**b**) $f(z) = \frac{\cos z}{z-1}$ at $z_0 = 1$. (**c**) $f(z) = \frac{\sinh z}{(z-i4)^2}$ at $z_0 = i4$.

Answer: The standard Laurent series expansion is given by Eq. 199 where the coefficients a_n are given by Eq. 200. So, in this answer we use these equations to generate the Laurent series of the given functions at the given points.

(**a**) Noting that $f = e^z/z^3$ and $z_0 = 0$, we have:

$$a_n = \frac{1}{i2\pi} \oint_C \frac{e^z/z^3}{z^{n+1}}\, dz = \frac{1}{i2\pi} \oint_C \frac{e^z}{z^{n+4}}\, dz$$

where C is a closed curve surrounding z_0. Now, for $n < -3$ the integrand of the last integral is analytic over C and the entire region surrounded by C and hence by Cauchy's theorem (see § 4.2) the integral is zero, i.e. all the terms of the Laurent series corresponding to $n < -3$ are zero. Regrading the remaining terms (i.e. those corresponding to $n \ge -3$) $a_n \ne 0$ in general since C encloses a singularity so we need to calculate a_n for these terms. Now, from Cauchy's derivative formula (Eq. 181) we have (noting the meaning of f here):

$$\oint_C \frac{e^z}{z^{n+4}}\, dz = \frac{i2\pi}{(n+3)!} \times f^{(n+3)}(0) = \frac{i2\pi}{(n+3)!} \times e^0 = \frac{i2\pi}{(n+3)!}$$

Hence:

$$a_n = \frac{1}{i2\pi} \oint_C \frac{e^z}{z^{n+4}}\, dz = \frac{1}{i2\pi} \times \frac{i2\pi}{(n+3)!} = \frac{1}{(n+3)!}$$

Accordingly, for $n \ge -3$ we have:

$$a_{-3} = \frac{1}{0!} = 1 \qquad a_{-2} = \frac{1}{1!} = 1 \qquad a_{-1} = \frac{1}{2!} \qquad \cdots \qquad a_n = \frac{1}{(n+3)!}$$

On inserting these coefficients into Eq. 199 we get:

$$f(z) = \frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!z} + \frac{1}{3!} + \frac{z}{4!} + \frac{z^2}{5!} + \frac{z^3}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{z^{n-3}}{n!}$$

which is identical to the series of this function that we obtained in part (a) of Problem 3 by using known Taylor series.

(**b**) Noting that $f = \frac{\cos z}{z-1}$ and $z_0 = 1$, we have:

$$a_n = \frac{1}{i2\pi} \oint_C \frac{\cos z/(z-1)}{(z-1)^{n+1}}\, dz = \frac{1}{i2\pi} \oint_C \frac{\cos z}{(z-1)^{n+2}}\, dz$$

[233] Again, the condition $\rho < 1$ (which is justified by the existence of a singularity of f at $z = 1$) is inline with the radius of convergence of the series of $1/(1-z)$.

where C is a closed curve surrounding z_0. Now, for $n < -1$ the integrand of the last integral is analytic over C and the entire region surrounded by C and hence by Cauchy's theorem (see § 4.2) the integral is zero, i.e. all the terms of the Laurent series corresponding to $n < -1$ are zero. Regrading the remaining terms (i.e. those corresponding to $n \ge -1$) $a_n \ne 0$ in general since C encloses a singularity so we need to calculate a_n for these terms. Now, from Cauchy's derivative formula (Eq. 181) we have (noting the meaning of f here):

$$\oint_C \frac{\cos z}{(z-1)^{n+2}}\,dz = \frac{i2\pi}{(n+1)!} \times f^{(n+1)}(1) = \frac{i2\pi}{(n+1)!}\left[\sin\left(\frac{3n\pi}{2}\right)\cos 1 - \cos\left(\frac{3n\pi}{2}\right)\sin 1\right]$$

Hence:

$$\begin{aligned} a_n &= \frac{1}{i2\pi}\oint_C \frac{\cos z}{(z-1)^{n+2}}\,dz = \frac{1}{i2\pi} \times \frac{i2\pi}{(n+1)!}\left[\sin\left(\frac{3n\pi}{2}\right)\cos 1 - \cos\left(\frac{3n\pi}{2}\right)\sin 1\right] \\ &= \frac{\sin\left(\frac{3n\pi}{2}\right)\cos 1 - \cos\left(\frac{3n\pi}{2}\right)\sin 1}{(n+1)!} \end{aligned}$$

Accordingly, for $n \ge -1$ we have:

$$a_{-1} = \cos 1 \qquad a_0 = -\sin 1 \qquad a_1 = -\frac{\cos 1}{2!} \qquad a_2 = \frac{\sin 1}{3!} \qquad a_3 = \frac{\cos 1}{4!} \qquad \cdots$$

On inserting these coefficients into Eq. 199 we get:

$$f(z) = \frac{\cos 1}{z-1} - \sin 1 - \frac{\cos 1}{2!}(z-1) + \frac{\sin 1}{3!}(z-1)^2 + \frac{\cos 1}{4!}(z-1)^3 - \cdots$$

which is identical to the series of this function that we obtained in part (b) of Problem 3 by using known Taylor series.

(**c**) Noting that $f = \frac{\sinh z}{(z-i4)^2}$ and $z_0 = i4$, we have:

$$a_n = \frac{1}{i2\pi}\oint_C \frac{\sinh z/(z-i4)^2}{(z-i4)^{n+1}}\,dz = \frac{1}{i2\pi}\oint_C \frac{\sinh z}{(z-i4)^{n+3}}\,dz$$

where C is a closed curve surrounding z_0. Now, for $n < -2$ the integrand of the last integral is analytic over C and the entire region surrounded by C and hence by Cauchy's theorem (see § 4.2) the integral is zero, i.e. all the terms of the Laurent series corresponding to $n < -2$ are zero. Regrading the remaining terms (i.e. those corresponding to $n \ge -2$) $a_n \ne 0$ in general since C encloses a singularity so we need to calculate a_n for these terms. Now, from Cauchy's derivative formula (Eq. 181) we have (noting the meaning of f here):

$$\oint_C \frac{\sinh z}{(z-i4)^{n+3}}\,dz = \frac{i2\pi}{(n+2)!} \times f^{(n+2)}(i4) = \frac{i2\pi}{(n+2)!}\left[\left|\cos\frac{n\pi}{2}\right|\sinh(i4) + \left|\sin\frac{n\pi}{2}\right|\cosh(i4)\right]$$

Hence:

$$\begin{aligned} a_n &= \frac{1}{i2\pi}\oint_C \frac{\sinh z}{(z-i4)^{n+3}}\,dz = \frac{1}{i2\pi} \times \frac{i2\pi}{(n+2)!}\left[\left|\cos\frac{n\pi}{2}\right|\sinh(i4) + \left|\sin\frac{n\pi}{2}\right|\cosh(i4)\right] \\ &= \frac{\left|\cos\frac{n\pi}{2}\right|\sinh(i4) + \left|\sin\frac{n\pi}{2}\right|\cosh(i4)}{(n+2)!} = \frac{\left|\cos\frac{n\pi}{2}\right| i\sin 4 + \left|\sin\frac{n\pi}{2}\right|\cos 4}{(n+2)!} \end{aligned}$$

Accordingly, for $n \ge -2$ we have:

$$a_{-2} = i\sin 4 \qquad a_{-1} = \cos 4 \qquad a_0 = i\frac{\sin 4}{2!} \qquad a_1 = \frac{\cos 4}{3!} \qquad a_2 = i\frac{\sin 4}{4!} \qquad \cdots$$

On inserting these coefficients into Eq. 199 we get:

$$f(z) = i\frac{\sin 4}{(z-i4)^2} + \frac{\cos 4}{(z-i4)} + i\frac{\sin 4}{2!} + \frac{\cos 4}{3!}(z-i4) + i\frac{\sin 4}{4!}(z-i4)^2 + \cdots$$

which is identical to the series of this function that we obtained in part (c) of Problem 3 by using known Taylor series.

6. Define the three types of singularity (i.e. removable, pole and essential which were investigated in § 3.3) in terms of the Laurent series of their functions. Also, give an example of each type.
Answer: Let z_0 be an isolated singular point (or singularity) of a given function $f(z)$ which has a Laurent series expansion around z_0. Now, we have three main cases (considering the Laurent series of f in the annulus of analyticity next to z_0):
(**a**) The Laurent series of f around z_0 has no principal part (i.e. it does not contain any negative-power terms). In this case, we have a removable singularity. An example of this type of singularity (at $z_0 = 0$) is:[234]

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

(**b**) The Laurent series of f around z_0 contains a finite number of negative-power terms (i.e. the principal part of the series has a finite number of terms). In this case, we have a pole of order m (where m is a positive integer with $-m$ being the lowest power in the Laurent series). An example of this type of singularity (i.e. a pole of order 2 at $z_0 = 0$) is:

$$\frac{\sin z}{z^3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-2} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \cdots$$

(**c**) The Laurent series of f around z_0 contains an infinite number of negative-power terms (i.e. the principal part of the series has an infinite number of terms). In this case, we have an essential singularity (or a "pole of infinite order"). An example of this type of singularity (at $z_0 = 0$) is:

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} = 1 + \frac{1}{z} + \frac{1}{2!\, z^2} + \frac{1}{3!\, z^3} + \frac{1}{4!\, z^4} + \cdots$$

7. Given that a complex function f is analytic on the punctured disk $0 < |z| < \rho$ and its modulus $|f|$ is bounded on this punctured disk, does f have a limit at the origin?
Answer: Since $|f|$ is bounded on the punctured disk then the Laurent series of f around the origin does not include negative-power terms.[235] So, any potential singularity at the origin cannot be a pole or an essential singularity (see Problem 6). In other words, the origin is either a removable singularity or an analytic point. Accordingly, f has a limit at the origin (see § 3.3).
8. Determine the location and type of the singularities of the following functions $f(z)$ using their Laurent series expansion around these singularities:
(**a**) $f(z) = \sinh\left(\frac{1}{z^2}\right)$. (**b**) $f(z) = \frac{\cos z - 1}{z^3}$. (**c**) $f(z) = \frac{\ln(1+z)}{z}$ $(|z| < 1)$.
(**d**) $f(z) = \frac{\sin z}{z^4}$. (**e**) $f(z) = \frac{\cos z}{z-1}$. (**f**) $f(z) = \frac{\sinh z}{(z-i4)^2}$.
Answer:
(**a**) f has a singularity at $z = 0$. The Laurent series of f around $z = 0$ is (see Eq. 197):

$$\sinh\left(\frac{1}{z^2}\right) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(\frac{1}{z^2}\right)^{2n+1} = \frac{1}{z^2} + \frac{1}{3! z^6} + \frac{1}{5! z^{10}} + \cdots$$

[234] As we see, this actually is a Taylor series and hence $\frac{\sin z}{z}$ is *effectively* "analytic" at $z_0 = 0$ despite this singularity.
[235] This is because as we approach the origin from any direction, any negative-power terms in the series of f will cause f to shoot up causing $|f|$ to exceed any bound (when we are sufficiently close to the origin). In fact, the Laurent series of f should have only an analytic part (with no principal part) and hence in effect it is a Taylor (or rather Maclaurin) series.

As we see, the principal part of the series has an infinite number of terms. Hence, f has an essential singularity at the origin.
(**b**) f has a singularity at $z = 0$. The Laurent series of f around $z = 0$ is (see Eq. 192):

$$\frac{\cos z - 1}{z^3} = \frac{1}{z^3}\left(\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}\right] - 1\right) = -\frac{1}{2!z} + \frac{z}{4!} - \frac{z^3}{6!} + \cdots$$

As we see, the principal part of the series has a single term whose z-power is -1. Hence, f has a simple pole at the origin.
(**c**) f has a singularity at $z = 0$. The Laurent series of f around $z = 0$ is (see Eq. 195 noting the condition $|z| < 1$):

$$\frac{\ln(1+z)}{z} = \frac{1}{z}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}z^n = 1 - \frac{z}{2} + \frac{z^2}{3} - \cdots$$

As we see, the series has no principal part. Hence, f has a removable singularity at the origin. Also, see part (d) of Problem 2 of § 5.3.
(**d**) f has a singularity at $z = 0$. The Laurent series of f around $z = 0$ is (see Eq. 193):

$$\frac{\sin z}{z^4} = \frac{1}{z^4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \cdots$$

As we see, the lowest z-power in the principal part of the series is -3. Hence, f has a triple pole (i.e. a pole of order 3) at the origin.
(**e**) f has a singularity at $z = 1$. The Laurent series of this function around $z = 1$ was obtained earlier (see part b of Problems 3 and 5). On inspecting the Laurent series we see that the principal part of the series has a finite number of terms and the lowest power of $(z - 1)$ in the principal part is -1. Hence, f has a simple pole at $z = 1$.
(**f**) f has a singularity at $z = i4$. The Laurent series of this function around $z = i4$ was obtained earlier (see part c of Problems 3 and 5). On inspecting the Laurent series we see that the principal part of the series has a finite number of terms and the lowest power of $(z - i4)$ in the principal part is -2. Hence, f has a double pole at $z = i4$.

9. Find the Laurent series of the following functions $f(z)$ around the given points in the different annuli of analyticity around these points:
(**a**) $f(z) = \frac{1}{1-z}$ around $z = 0$. (**b**) $f(z) = \frac{-3z-34}{(z+4)(z-7)}$ around $z = 0$.
(**c**) $f(z) = \frac{1}{z^2-z}$ around $z = 0$. (**d**) $f(z) = \frac{1}{z(z-2)^3}$ around $z = 2$.
Answer:
(**a**) f has only one singularity at $z = 1$ and hence it is analytic on the open disk $|z| < 1$ and on the annulus $1 < |z| < \infty$. So, we need to consider these two regions separately. As for the disk $|z| < 1$, the function is analytic on the entire disk and hence it is represented by its Maclaurin series (as given by Eq. 194) which is its "Laurent" series on this "annulus". As for the annulus $1 < |z| < \infty$, we manipulate the function and use Eq. 194 (with the replacement of z by $1/z$), that is:

$$\frac{1}{1-z} = -\frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \cdots \qquad (|z| > 1)$$

(**b**) f has a singularity at $z = -4$ and a singularity at $z = 7$ and hence it is analytic on the disk $|z| < 4$, on the annulus $4 < |z| < 7$, and on the annulus $7 < |z| < \infty$. So, we need to consider these three regions separately. On splitting f by partial fractions we get:

$$f(z) = \frac{-3z - 34}{(z+4)(z-7)} = \frac{2}{z+4} - \frac{5}{z-7} = f_1 + f_2$$

Accordingly, the Laurent series of f around $z = 0$ is the sum of the Laurent series of $f_1 = \frac{2}{z+4}$ around $z = 0$ and the Laurent series of $f_2 = -\frac{5}{z-7}$ around $z = 0$.

As for the disk $|z| < 4$, f_1 and f_2 are analytic (because they have no singularity in this disk) and hence the Laurent series of f_1 and f_2 are their Taylor series (or Maclaurin series noting that the center of the series is $z = 0$) that is:

$$\begin{aligned} f(z) &= \frac{2}{z+4} - \frac{5}{z-7} = \frac{1/2}{\frac{z}{4}+1} - \frac{5/7}{\frac{z}{7}-1} = \frac{1/2}{1-\left(-\frac{z}{4}\right)} + \frac{5/7}{1-\frac{z}{7}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{4}\right)^n + \frac{5}{7}\sum_{n=0}^{\infty}\left(\frac{z}{7}\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n 2^{-(2n+1)} z^n + 5\sum_{n=0}^{\infty} 7^{-(n+1)} z^n = \sum_{n=0}^{\infty}\left[(-1)^n 2^{-(2n+1)} + 5\times 7^{-(n+1)}\right] z^n \end{aligned}$$

where we used the Maclaurin series for $\frac{1}{1-z}$ (see Eq. 194) in the fourth equality. As we see, this is a Maclaurin series (as it should be) noting that f is analytic on the entire disk $|z| < 4$.
As for the annulus $4 < |z| < 7$, f_1 has a singularity inside this annulus (i.e. at $z = -4$) and hence we use its Laurent series while f_2 is analytic on the entirety of $|z| < 7$ and hence we use its Maclaurin series (which is its "Laurent" series), that is:

$$\begin{aligned} f(z) &= \frac{2}{z+4} - \frac{5}{z-7} = \frac{2}{z}\left[\frac{1}{1+\frac{4}{z}}\right] + \frac{5}{7}\left[\frac{1}{1-\frac{z}{7}}\right] = \frac{2}{z}\sum_{n=0}^{\infty}\left(-\frac{4}{z}\right)^n + \frac{5}{7}\sum_{n=0}^{\infty}\left(\frac{z}{7}\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n 2^{2n+1} z^{-(n+1)} + 5\sum_{n=0}^{\infty} 7^{-(n+1)} z^n \end{aligned}$$

where we used the Maclaurin series for $\frac{1}{1-z}$ (see Eq. 194) in the third equality.
As for the annulus $7 < |z| < \infty$, both f_1 and f_2 have a singularity inside this annulus (i.e. at $z = -4$ for f_1 and at $z = 7$ for f_2) and hence we use their Laurent series there, that is:

$$\begin{aligned} f(z) &= \frac{2}{z+4} - \frac{5}{z-7} = \frac{2}{z}\left[\frac{1}{1+\frac{4}{z}}\right] - \frac{5}{z}\left[\frac{1}{1-\frac{7}{z}}\right] = \frac{2}{z}\sum_{n=0}^{\infty}\left(-\frac{4}{z}\right)^n - \frac{5}{z}\sum_{n=0}^{\infty}\left(\frac{7}{z}\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n 2^{2n+1} z^{-(n+1)} - 5\sum_{n=0}^{\infty} 7^n z^{-(n+1)} = \sum_{n=0}^{\infty}\left[(-1)^n 2^{2n+1} - 5\times 7^n\right] z^{-(n+1)} \end{aligned}$$

where we used the Maclaurin series for $\frac{1}{1-z}$ (see Eq. 194) in the third equality.
(**c**) f has a singularity at $z = 0$ and a singularity at $z = 1$ and hence it is analytic over the annulus $0 < |z| < 1$ and over the annulus $1 < |z| < \infty$. So, we need to consider these annuli separately. On splitting f by partial fractions we get:

$$\frac{1}{z^2 - z} = \frac{1}{z(z-1)} = \frac{-1}{z(1-z)} = -\frac{1}{z} - \frac{1}{1-z} = f_1 + f_2$$

Accordingly, the Laurent series of f around $z = 0$ is the sum of the Laurent series of $f_1 = -\frac{1}{z}$ around $z = 0$ and the Laurent series of $f_2 = -\frac{1}{1-z}$ around $z = 0$.
As for the annulus $0 < |z| < 1$, f_1 has a singularity inside this annulus (i.e. at $z = 0$) and hence we use its Laurent series while f_2 is analytic on the entirety of $|z| < 1$ and hence we use its Maclaurin series (which is its "Laurent" series). Now, if we note that the Laurent series of f_1 around $z = 0$ is itself then we have:

$$f(z) = -\frac{1}{z} - \frac{1}{1-z} = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n = -\sum_{n=-1}^{\infty} z^n = -\sum_{n=0}^{\infty} z^{n-1} = -z^{-1} - 1 - z - z^2 - z^3 - \cdots$$

where we used the Maclaurin series for $\frac{1}{1-z}$ (see Eq. 194) in the second equality.
As for the annulus $1 < |z| < \infty$, both f_1 and f_2 have singularities inside this annulus (i.e. at $z = 0$ for f_1 and at $z = 1$ for f_2) and hence we use their Laurent series there, that is:

$$f(z) = -\frac{1}{z} - \frac{1}{1-z} = -\frac{1}{z} + \frac{1}{z}\left[\frac{1}{1-z^{-1}}\right] = -\frac{1}{z} + \frac{1}{z}\sum_{n=0}^{\infty} z^{-n} = \frac{1}{z}\left(-1 + \sum_{n=0}^{\infty} z^{-n}\right) = \frac{1}{z}\sum_{n=1}^{\infty} z^{-n}$$

$$= \sum_{n=1}^{\infty} z^{-(n+1)} = \sum_{n=2}^{\infty} z^{-n} = z^{-2} + z^{-3} + z^{-4} + z^{-5} + \cdots$$

where we used the Maclaurin series for $\frac{1}{1-z}$ (see Eq. 194) in the third equality.
(**d**) f has a singularity at $z = 0$ and a singularity at $z = 2$ and hence it is analytic over the annulus $0 < |z-2| < 2$ and over the annulus $2 < |z-2| < \infty$. So, we need to consider these annuli separately. On splitting f by partial fractions we get:

$$\frac{1}{z(z-2)^3} = \frac{1}{2(z-2)^3} - \frac{1}{4(z-2)^2} + \frac{1}{8(z-2)} - \frac{1}{8z} = f_1 + f_2 + f_3 + f_4$$

Accordingly, the Laurent series of f around $z = 2$ is the sum of the Laurent series of f_1, f_2, f_3, f_4 around $z = 2$.
As for the annulus $0 < |z-2| < 2$, f_1, f_2, f_3 have a singularity inside this annulus (i.e. at $z = 2$) and hence we use their Laurent series while f_4 is analytic on the entirety of this annulus and inside it and hence we use its Taylor series (which is its "Laurent" series). Now, if we note that the Laurent series of f_1, f_2, f_3 at $z = 2$ are themselves then we have:

$$\begin{aligned}
f(z) &= f_1 + f_2 + f_3 - \frac{1}{8z} = f_1 + f_2 + f_3 - \frac{1}{16}\left(\frac{1}{1-\left(\frac{z-2}{-2}\right)}\right) = f_1 + f_2 + f_3 - \frac{1}{16}\sum_{n=0}^{\infty}\left(\frac{z-2}{-2}\right)^n \\
&= \frac{1}{2(z-2)^3} - \frac{1}{4(z-2)^2} + \frac{1}{8(z-2)} + \sum_{n=0}^{\infty}(-1)^{n+1}2^{-(n+4)}(z-2)^n \\
&= \sum_{n=-3}^{\infty}(-1)^{n+1}2^{-(n+4)}(z-2)^n \\
&= \frac{1}{2(z-2)^3} - \frac{1}{4(z-2)^2} + \frac{1}{8(z-2)} - \frac{1}{16} + \frac{(z-2)}{32} - \frac{(z-2)^2}{64} + \frac{(z-2)^3}{128} - \cdots
\end{aligned}$$

As for the annulus $2 < |z-2| < \infty$, all f_1, f_2, f_3, f_4 have singularities inside this annulus (i.e. at $z = 2$ for f_1, f_2, f_3 and at $z = 0$ for f_4) and hence we use their Laurent series there. Now, the Laurent series for f_1, f_2, f_3 at $z = 2$ are themselves while the Laurent series for f_4 at $z = 2$ is:

$$-\frac{1}{8z} = -\frac{1}{8(z-2)}\left(\frac{1}{1-\left(\frac{-2}{z-2}\right)}\right) = -\frac{1}{8(z-2)}\sum_{n=0}^{\infty}\left(\frac{-2}{z-2}\right)^n = \sum_{n=0}^{\infty}(-1)^{n+1}\, 2^{n-3}(z-2)^{-(n+1)}$$

Hence:

$$\begin{aligned}
f(z) &= \frac{1}{2(z-2)^3} - \frac{1}{4(z-2)^2} + \frac{1}{8(z-2)} + \sum_{n=0}^{\infty}(-1)^{n+1}\, 2^{n-3}(z-2)^{-(n+1)} \\
&= \frac{1}{2(z-2)^3} - \frac{1}{4(z-2)^2} + \frac{1}{8(z-2)} - \frac{1}{8(z-2)} + \frac{1}{4(z-2)^2} - \frac{1}{2(z-2)^3} + \frac{1}{(z-2)^4} - \cdots \\
&= \frac{1}{(z-2)^4} - \frac{2}{(z-2)^5} + \frac{4}{(z-2)^6} - \frac{8}{(z-2)^7} + \cdots \\
&= \sum_{n=3}^{\infty}(-1)^{n+1}\, 2^{n-3}(z-2)^{-(n+1)}
\end{aligned}$$

10. Outline the method used in Problem 9 to generate the Laurent series of a given function around a given point over different annular regions of analyticity.
Answer: We do the following:

• We find the singularities of the function and hence find the annular regions (separated by the singularities) over which the function is analytic.
• We split the function into its partial fractions form (if this split is required) noting that the Laurent series of the function in each annular region is the sum of the Laurent series of the individual terms of the partial fractions in that annular region.
• We find the Laurent series (i.e. around the given point) of the individual terms of the partial fractions for each annular region noting that if the term is analytic over the entirety of the annular region and its interior (i.e. the annular region does not contain a singularity for that term inside it) then we use the Taylor series of that term (i.e. around the center of the series).
• We add the Laurent series that we found in the previous point and simplify the final expression of the series as far as we can (i.e. we put the result in its simplest form) to obtain the Laurent series of the function in that annular region around the given point.

5.3 Radius of Convergence of Complex Power Series

As indicated earlier, the radius of convergence of a complex power series (which represents a given complex function f) is the radius of the disk (or circle) in the complex plane inside which the series is valid (i.e. it converges to the function represented by this series).[236] For example, the radius of convergence of the complex (Maclaurin) series $\sum_{n=0}^{\infty} z^n$ is 1 because this series converges to the complex function $\frac{1}{1-z}$ (which this series represents according to Eq. 194) only inside the origin-centered unit disk $|z| < 1$. On the other hand, the radius of convergence of the complex (Maclaurin) series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is infinite because this series converges to the complex function e^z (which this series represents according to Eq. 191) over the entire complex plane. Some power series do not converge at all (except at their center such as $\sum_{n=1}^{\infty} n!\, z^n$ which "converges" only at $z = 0$; see part f of Problem 1) and hence their radius of convergence is zero.

It is important to take notice of the following remarks about the radius of convergence (and some related issues):

• The power series of a complex function converges inside its circle of convergence[237] and diverges outside it. However, it may (or may not) converge on its circle of convergence (i.e. the perimeter of the disk of convergence).[238] For example, the Maclaurin series of $f(z) = \ln(1 + z)$ converges for all values of z with $|z| < 1$ (see Eq. 195 and part e of Problem 12 of § 5.1). Moreover, it converges at $z = 1$ (which is on the circle of convergence) as well but it does not converge at $z = -1$ (which is also on the circle of convergence).[239]
• The disk (or circle) of convergence of the Maclaurin series of a function is centered on the origin, while the disk of convergence of the Taylor series of a function f is centered on the point which the series is generated at, i.e. the point z_0 in Eq. 189 where f and its derivatives are evaluated for the generation of the series.[240] For example, the disk of convergence of the series $\sum_{n=0}^{\infty} z^n$ (which represents the function $\frac{1}{1-z}$ according to Eq. 194) is centered on the origin because it is a Maclaurin series, while the disk of convergence of the series $\sum_{n=0}^{\infty} -(z + 1)^n$ (which represents the function $f = \frac{-1}{1-(z+1)} = \frac{1}{z}$ according to Eq. 198) is centered on the point -1 because it is a Taylor series of f at the point $z_0 = -1$ (see part a of Problem 9 of § 5.1).
• For the power series of a complex function that has only one singularity, the radius of convergence of the series is the distance between this singularity and the point which the series is generated at. For example, the function $\frac{1}{1-z}$ has only one singularity (which is at $z = 1$) and hence the radius of convergence of the

[236] Accordingly, this disk (or circle) may be called the disk (or circle) of convergence. We may also call the center of the disk of convergence the center of convergence (and this center is the point around which the series expansion is obtained). As we will see, the radius of convergence is a non-negative real number (which could be zero when the series does not converge at all except at its center and could be infinite when the series converges over the entire complex plane).
[237] If the radius of convergence is zero then the circle of convergence is the center of the series only, i.e. the series converges only at its center.
[238] This usually occurs at individual points on the circle of convergence.
[239] In fact, $f(-1) = \ln(0)$ is not defined regardless of the series.
[240] The point which the series is generated at may be called the expansion point (or the point of expansion).

Maclaurin series (which by definition is generated at $z_0 = 0$) of this function is the distance between these points (which is 1).

• For the power series of a complex function that has more than one singularity, the radius of convergence of the series is the distance between the closest singularity and the point which the series is generated at. In other words, the radius of convergence is the smallest of the potential radii of convergence (where each radius is determined by the distance to one of the singularities). For example, the function $f(z) = \frac{1}{z^2-3z}$ has one singularity at $z = 0$ and another singularity at $z = 3$, and so if we have a Taylor series expansion of f at $z = -1$ (i.e. -1 is the center of convergence of the series) then its radius of convergence is 1 because $z = 0$ is the closest singularity to -1 and hence the radius of convergence is determined by the distance between -1 and this singularity (which is 1).[241]

• The radius of convergence of a power series that represents an algebraic sum or a product of complex functions is the smallest of the radii of convergence of the series representing the individual functions involved, i.e. it is determined by the distance to the singularity nearest to the center of convergence.[242] For example, the function $f = \frac{1}{z-i2} - \frac{1}{z+5}$ (which is an algebraic sum) has one singularity at $z = i2$ and another singularity at $z = -5$, and so if we have a Taylor series expansion of f at $z = 1+i$ then its radius of convergence is $\sqrt{2}$ because $z = i2$ is the closest singularity to $1+i$ and hence the radius of convergence is determined by the distance between $1+i$ and this singularity (which is $\sqrt{2}$).

• The radius of convergence of a power series representing the Cauchy product of two power series is the smallest of the radii of convergence of the multiplied series.

• The radius of convergence of a power series obtained by differentiating or integrating another power series is the same as the radius of convergence of the original series.

• External extensions (based on the particular cases and circumstances) may expand the radii of convergence as identified above.

• Laurent series (as opposite to Taylor and Maclaurin series) converge in their annuli of analyticity according to the rules and conditions that we investigated in § 5.2, and hence they do not have a radius of convergence in the sense of power series (i.e. Taylor and Maclaurin series).

Problems

1. Investigate the radius and disk of convergence of the following power series:[243]

   (a) $\sum_{n=0}^{\infty} (2-i)^{n+1} z^{n+2}$. **(b)** $\sum_{n=0}^{\infty} i^n z^{2n}$. **(c)** $\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}$.

   (d) $\sum_{n=0}^{\infty} \frac{(z-2+i6)^n}{(n-i)^2}$. **(e)** $\sum_{n=0}^{\infty} \frac{z^n}{n!}$. **(f)** $\sum_{n=1}^{\infty} n!\, z^n$.

   Answer: We use the ratio test (see Problem 10 of § 5.1) for this investigation.

   (a)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2-i)^{n+2}z^{n+3}}{(2-i)^{n+1}z^{n+2}}\right| = \lim_{n\to\infty}|(2-i)z| = |(2-i)z| = |2-i|\,|z| < 1$$

   Hence, this series converges for all values of z with $|z| < \frac{1}{|2-i|} = \frac{1}{\sqrt{5}}$, i.e. on the interior of the origin-centered disk with radius $\frac{1}{\sqrt{5}}$.

   (b)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{i^{n+1}z^{2(n+1)}}{i^n z^{2n}}\right| = \lim_{n\to\infty}\left|iz^2\right| = \left|z^2\right| < 1$$

   Hence, this series converges for all values of z with $|z| < 1$, i.e. on the interior of the origin-centered

[241] However, removable singularities do not count in this regard. For example, a function whose radius of convergence with respect to a removable singularity is 1 and with respect to another (non-removable) singularity is 2 has a radius of convergence 2. This is because removable singularities are *effectively* "analytic points" (see § 3.3).

[242] The individual series of the functions involved are supposed to have the same center of convergence.

[243] In questions like this (where a series is given without associating it with a function), the convergence means the convergence of the given series to a definite finite value which represents the value of the function that the series supposedly represents (noting that a convergent series represents a function regardless of having a known standard closed form or not; see § 5.1). Anyway, the purpose of questions like this is practicing the application of the techniques of complex series such as using the convergence tests (like the ratio test) regardless of any other consideration.

unit disk.[244]

(**c**)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{z^{n+2}}{n+2}\times\frac{n+1}{z^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n+2}z\right| = |z| < 1$$

Hence, this series converges for all values of z with $|z| < 1$, i.e. on the interior of the origin-centered unit disk.

(**d**)

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(z-2+i6)^{n+1}}{(n+1-i)^2}\times\frac{(n-i)^2}{(z-2+i6)^n}\right| = \lim_{n\to\infty}\left|\frac{(n-i)^2(z-2+i6)}{(n+1-i)^2}\right| \\ &= \lim_{n\to\infty}\left|\left[\frac{n-i}{n+1-i}\right]^2(z-2+i6)\right| = \lim_{n\to\infty}\left|\left[\frac{1-(i/n)}{1+(1/n)-(i/n)}\right]^2(z-2+i6)\right| \\ &= |z-2+i6| < 1\end{aligned}$$

Hence, this series converges on the interior of the unit disk centered on the point $2-i6$.

(**e**)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{z^{n+1}}{(n+1)!}\times\frac{n!}{z^n}\right| = \lim_{n\to\infty}\left|\frac{z}{n+1}\right| = \left|\frac{z}{\infty}\right| < 1$$

Hence, this series converges for all values of z with $|z| < \infty$, i.e. on the entire (finite) z plane (which is consistent with the fact that this is the Maclaurin series of the exponential function e^z which is entire; see Eq. 191).

(**f**)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,z^{n+1}}{n!\,z^n}\right| = \lim_{n\to\infty}|(n+1)\,z| = |\infty\, z| < 1$$

Hence, this series converges for all values of z with $|z| < \frac{1}{\infty} = 0$, i.e. nowhere in the z plane (according to this test). However, it should be convergent at its center $z = 0$ (see Problem 3), as can be verified by direct test:

$$\sum_{n=1}^{\infty} n!\,z^n\bigg|_{z=0} = (1\times 0^1) + (2\times 0^2) + (6\times 0^3) + \cdots = 0+0+0+\cdots = 0$$

2. Find the series expansion (around the given center z_0) and the radius and disk of convergence of the power series representing the following complex functions:

   (**a**) $f(z) = \frac{3z^2}{1-3z}$ around $z_0 = 0$. (**b**) $f(z) = (z-1)\ln z$ around $z_0 = 1$.

   (**c**) $f(z) = z\ln z$ around $z_0 = 1$. (**d**) $f(z) = \frac{\ln(1+z)}{z}$ around $z_0 = 0$.

   Answer:

   (**a**) Here, f is a product of $3z^2$ and $\frac{1}{1-3z}$. Now, the (Maclaurin) series expansion of $3z^2$ (which is a polynomial) is itself (see Problem 7 of § 5.1) while the series expansion of $\frac{1}{1-3z}$ can be obtained from the Maclaurin series expansion of $\frac{1}{1-z}$ (see Eq. 194 and part a of Problem 8 of § 5.1) by replacing z by $3z$. Hence:

$$f(z) = \frac{3z^2}{1-3z} = 3z^2\left(\sum_{n=0}^{\infty}(3z)^n\right) = \sum_{n=0}^{\infty}3^{n+1}z^{n+2} = 3z^2 + 3^2z^3 + 3^3z^4 + 3^4z^5 + \cdots$$

   Regarding the radius of convergence, $3z^2$ has infinite radius of convergence (because it is a polynomial which converges to itself over the entire complex plane), while the radius of convergence of $\frac{1}{1-3z}$ is

[244] We have $\left|z^2\right| = |z|^2$ (see part a of Problem 4 of § 1.8.7) and hence from $\left|z^2\right| < 1$ we get $|z|^2 < 1$ which on taking the square root of its sides becomes $|z| < 1$ which represents the interior of the origin-centered unit disk.

determined by $|3z| < 1$ (see Eq. 194 noting that z there corresponds to $3z$ here), i.e. $|z| < 1/3$. Hence, the radius of convergence of the series of f is $1/3$ (which is the smallest of the two radii). This can also be obtained from the ratio test, that is:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+2}z^{n+3}}{3^{n+1}z^{n+2}}\right| = \lim_{n\to\infty}|3z| < 1$$

Hence, the complex series of f converges for all values of z with $|z| < 1/3$, i.e. on the interior of the disk with center $z = 0$ and radius $R = 1/3$.

(**b**) Here, f is a product of $(z-1)$ and $\ln z$. Now, the series expansion of the polynomial $(z-1)$ is itself (see the upcoming note 1) while the series expansion of $\ln z = \ln(1+z-1)$ can be obtained from the series expansion of $\ln(1+z)$ (see Eq. 195) by replacing z by $z-1$. Hence:

$$\begin{aligned} f(z) &= (z-1)\ln z = (z-1)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^{n+1} \\ &= (z-1)^2 - \frac{(z-1)^3}{2} + \frac{(z-1)^4}{3} - \frac{(z-1)^5}{4} + \frac{(z-1)^6}{5} - \frac{(z-1)^7}{6} + \cdots \end{aligned}$$

Regarding the radius of convergence, $(z-1)$ has infinite radius of convergence (because it is a polynomial which is entire), while the radius of convergence of $\ln z$ is determined by $|z-1| < 1$ (see Eq. 195 noting that z there corresponds to $z-1$ here). Hence, the radius of convergence of the series of f is 1 (which is the smallest of the two radii). This can also be obtained from the ratio test, that is:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}(z-1)^{n+2}}{n+1}\;\frac{n}{(-1)^{n+1}(z-1)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n(z-1)}{n+1}\right| = |z-1| < 1$$

Hence, the complex series of f converges for all values of z with $|z-1| < 1$, i.e. on the interior of the disk with center $z = 1$ and radius $R = 1$.

Note 1: unlike part (a) where we considered the Maclaurin series of the polynomial (following Problem 7 of § 5.1), in the present part (i.e. part b) we should consider the Taylor series of the polynomial (noting that $z_0 = 1$) because the center of convergence of the series of both $(z-1)$ and $\ln z$ should be the same. In fact, the Taylor series of $(z-1)$ around $z_0 = 1$ is itself. This can be easily obtained by applying the standard form of the Taylor series (as given by Eq. 189 with $z_0 = 1$), that is:

$$f(1) = 0 \qquad\qquad f'(1) = 1 \qquad\qquad f^{(m)}(1) = 0 \quad (m > 1)$$

where f here represents the polynomial $(z-1)$. Hence, from Eq. 189 we get $f(z) = (z-1)$.

(**c**) Here, f is a product of z and $\ln z$. Now, the series expansion of the polynomial z around $z_0 = 1$ is $1 + (z-1)$ (see the upcoming note 2) while the series expansion of $\ln z = \ln(1+z-1)$ can be obtained from the series expansion of $\ln(1+z)$ (see Eq. 195) by replacing z by $z-1$. Hence:

$$\begin{aligned} f(z) &= z\ln z \\ &= (1+z-1)\ln(1+z-1) \\ &= \big[1+(z-1)\big]\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \\ &= 1\times\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n + (z-1)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^{n+1} \\ &= \frac{(-1)^{1+1}}{1}(z-1)^1 + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^{n+1} \end{aligned}$$

$$
\begin{aligned}
&= (z-1) + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n + \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}(z-1)^n \\
&= (z-1) + \sum_{n=2}^{\infty} \left[\frac{(-1)^{n+1}}{n}(z-1)^n + \frac{(-1)^n}{n-1}(z-1)^n \right] \\
&= (z-1) + \sum_{n=2}^{\infty} \left[\frac{(-1)^{n+1}}{n} + \frac{(-1)^n}{n-1} \right](z-1)^n \\
&= (z-1) + \sum_{n=2}^{\infty} \left[\frac{(-1)^{n+1}(n-1) + (-1)^n n}{n(n-1)} \right](z-1)^n \\
&= (z-1) + \sum_{n=2}^{\infty} \left[\frac{(-1)^{n+1} n - (-1)^{n+1} + (-1)^n n}{n(n-1)} \right](z-1)^n \\
&= (z-1) + \sum_{n=2}^{\infty} \left[\frac{-(-1)^{n} n - (-1)^{n+1} + (-1)^n n}{n(n-1)} \right](z-1)^n \\
&= (z-1) + \sum_{n=2}^{\infty} \frac{-(-1)^{n+1}}{n(n-1)}(z-1)^n \\
&= (z-1) + \sum_{n=2}^{\infty} \frac{(-1)^{n}}{n(n-1)}(z-1)^n \\
&= (z-1) + \frac{(z-1)^2}{2} - \frac{(z-1)^3}{6} + \frac{(z-1)^4}{12} - \frac{(z-1)^5}{20} + \cdots
\end{aligned}
$$

Regarding the radius of convergence, z has infinite radius of convergence (because it is a polynomial), while the radius of convergence of $\ln z$ is determined by $|z-1| < 1$ (see Eq. 195 noting that z there corresponds to $z-1$ here). Hence, the radius of convergence of the series of f is 1 (which is the smallest of the two radii). This can also be obtained from the ratio test, that is:

$$
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(z-1)^{n+1}}{n(n+1)} \frac{n(n-1)}{(-1)^n (z-1)^n} \right| = \lim_{n\to\infty} \left| \frac{(n-1)(z-1)}{(n+1)} \right| = |z-1| < 1
$$

Hence, the complex series of f converges for all values of z with $|z-1| < 1$, i.e. on the interior of the disk with center $z = 1$ and radius $R = 1$.
Note 2: as indicated in note 1 of part (b), the center of convergence of both series (i.e. of z and $\ln z$) should be the same and that is why we considered the Taylor series of z which is $1 + (z-1)$. This can be easily obtained by applying the standard form of the Taylor series (as given by Eq. 189 with $z_0 = 1$), that is:

$$
f(1) = 1 \qquad\qquad f'(1) = 1 \qquad\qquad f^{(m)}(1) = 0 \quad (m > 1)
$$

where f here represents the polynomial z. Hence, from Eq. 189 we get $f(z) = 1 + (z-1)$.
(**d**) Here, f is just $\ln(1+z)$ divided by z and hence the series of f is the Maclaurin series of $\ln(1+z)$ divided by z (noting that $z_0 = 0$), that is (see Eq. 195):

$$
f(z) = \frac{\ln(1+z)}{z} = \frac{1}{z}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^{n-1} = 1 - \frac{z}{2} + \frac{z^2}{3} - \cdots
$$

Regarding the radius and disk of convergence, they are the same as those of $\ln(1+z)$ but with a possible exclusion of $z = 0$ (because $z = 0$ is a singularity of f). However, as we see $z = 0$ is a removable singularity and hence if we define f at $z = 0$ by its limit (which is 1) then the above series converges to f even at $z = 0$. As a result, the complex series of f converges for all values of z with $|z| < 1$, i.e. on the interior of the disk with center $z = 0$ and radius $R = 1$. Also, see part (c) of Problem 8 of § 5.2.

3. Show that every Taylor series (including Maclaurin) should converge somewhere in the complex plane, i.e. at least at its center.
Answer: According to the Taylor (or Maclaurin) series expansion (see Eqs. 189 and 190), the series at its center z_0 (or 0) reduces to $f(z_0)$ $\big[$or $f(0)\big]$ and hence it should be convergent there noting that $f(z)$ should be defined at least at the center, i.e. $f(z_0)$ $\big[$or $f(0)\big]$ should have a definite finite value.
4. Give an example of a complex series that converges nowhere in the complex plane (including its center).
Answer: For example, the series $\sum_{n=1}^{\infty} \frac{n!}{z^n}$ converges nowhere even at its center (which is 0). However, we should note that this is a Laurent (or Laurent-like) series and not a Taylor or Maclaurin series.

5.4 The Calculus of Residues

The coefficient a_{-1} in the Laurent series of Eq. 199 is called the residue of $f(z)$ (i.e. corresponding to the Laurent series expansion of f around z_0). The reason is that if we contour-integrate $f(z)$ around a simple closed curve C that encloses z_0 (with f being analytic on C and inside it except at z_0 which is a singularity of f)[245] then the integral evaluates to $i2\pi a_{-1}$, that is (see Problem 1):

$$\oint_C f(z)\,dz = i2\pi a_{-1} \tag{202}$$

This equation (or rather the statement that associates this equation) is commonly known as the **residue theorem** (see § 4.2.2). This theorem can be easily extended to the case where C encloses a finite number of singularities of f (with f being analytic on C and inside it except at these singularities) where in this case we have:

$$\oint_C f(z)\,dz = i2\pi \sum_{k=1}^{n} {}_k a_{-1} = i2\pi \left({}_1a_{-1} + {}_2a_{-1} + \cdots + {}_na_{-1} \right) \tag{203}$$

where n is the number of singularities inside C and ${}_ka_{-1}$ is the residue of f corresponding to its Laurent series expansion around the k^{th} singularity. It is noteworthy that Cauchy's theorem (i.e. $\oint_C f\,dz = 0$ when f has no singularity inside C) is a special case of the residue theorem (as represented by Eqs. 202 and 203) since the residue of analytic function (i.e. a function that is analytic at the expansion point of its series) is zero because its series expansion has no principal part since it is a Taylor series (see § 5.1 and § 5.2).[246]

As we will see, the residue theorem is very powerful and is commonly used to evaluate difficult definite integrals (where the use of this theorem reduces the difficulty of evaluating these integrals).[247] In fact, the residue theorem is used even to evaluate real integrals that cannot be evaluated by the techniques of real analysis. However, a major handicap in using the residue theorem is the requirement of having the Laurent series of the function f (due to the requirement of having the residue a_{-1} which is contained in the series) and the series in most circumstances is not easy to obtain. This difficulty is alleviated by the fact that the residue a_{-1} of a function f can be obtained without the calculation of the Laurent series of that function. In brief, if C is a simple closed curve and $f(z)$ is an analytic function on and inside C except at a point z_0 inside C (where f has a pole of order n there), then the residue of f at z_0 can be calculated from the following limit:

$$a_{-1} = \lim_{z \to z_0} \left[\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \Big\{ (z - z_0)^n f \Big\} \right] \tag{204}$$

where ! symbolizes factorial and $\frac{d^{n-1}}{dz^{n-1}}$ symbolizes the $(n-1)^{th}$ derivative (noting that in the case of $n = 1$ we have 0! which equals 1 and $\frac{d^0}{dz^0}$ which means no derivative should be taken).

[245] In fact, being a singularity is because we are considering this case noting that this may be extended to non-singularity (where $a_{-1} = 0$ in this case) and hence we retrieve Cauchy's integral theorem.

[246] In this regard, removable singularities are like "analytic points" (see Problem 3).

[247] This should justify the use of "The *Calculus* of Residues" in the title of this section. We should also note that the residue theorem is also commonly used to evaluate contour integrals in general (see for instance Problem 4).

It should be noted that as the residue theorem is extended (i.e. by moving from Eq. 202 to Eq. 203), the above method for computing the residue (as given by Eq. 204) is extended to the case when C encloses a finite number m of poles each of order n_k ($k = 1, \cdots, m$) where in this case the m residues ${}_k a_{-1}$ ($k = 1, \cdots, m$) corresponding to the m poles at z_k ($k = 1, \cdots, m$) are computed from the following limits:

$$ {}_k a_{-1} = \lim_{z \to z_k} \left[\frac{1}{(n_k - 1)!} \frac{d^{n_k - 1}}{dz^{n_k - 1}} \left\{ (z - z_k)^{n_k} f \right\} \right] \qquad (k = 1, \cdots, m) \tag{205} $$

Some examples will be given in the Problems to clarify these issues further.

We should now draw the attention to the following useful remarks:

• The residue(s) of a function can be evaluated by different methods (apart from being obtained directly form its Laurent expansion). In fact, in some cases Eqs. 204 and 205 may not be the best (or even viable such as when the singularity is essential) for calculating the residue(s) although they have the advantage of being more general in application than other methods (and hence we use them persistently despite being non-ideal occasionally).

• If the singularity of f at z_0 is removable the residue of f at z_0 is zero (since the principal part of the Laurent series of f around z_0 is missing and the function is effectively analytic at that point; see Problem 3 as well as Problem 6 of § 5.2).

• The result that we obtain from evaluating any real integral by the calculus of residue (or by any other complex technique) should be real and this should be taken as a first initial check on the validity of the obtained result (i.e. the result should be discarded immediately if it is complex or imaginary).

Problems

1. Verify Eq. 202.

   Answer: If we represent f by its Laurent series around z_0 (as given by Eq. 199) then we have:

$$
\begin{aligned}
\oint_C f(z)\,dz &= \oint_C \left[\cdots + a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \right] dz \\
&= \cdots + \oint_C \frac{a_{-2}}{(z-z_0)^2}\,dz + \oint_C \frac{a_{-1}}{(z-z_0)^1}\,dz + \oint_C \frac{a_0}{(z-z_0)^0}dz + \\
&\qquad \oint_C \frac{a_1}{(z-z_0)^{-1}}\,dz + \oint_C \frac{a_2}{(z-z_0)^{-2}}\,dz + \cdots \\
&= \cdots + a_{-2}\oint_C \frac{1}{(z-z_0)^2}\,dz + a_{-1}\oint_C \frac{1}{(z-z_0)^1}\,dz + a_0\oint_C \frac{1}{(z-z_0)^0}dz + \\
&\qquad a_1\oint_C \frac{1}{(z-z_0)^{-1}}\,dz + a_2\oint_C \frac{1}{(z-z_0)^{-2}}\,dz + \cdots \\
&= \cdots + (a_{-2}\times 0) + (a_{-1}\times i2\pi) + (a_0\times 0) + (a_1\times 0) + (a_2\times 0) + \cdots \\
&= \cdots + 0 + i2\pi\,a_{-1} + 0 + 0 + 0 + \cdots \\
&= i2\pi\,a_{-1}
\end{aligned}
$$

   where the integrals of the third equality are evaluated by the result of Problem 6 of § 4.2.1 which is based on the extended Cauchy's theorem. As we see, the integral of f removes everything except a_{-1} (multiplied by $i2\pi$) which resides and hence it is the residue of f (when integrated as above).

   Note: it is obvious that Eq. 203 is a simple extension of Eq. 202 when C encloses n singularities since by the extended Cauchy's theorem the integral over C is equal to the sum of n integrals over closed contours each of which encloses one (and only one) of the n singularities (see Eq. 175). In fact, Eq. 202 is a special case of Eq. 203 corresponding to $n = 1$.

2. What you notice about the residue formula (i.e. Eq. 204)?

   Answer: We note the following:

   • This formula is restricted to poles. In other words, the order of the pole n should be finite and hence this formula is not usable for "poles of infinite order" (i.e. essential singularities) where in this case the residue should be determined from the Laurent series of the function or by another method.

• The formula should become increasingly impractical as the order of the pole increases due to the fact that obtaining the $(n-1)^{th}$ derivative usually becomes harder and harder (although this does not apply to functions whose derivatives of various orders have a certain simple pattern, e.g. the exponential or the cosine and sine functions). So again, the residue in these cases should be determined from the Laurent series of the function or by other methods which are more practical to use with poles of high order.

3. At which type of singularity the residue of a function is necessarily zero?
Answer: The residue of a function is necessarily zero at its removable singularities because the Laurent series of a function around a removable singularity has no principal part (see Problem 6 of § 5.2). At the other two types of singularity (i.e. pole and essential) the function must have a principal part which may contain a $1/z$ term and hence its residue is not zero (although if the principal part has no $1/z$ term then the residue is also zero). So in brief, the residue at a removable singularity is necessarily zero while the residue at a non-removable singularity (i.e. pole and essential) may and may not be zero.
Note: it should be obvious that the "residue at an analytic point" (assuming this terminology applies) is zero because at an analytic point the "Laurent" series around the point (and in its immediate neighborhood) is actually a Taylor series (with no principal part). See Figure 33.

4. Evaluate the following contour integrals around the given curves C using the residue theorem:
(**a**) $\oint_C \frac{e^z}{z^3}\,dz$ where C is the circle $|z| = 1$.
(**b**) $\oint_C \frac{\cos z}{z-1}\,dz$ where C is the circle $|z - i| = 2$.
(**c**) $\oint_C \frac{\sinh z}{(z-i4)^2}\,dz$ where C is the circle $|z - 1 - i3| = \pi$.
Answer: In all parts of this Problem C is a simple closed curve enclosing a single singularity and f is analytic on C and inside the region surrounded by C (except at the singularity) and hence the residue theorem is applicable.
(**a**) From part (a) of Problem 3 of § 5.2 (as well as part a of Problem 5 of § 5.2) we have $a_{-1} = 1/2$. Hence, from the residue theorem we get:

$$\oint_C \frac{e^z}{z^3}\,dz = i2\pi\frac{1}{2} = i\pi$$

(**b**) From part (b) of Problem 3 of § 5.2 (as well as part b of Problem 5 of § 5.2) we have $a_{-1} = \cos 1$. Hence, from the residue theorem we get:

$$\oint_C \frac{\cos z}{z-1}\,dz = i2\pi\cos 1 \simeq i3.3948$$

(**c**) From part (c) of Problem 3 of § 5.2 (as well as part c of Problem 5 of § 5.2) we have $a_{-1} = \cos 4$. Hence, from the residue theorem we get:

$$\oint_C \frac{\sinh z}{(z-i4)^2}\,dz = i2\pi\cos 4 \simeq -i4.1070$$

Note: as we see, the evaluation of contour integral is trivial if we have the residue. For example, let evaluate the seemingly easiest of the above contour integrals (i.e. the integral of part a) by one of the conventional methods of contour integration (which were investigated in § 3.2) and compare the effort required by the two methods. If we use a parameterization approach then the curve C (which is the origin-centered unit circle) can be parameterized as $z = e^{i\theta}$ (where $0 \le \theta < 2\pi$) with $dz = ie^{i\theta}d\theta$ and we have:

$$\oint_C \frac{e^z}{z^3}\,dz = \int_0^{2\pi} \frac{e^{e^{i\theta}}}{e^{i3\theta}}\,ie^{i\theta}d\theta = i\int_0^{2\pi} \frac{e^{e^{i\theta}}}{e^{i2\theta}}\,d\theta$$

As we see, this is a complicated integral and difficult to evaluate analytically (as well as numerically) as it needs special functions. So, assuming that we have the residue or we can obtain it easily, the residue

theorem method for evaluating contour integrals is superior compared to the conventional methods. This also applies to other types of integrals (e.g. certain types of real definite integrals) some of which cannot be evaluated analytically by any method other than the residue theorem (or similar methods of complex analysis).

5. Verify the residues of the functions $f(z)$ of Problem 3 of § 5.2 (as well as Problem 5 of § 5.2) by using Eq. 204.

Answer:

(**a**) Here, f has a pole of order $n = 3$ at $z_0 = 0$ and hence from Eq. 204 we have:

$$a_{-1} = \lim_{z\to 0}\left[\frac{1}{2!}\frac{d^2}{dz^2}\left\{z^3\frac{e^z}{z^3}\right\}\right] = \lim_{z\to 0}\left[\frac{1}{2}\frac{d^2}{dz^2}\left\{e^z\right\}\right] = \lim_{z\to 0}\left[\frac{1}{2}e^z\right] = \frac{1}{2}e^0 = \frac{1}{2}$$

which agrees with what we obtained in part (a) of Problems 3 and 5 of § 5.2.

(**b**) Here, f has a pole of order $n = 1$ at $z_0 = 1$ and hence from Eq. 204 we have:

$$a_{-1} = \lim_{z\to 1}\left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{(z-1)\frac{\cos z}{z-1}\right\}\right] = \lim_{z\to 1}\left[\cos z\right] = \cos 1$$

which agrees with what we obtained in part (b) of Problems 3 and 5 of § 5.2.

(**c**) Here, f has a pole of order $n = 2$ at $z_0 = i4$ and hence from Eq. 204 we have:

$$a_{-1} = \lim_{z\to i4}\left[\frac{1}{1!}\frac{d}{dz}\left\{(z-i4)^2\frac{\sinh z}{(z-i4)^2}\right\}\right] = \lim_{z\to i4}\left[\frac{d}{dz}\left\{\sinh z\right\}\right] = \lim_{z\to i4}\left[\cosh z\right] = \cosh(i4) = \cos 4$$

which agrees with what we obtained in part (c) of Problems 3 and 5 of § 5.2.

6. Evaluate the following contour integrals around the given curves C using the residue theorem:

(**a**) $\oint_C \frac{1}{z^2-1}\,dz$ where C is the origin-centered circle with radius $\rho > 1$.

(**b**) $\oint_C \frac{7z-23+i2z}{(z-3)(z+i)}\,dz$ where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

(**c**) $\oint_C \frac{-z^4+4z^3-3z^2+z+i(4z^2-4z+2)}{z^2(z-1)^2(z+i2)}\,dz$ where C is the triangle with vertices at $z_a = -i5$, $z_b = -2+i$ and $z_c = 2 - i\frac{3}{4}$.

Answer: In all parts of this Problem C is a simple closed curve enclosing some singularities and f is analytic on C and inside the region surrounded by C (except at the surrounded singularities) and hence the residue theorem is applicable.

(**a**) The integrand f has two simple poles (i.e. at $z_1 = -1$ and at $z_2 = +1$) and C encloses both of them. The residues of f corresponding to these poles (according to Eq. 205) are:

$$\begin{aligned} {}_1a_{-1} &= \lim_{z\to -1}\left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{(z+1)\frac{1}{z^2-1}\right\}\right] = \lim_{z\to -1}\left[\frac{1}{z-1}\right] = -\frac{1}{2} \\ {}_2a_{-1} &= \lim_{z\to +1}\left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{(z-1)\frac{1}{z^2-1}\right\}\right] = \lim_{z\to +1}\left[\frac{1}{z+1}\right] = +\frac{1}{2} \end{aligned}$$

Hence, from Eq. 203 we get:

$$\oint_C \frac{1}{z^2-1}\,dz = i2\pi\left({}_1a_{-1} + {}_2a_{-1}\right) = i2\pi\left(-\frac{1}{2} + \frac{1}{2}\right) = 0$$

This result is identical to the result that we obtained for this contour integral in part (a) of Problem 8 of § 4.2.1 using the extended Cauchy's theorem (i.e. directly).

(**b**) The integrand f has two simple poles, i.e. at $z_1 = 3$ and at $z_2 = -i$. However, C encloses only the pole at z_2. The residue of f corresponding to this pole (according to Eq. 204) is:

$$a_{-1} = \lim_{z\to -i}\left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{(z+i)\frac{7z-23+i2z}{(z-3)(z+i)}\right\}\right] = \lim_{z\to -i}\left[\frac{7z-23+i2z}{z-3}\right] = \frac{-i7-23+2}{-i-3} = 7$$

Hence, from Eq. 202 we get:

$$\oint_C \frac{7z-23+i2z}{(z-3)(z+i)}\,dz = i2\pi a_{-1} = i2\pi \times 7 = i14\pi$$

This result is identical to the result that we obtained for this contour integral in part (b) of Problem 8 of § 4.2.1 using the extended Cauchy's theorem (i.e. directly).
(**c**) The integrand f has a double pole at $z_1 = 0$, a double pole at $z_2 = 1$, and a simple pole at $z_3 = -i2$. However, C encloses only the poles at z_1 and z_3. The residues of f corresponding to the poles at z_1 and z_3 (according to Eq. 205) are:

$$\begin{aligned}
{}_1a_{-1} &= \lim_{z\to 0}\left[\frac{1}{1!}\frac{d^1}{dz^1}\left\{z^2\frac{-z^4+4z^3-3z^2+z+i\left(4z^2-4z+2\right)}{z^2(z-1)^2(z+i2)}\right\}\right]\\
&= \lim_{z\to 0}\left[\frac{-z^5+(3-i4)z^4-(5-i12)z^3+(1-i20)z^2+(8+i4)z}{(z-1)^3(z+i2)^2}\right] = 0\\
{}_3a_{-1} &= \lim_{z\to -i2}\left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{(z+i2)\frac{-z^4+4z^3-3z^2+z+i\left(4z^2-4z+2\right)}{z^2(z-1)^2(z+i2)}\right\}\right]\\
&= \lim_{z\to -i2}\left[\frac{-z^4+4z^3-3z^2+z+i\left(4z^2-4z+2\right)}{z^2(z-1)^2}\right] = -1
\end{aligned}$$

Hence, from Eq. 203 we get:

$$\oint_C \frac{-z^4+4z^3-3z^2+z+i\left(4z^2-4z+2\right)}{z^2(z-1)^2(z+i2)}\,dz = i2\pi\left({}_1a_{-1}+{}_3a_{-1}\right) = i2\pi(0-1) = -i2\pi$$

This result is identical to the result that we obtained for this contour integral in part (c) of Problem 8 of § 4.2.1 using the extended Cauchy's theorem (i.e. directly).

7. Find the residues of the following functions that correspond to their singularities:

(**a**) $f(z) = \frac{1}{z^3-9z}$. (**b**) $f(z) = \frac{e^{iz}}{(z+i)^2(z-4)^2}$. (**c**) $f(z) = \frac{\sinh(3z)}{(z-1+i2)(z+5+i)^3}$.

(**d**) $f(z) = \cot z$. (**e**) $f(z) = \csc z$.

Answer:
(**a**) We have $f(z) = \frac{1}{z^3-9z} = \frac{1}{z(z+3)(z-3)}$. Hence, f has a pole of order $n_1 = 1$ at $z_1 = 0$, a pole of order $n_2 = 1$ at $z_2 = -3$ and a pole of order $n_3 = 1$ at $z_3 = 3$. So, from Eq. 205 the residues of f that correspond to the poles at z_1, z_2, z_3 are:

$$\begin{aligned}
{}_1a_{-1} &= \lim_{z\to 0}\left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{z\frac{1}{z^3-9z}\right\}\right] = \lim_{z\to 0}\left[\frac{1}{z^2-9}\right] = -\frac{1}{9}\\
{}_2a_{-1} &= \lim_{z\to -3}\left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{(z+3)\frac{1}{z^3-9z}\right\}\right] = \lim_{z\to -3}\left[\frac{1}{z^2-3z}\right] = \frac{1}{18}\\
{}_3a_{-1} &= \lim_{z\to +3}\left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{(z-3)\frac{1}{z^3-9z}\right\}\right] = \lim_{z\to +3}\left[\frac{1}{z^2+3z}\right] = \frac{1}{18}
\end{aligned}$$

(**b**) Here, f has a pole of order $n_1 = 2$ at $z_1 = -i$ and another pole of order $n_2 = 2$ at $z_2 = 4$. So, from Eq. 205 the residues of f that correspond to the poles at z_1, z_2 are:

$$\begin{aligned}
{}_1a_{-1} &= \lim_{z\to -i}\left[\frac{1}{1!}\frac{d}{dz}\left\{(z+i)^2\frac{e^{iz}}{(z+i)^2(z-4)^2}\right\}\right] = \lim_{z\to -i}\left[\frac{d}{dz}\left\{\frac{e^{iz}}{(z-4)^2}\right\}\right]\\
&= \lim_{z\to -i}\left[\frac{ie^{iz}(z-4)^2-2(z-4)e^{iz}}{(z-4)^4}\right] = \frac{ie^1(-i-4)^2-2(-i-4)e^1}{(-i-4)^4} = \frac{(240+i161)e}{4913}
\end{aligned}$$

$$
\begin{aligned}
{}_2a_{-1} &= \lim_{z\to+4}\left[\frac{1}{1!}\frac{d}{dz}\left\{(z-4)^2\frac{e^{iz}}{(z+i)^2(z-4)^2}\right\}\right] = \lim_{z\to+4}\left[\frac{d}{dz}\left\{\frac{e^{iz}}{(z+i)^2}\right\}\right] \\
&= \lim_{z\to+4}\left[\frac{ie^{iz}(z+i)^2-2(z+i)e^{iz}}{(z+i)^4}\right] = \frac{ie^{i4}(4+i)^2-2(4+i)e^{i4}}{(4+i)^4} = \frac{(32+i349)e^{i4}}{4913}
\end{aligned}
$$

(**c**) Here, f has a pole of order $n_1 = 1$ at $z_1 = 1 - i2$ and another pole of order $n_2 = 3$ at $z_2 = -5 - i$. So, from Eq. 205 the residues of f that correspond to the poles at z_1, z_2 are:

$$
\begin{aligned}
{}_1a_{-1} &= \lim_{z\to1-i2}\left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{(z-1+i2)\frac{\sinh(3z)}{(z-1+i2)(z+5+i)^3}\right\}\right] = \lim_{z\to1-i2}\left[\frac{\sinh(3z)}{(z+5+i)^3}\right] \\
&= \frac{\sinh(3-i6)}{(1-i2+5+i)^3} = \frac{\sinh(3-i6)}{(6-i)^3} = \frac{-i\sin(6+i3)}{(6-i)^3} = \frac{107-i198}{50653}\sin(6+i3) \\
&\simeq 0.03166+i0.03132 \\
{}_2a_{-1} &= \lim_{z\to-5-i}\left[\frac{1}{2!}\frac{d^2}{dz^2}\left\{(z+5+i)^3\frac{\sinh(3z)}{(z-1+i2)(z+5+i)^3}\right\}\right] = \lim_{z\to-5-i}\left[\frac{1}{2!}\frac{d^2}{dz^2}\left\{\frac{\sinh(3z)}{z-1+i2}\right\}\right] \\
&= \lim_{z\to-5-i}\left[\frac{9(z-1+i2)^2\sinh(3z)-6(z-1+i2)\cosh(3z)+2\sinh(3z)}{2\,(z-1+i2)^3}\right] \\
&= \frac{9(6-i)^2\sinh(15+3i)-(36-i6)\cosh(15+3i)+2\sinh(15+3i)}{2\,(6-i)^3} \\
&\simeq -1085504.02794-i6137.4692
\end{aligned}
$$

(**d**) Here, f has an infinite number of simple poles at $z = k\pi$ where k is an integer (see part i of Problem 3 of § 3.3). So, from Eq. 205 the residues of f that correspond to these poles are:

$$
\begin{aligned}
{}_ka_{-1} &= \lim_{z\to k\pi}\left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{(z-k\pi)\cot(z)\right\}\right] \qquad (k=0,\pm1,\pm2,\cdots) \\
&= \lim_{z\to k\pi}\left[(z-k\pi)\cot(z)\right] = \lim_{z\to k\pi}\left[\frac{\cos(z)}{\frac{\sin(z)}{z-k\pi}}\right] = \lim_{z\to k\pi}\left[\frac{\cos(z)}{\frac{\sin(z)-0}{z-k\pi}}\right] \\
&= \lim_{z\to k\pi}\left[\frac{\cos(z)}{\frac{\sin(z)-\sin(k\pi)}{z-k\pi}}\right] = \frac{\lim_{z\to k\pi}\cos(z)}{\lim_{z\to k\pi}\frac{\sin(z)-\sin(k\pi)}{z-k\pi}} = \frac{\cos(k\pi)}{\left.\frac{d\sin(z)}{dz}\right|_{z=k\pi}} \\
&= \frac{\cos(k\pi)}{\left.\cos(z)\right|_{z=k\pi}} = \frac{\cos(k\pi)}{\cos(k\pi)} = 1
\end{aligned}
$$

So, the residue of $\cot z$ at each singularity (i.e. at $z = k\pi = 0, \pm\pi, \pm2\pi, \cdots$) is 1.
(**e**) We have $f = \csc z = 1/\sin z$ and hence f has an infinite number of simple poles at $z = k\pi$ where k is an integer (noting that $\sin z$ has zeros at these points according to the result of part b of Problem 14 of § 2.3 and these zeros are simple since the derivative of $\sin z$ does not vanish at these points). So, from Eq. 205 the residues of f that correspond to these poles are:

$$
\begin{aligned}
{}_ka_{-1} &= \lim_{z\to k\pi}\left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{(z-k\pi)\csc(z)\right\}\right] \qquad (k=0,\pm1,\pm2,\cdots) \\
&= \lim_{z\to k\pi}\left[(z-k\pi)\csc(z)\right] = \lim_{z\to k\pi}\left[\frac{1}{\frac{\sin(z)}{z-k\pi}}\right] = \lim_{z\to k\pi}\left[\frac{1}{\frac{\sin(z)-0}{z-k\pi}}\right] \\
&= \lim_{z\to k\pi}\left[\frac{1}{\frac{\sin(z)-\sin(k\pi)}{z-k\pi}}\right] = \frac{1}{\lim_{z\to k\pi}\frac{\sin(z)-\sin(k\pi)}{z-k\pi}} = \frac{1}{\left.\frac{d\sin(z)}{dz}\right|_{z=k\pi}}
\end{aligned}
$$

$$= \frac{1}{\cos(z)\Big|_{z=k\pi}} = \frac{1}{\cos(k\pi)} = (-1)^k$$

So, the residue of $\csc z$ at each singularity (i.e. at $z = k\pi = 0, \pm\pi, \pm 2\pi, \cdots$) is $(-1)^k$.

8. Verify Eq. 204.
Answer: If f has a pole of order n at z_0 then $g(z) = (z - z_0)^n f$ is *analytic* at z_0 and hence it should have a *Taylor* series:[248]

$$\begin{aligned} g(z) &= (z - z_0)^n f = \alpha_0 + \alpha_1(z - z_0) + \cdots + \alpha_{n-1}(z - z_0)^{n-1} + \alpha_n(z - z_0)^n + \cdots \quad (206) \\ \text{where} \quad \alpha_k &= \frac{1}{k!}\frac{d^k g}{dz^k}\Big|_{z=z_0} \qquad (k = 0, 1, \cdots) \end{aligned}$$

Also, from the Laurent series of f (see Eq. 199) we have:

$$\begin{aligned} (z - z_0)^n f &= (z - z_0)^n \Big[a_{-n}(z - z_0)^{-n} + a_{-n+1}(z - z_0)^{-n+1} + \cdots + \\ &\qquad a_{-2}(z - z_0)^{-2} + a_{-1}(z - z_0)^{-1} + a_0 + a_1(z - z_0) + \cdots\Big] \\ &= a_{-n} + a_{-n+1}(z - z_0) + \cdots + \\ &\qquad a_{-2}(z - z_0)^{n-2} + a_{-1}(z - z_0)^{n-1} + a_0(z - z_0)^n + a_1(z - z_0)^{n+1} + \cdots \quad (207) \end{aligned}$$

On comparing the coefficients of $(z - z_0)^{n-1}$ in Eqs. 206 and 207 we get:

$$a_{-1} = \alpha_{n-1} = \frac{1}{(n-1)!}\frac{d^{n-1}g}{dz^{n-1}}\Big|_{z=z_0} = \lim_{z\to z_0}\left[\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\Big\{(z - z_0)^n f\Big\}\right]$$

where we took the limit because z_0 is actually a (removable) singularity of g (since z_0 is a pole of f) noting that this limit does exist (see § 3.3).

9. Evaluate the following real definite integrals using the residue theorem and check the results by the techniques of real analysis (i.e. analytically or numerically).
(**a**) $I = \int_0^{2\pi} \frac{1}{\cos\theta + 2} d\theta$. (**b**) $I = \int_0^{2\pi} \frac{1}{\sin\theta + \pi} d\theta$. (**c**) $I = \int_0^{2\pi} \frac{\sin^2\theta}{1 - \cos\theta} d\theta$.
Answer:[249]
(**a**) We link this integral to a contour integral around the origin-centered unit circle $|z| = 1$ (which we label C). Accordingly, on this circle we have $z = e^{i\theta}$ $(0 \le \theta < 2\pi)$ with $dz = ie^{i\theta}d\theta = iz\,d\theta$. Now, if we use the definition of $\cos\theta$ (as given by Eq. 131) then we get:

$$\begin{aligned} \int_0^{2\pi} \frac{1}{\cos\theta + 2}\,d\theta &= \int_0^{2\pi} \frac{1}{\frac{e^{i\theta}+e^{-i\theta}}{2} + 2}\,d\theta = \int_0^{2\pi} \frac{2}{e^{i\theta} + e^{-i\theta} + 4}\,d\theta = \oint_C \frac{2}{(z + z^{-1} + 4)}\frac{dz}{iz} \\ &= \frac{2}{i}\oint_C \frac{1}{z^2 + 4z + 1}\,dz = \frac{2}{i}\oint_C \frac{1}{(z + 2 - \sqrt{3})(z + 2 + \sqrt{3})}\,dz \quad (208) \end{aligned}$$

As we see, the integrand of this complex contour integral has two poles of order 1: one at $z_0 = -2 + \sqrt{3} \simeq -0.2679$ (which is inside C) and one at $z_1 = -2 - \sqrt{3} \simeq -3.7321$ (which is outside C). So, if we have to apply the residue theorem to evaluate the integral in the last equality of Eq. 208 then we should use Eq. 202 (since C encloses only one pole). As we see, the use of Eq. 202 requires the

[248] To be more precise, z_0 is a removable singularity of g and hence g is *effectively* analytic at z_0 and thus g has a (Taylor-like) power series at z_0.

[249] The method that we use here to evaluate the integrals in parts (a) and (b) applies to any integral of a rational function of $\cos\theta$ or $\sin\theta$ with θ ranging between 0 and 2π and the denominator does not vanish at any point within this range. This method can also be used to evaluate integrals of this type over the range $0 \le \theta \le \pi$ if the integrand is symmetric with respect to $\theta = \pi$. Regarding part (c), the integrand has removable singularities at 0 and 2π and hence the integrand remains bounded over the stated range.

calculation of the residue (corresponding to the pole at z_0) of the integrand by using Eq. 204, that is:

$$a_{-1} = \lim_{z\to -2+\sqrt{3}} \left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{\frac{(z+2-\sqrt{3})}{(z+2-\sqrt{3})(z+2+\sqrt{3})}\right\}\right] = \lim_{z\to -2+\sqrt{3}} \left[\frac{1}{z+2+\sqrt{3}}\right] = \frac{1}{2\sqrt{3}}$$

Accordingly, from the last equality of Eq. 208 (in conjunction with Eq. 202 and the value of a_{-1} that we just obtained) we get:

$$\int_0^{2\pi} \frac{1}{\cos\theta + 2} d\theta = \frac{2}{i}\left(i2\pi\frac{1}{2\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}} \simeq 3.62759872846844$$

On evaluating the real integral numerically (using a numerical integrator) we get $I \simeq 3.627598728468$ which confirms the result obtained by the residue theorem.

(**b**) We link this integral to a contour integral around the origin-centered unit circle $|z| = 1$ (which we label C). Accordingly, on this circle we have $z = e^{i\theta}$ $(0 \le \theta < 2\pi)$ with $dz = ie^{i\theta}d\theta = iz\,d\theta$. Now, if we use the definition of $\sin\theta$ (as given by Eq. 131) then we get:

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{\sin\theta + \pi} d\theta &= \int_0^{2\pi} \frac{1}{\frac{e^{i\theta}-e^{-i\theta}}{i2} + \pi} d\theta = \int_0^{2\pi} \frac{i2}{e^{i\theta} - e^{-i\theta} + i2\pi} d\theta \\
&= \oint_C \frac{i2}{(z - z^{-1} + i2\pi)}\frac{dz}{iz} = \oint_C \frac{2}{z^2 + i2\pi z - 1} dz \\
&= \oint_C \frac{2}{\left[z - i\left(-\pi + \sqrt{\pi^2-1}\right)\right]\left[z - i\left(-\pi - \sqrt{\pi^2-1}\right)\right]} dz
\end{aligned} \quad (209)$$

As we see, the integrand of this complex contour integral has two poles of order 1: one at $z_0 = i\left(-\pi + \sqrt{\pi^2-1}\right) \simeq -i0.1634$ (which is inside C) and one at $z_1 = i\left(-\pi - \sqrt{\pi^2-1}\right) \simeq -i6.1198$ (which is outside C). So, if we have to apply the residue theorem to evaluate the integral in the last equality of Eq. 209 then we should use Eq. 202 (since C encloses only one pole). As we see, the use of Eq. 202 requires the calculation of the residue (corresponding to the pole at z_0) of the integrand by using Eq. 204, that is:

$$\begin{aligned}
a_{-1} &= \lim_{z\to i\left(-\pi+\sqrt{\pi^2-1}\right)} \left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{\frac{2\left[z - i\left(-\pi + \sqrt{\pi^2-1}\right)\right]}{\left[z - i\left(-\pi + \sqrt{\pi^2-1}\right)\right]\left[z - i\left(-\pi - \sqrt{\pi^2-1}\right)\right]}\right\}\right] \\
&= \lim_{z\to i\left(-\pi+\sqrt{\pi^2-1}\right)} \left[\frac{2}{z - i\left(-\pi - \sqrt{\pi^2-1}\right)}\right] = \frac{-i}{\sqrt{\pi^2-1}}
\end{aligned}$$

Accordingly, from the last equality of Eq. 209 (in conjunction with Eq. 202 and the value of a_{-1} that we just obtained) we get:

$$\int_0^{2\pi} \frac{1}{\sin\theta + \pi} d\theta = i2\pi\frac{-i}{\sqrt{\pi^2-1}} = \frac{2\pi}{\sqrt{\pi^2-1}} \simeq 2.10973420122964$$

On evaluating the real integral numerically (using a numerical integrator) we get $I \simeq 2.1097342012296$ which confirms the result obtained by the residue theorem.

(**c**) We link this integral to a contour integral around the origin-centered unit circle $|z| = 1$ (which we label C). Accordingly, on this circle we have $z = e^{i\theta}$ $(0 \le \theta < 2\pi)$ with $dz = ie^{i\theta}d\theta = iz\,d\theta$. Now, if we simplify the integrand[250] (by using the identity $\cos^2\theta + \sin^2\theta = 1$) and use the definition of $\cos\theta$ (as given by Eq. 131) then we get:

$$\int_0^{2\pi} \frac{\sin^2\theta}{1-\cos\theta} d\theta = \int_0^{2\pi} \frac{1-\cos^2\theta}{1-\cos\theta} d\theta = \int_0^{2\pi} (1+\cos\theta)\, d\theta = \int_0^{2\pi} \left(1 + \frac{e^{i\theta} + e^{-i\theta}}{2}\right) d\theta$$

[250] In fact, the integrand has removable singularities at the two limits (since it is 0/0) which are removed by this simplification.

$$= \frac{1}{2}\int_0^{2\pi}\left(2+e^{i\theta}+e^{-i\theta}\right)d\theta = \frac{1}{2}\oint_C\left(2+z+z^{-1}\right)\frac{dz}{iz} = \frac{1}{i2}\oint_C\left(\frac{2}{z}+1+\frac{1}{z^2}\right)dz$$

As we see, the integrand of this complex contour integral has one pole at $z_0 = 0$ (which is inside C). Moreover, the residue of the integrand (corresponding to this pole) is $a_{-1} = 2$ (because a_{-1} by definition is the coefficient of the $\frac{1}{z}$ term). Accordingly, we apply the residue theorem immediately (using Eq. 202) with no need for evaluating a_{-1} (by using Eq. 204), that is:

$$\int_0^{2\pi}\frac{\sin^2\theta}{1-\cos\theta}d\theta = \frac{1}{i2}\left(i2\pi a_{-1}\right) = \frac{i4\pi}{i2} = 2\pi \simeq 6.28318530717959$$

On evaluating the real integral numerically (using a numerical integrator) we get $I \simeq 6.2831853071795$ which confirms the result obtained by the residue theorem. We may also evaluate the real integral (in its simplified form) analytically, that is:

$$\int_0^{2\pi}(1+\cos\theta)d\theta = \Big[\theta+\sin\theta\Big]_0^{2\pi} = \Big[2\pi+\sin 2\pi\Big] - \Big[0+\sin 0\Big] = \Big[2\pi+0\Big] - \Big[0+0\Big] = 2\pi$$

10. Let $f(z)$ be an analytic function with a simple pole at a given point z_0 and let C be a z_0-centered circle (of radius ρ) on and inside which there is no singularity of f other than z_0. Obtain a formula for the value of the contour integral of f along a circular arc C_1 that is part of C.
Answer: The circle C is represented by $z = z_0 + \rho e^{i\theta}$ $(0 \le \theta < 2\pi)$ with $dz = i\rho e^{i\theta}d\theta$. Because inside C f has only one simple pole, f can be represented by a z_0-centered Laurent series as (see Eq. 199):

$$f(z) = a_{-1}(z-z_0)^{-1} + \sum_{n=0}^{\infty} a_n(z-z_0)^n = a_{-1}(z-z_0)^{-1} + A(z)$$

On contour-integrating f along C we get:

$$\oint_C f(z)\,dz = \oint_C\Big[a_{-1}(z-z_0)^{-1}+A(z)\Big]dz = a_{-1}\oint_C(z-z_0)^{-1}\,dz + \oint_C A(z)\,dz = a_{-1}\oint_C(z-z_0)^{-1}\,dz$$

where in the third equality we used the fact that $A(z)$ is analytic on and inside C and hence the integral $\oint_C A(z)\,dz$ is zero according to Cauchy's theorem (see § 4.2). Now, if C_2 is the other arc of the circle (i.e. $C = C_1 \cup C_2$) then we have:

$$\begin{aligned}\oint_C f(z)\,dz &= a_{-1}\oint_C(z-z_0)^{-1}\,dz = a_{-1}\oint_C(\rho e^{i\theta})^{-1}\,i\rho e^{i\theta}d\theta = ia_{-1}\oint_C d\theta \\ &= ia_{-1}\left[\int_{C_1}d\theta+\int_{C_2}d\theta\right] = ia_{-1}\int_{C_1}d\theta + ia_{-1}\int_{C_2}d\theta = ia_{-1}\theta_1 + ia_{-1}\theta_2\end{aligned}$$

where θ_1, θ_2 are the angular measurements (in radian) of C_1, C_2 (i.e. C_1, C_2 are the circular arcs subtending the central angles θ_1, θ_2). As we see, the value of the contour integral of f along C_1 (as well as along C_2) is proportional to its angular measurement with a proportionality factor of ia_{-1}, that is:[251]

$$\int_{C_1} f(z)\,dz = ia_{-1}\theta_1 \tag{210}$$

Note: the result of Problem 6 of 4.2.1 (with $n = 1$ and $a_{-1} = 1$) can be seen as an instance of the result of the present Problem corresponding to $\theta_1 = 2\pi$ and $\theta_2 = 0$ (when the circular arc C_1 represents the entire circle C).

[251] We note that the angular measurement of a circular arc should include even the sense of the angle (i.e. + for anticlockwise and − for clockwise).

11. Show that if $f(z)$ and $g(z)$ are two analytic functions with f having a simple pole with residue a_{-1} at a given point z_0 but g is analytic at z_0 then the residue A_{-1} of $f(z)g(z)$ at z_0 is given by $A_{-1} = a_{-1}\, g(z_0)$.
Answer: Because at z_0 f has a simple pole while g is analytic, their product fg should have a simple pole at z_0. Hence, from Eq. 204 $\big[$with A_{-1} replacing a_{-1} and with $f(z)g(z)$ replacing $f\big]$ we have:

$$\begin{aligned} A_{-1} &= \lim_{z\to z_0}\left[\frac{1}{(1-1)!}\frac{d^{1-1}}{dz^{1-1}}\Big\{(z-z_0)f(z)g(z)\Big\}\right] = \lim_{z\to z_0}\big[(z-z_0)f(z)g(z)\big] \\ &= \left(\lim_{z\to z_0}\big[(z-z_0)f(z)\big]\right)\times\left(\lim_{z\to z_0}\big[g(z)\big]\right) = a_{-1}\times g(z_0) = a_{-1}\, g(z_0) \end{aligned}$$

12. Evaluate the following contour integrals around the given contours C:
(**a**) $I = \oint_C \frac{z^2+2}{(z-4)(z-5)(z+i7)}\, dz$ where C is the (anticlockwise) circle $|z-3| = 5$.
(**b**) $I = \oint_C \frac{\sin z}{z^2(z^2+4)}\, dz$ where C is the (anticlockwise) circle $|z-i| = 2$.
Answer: For solving this Problem we apply a number of theorems and techniques as will be explained in the following.[252] It is noteworthy that these integrals satisfy the conditions and requirements of the theorems and techniques that we will apply (e.g. the integrands are analytic on and inside C except at the enclosed singularities, etc.) and hence to save space and time we will not mention these conditions and requirements in detail although by using these theorems and techniques we implicitly assume the satisfaction of these conditions and requirements.
(**a**) The integral has a simple pole at $z_1 = 4$, a simple pole at $z_2 = 5$ and a simple pole at $z_3 = -i7$. However, the pole at z_3 is not enclosed by C and hence it does not contribute to the integral. So, according to the extended Cauchy's theorem (see § 4.2.1 and Eq. 175 in particular) we can write:

$$\begin{aligned} I &= \oint_{C_1} \frac{z^2+2}{(z-4)(z-5)(z+i7)}\, dz + \oint_{C_2} \frac{z^2+2}{(z-4)(z-5)(z+i7)}\, dz \\ &= \oint_{C_1} \frac{\left(\frac{z^2+2}{(z-5)(z+i7)}\right)}{(z-4)}\, dz + \oint_{C_2} \frac{\left(\frac{z^2+2}{(z-4)(z+i7)}\right)}{(z-5)}\, dz = I_1 + I_2 \end{aligned}$$

where C_1 is a tiny circle surrounding z_1 only and C_2 is a tiny circle surrounding z_2 only and where in line 2 we just manipulated the integrands. So, all we need to do now is to evaluate I_1 and I_2 which can be done by using more than one technique. For example, we can use Cauchy's integral formula (see Eq. 179) and hence:[253]

$$\begin{aligned} I_1 &= i2\pi f(4) = i2\pi \frac{4^2+2}{(4-5)(4+i7)} = \frac{-i36\pi}{4+i7} \\ I_2 &= i2\pi f(5) = i2\pi \frac{5^2+2}{(5-4)(5+i7)} = \frac{i54\pi}{5+i7} \end{aligned}$$

We may also use the residue theorem where:

$$\begin{aligned} {}_1a_{-1} &= \lim_{z\to 4}\left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{\frac{z^2+2}{(z-5)(z+i7)}\right\}\right] = \lim_{z\to 4}\left[\frac{z^2+2}{(z-5)(z+i7)}\right] = \frac{-18}{4+i7} \\ {}_2a_{-1} &= \lim_{z\to 5}\left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{\frac{z^2+2}{(z-4)(z+i7)}\right\}\right] = \lim_{z\to 5}\left[\frac{z^2+2}{(z-4)(z+i7)}\right] = \frac{27}{5+i7} \end{aligned}$$

[252] In fact, this Problem is not specific to the calculus of residues. However, we put it here because (as indicated above) we use in the solution a number of theorems and techniques (including the calculus of residues) and hence it should come after full investigation of these theorems and techniques (i.e. in its current position).

[253] In the following, f in each line represents the function in the numerator of the integrand of the corresponding integral, i.e. $f = \frac{z^2+2}{(z-5)(z+i7)}$ for I_1 and $f = \frac{z^2+2}{(z-4)(z+i7)}$ for I_2.

Hence, $I_1 = i2\pi\,{}_1a_{-1} = \frac{-i36\pi}{4+i7}$ and $I_2 = i2\pi\,{}_2a_{-1} = \frac{i54\pi}{5+i7}$ (as before).
Accordingly:

$$I = I_1 + I_2 = \frac{-i36\pi}{4+i7} + \frac{i54\pi}{5+i7} = \frac{2961 + i3447}{2405}\pi \simeq 3.8679 + i4.5027$$

(**b**) The integral has a double pole at $z_1 = 0$, a simple pole at $z_2 = i2$ and a simple pole at $z_3 = -i2$. However, the pole at z_3 is not enclosed by C and hence it does not contribute to the integral. So, according to the extended Cauchy's theorem (see § 4.2.1) we can write:

$$\begin{aligned} I &= \oint_{C_1} \frac{\sin z}{z^2(z^2+4)}\, dz + \oint_{C_2} \frac{\sin z}{z^2(z^2+4)}\, dz \\ &= \oint_{C_1} \frac{\left(\frac{\sin z}{z^2+4}\right)}{z^2}\, dz + \oint_{C_2} \frac{\left(\frac{\sin z}{z^2(z+i2)}\right)}{z - i2}\, dz = I_1 + I_2 \end{aligned}$$

where C_1 is a tiny circle surrounding z_1 only and C_2 is a tiny circle surrounding z_2 only and where in line 2 we just manipulated the integrands. So, all we need to do now is to evaluate I_1 and I_2 which can be done by using more than one technique. For example, we can use Cauchy's integral formula for derivatives (as given by Eq. 181 including $n = 0$) and hence:[254]

$$\begin{aligned} I_1 &= i2\pi f^{(1)}(0) = i2\pi \frac{(z^2+4)\cos z - 2z\sin z}{(z^2+4)^2}\Big|_{z=0} = i2\pi\frac{4}{16} = i\frac{\pi}{2} \\ I_2 &= i2\pi f^{(0)}(i2) = i2\pi \frac{\sin z}{z^2(z+i2)}\Big|_{z=i2} = i2\pi\frac{\sin(i2)}{(i2)^2(i2+i2)} = -\frac{\pi}{8}\sin(i2) = -i\frac{\pi}{8}\sinh 2 \end{aligned}$$

We may also use the residue theorem where:

$$\begin{aligned} {}_1a_{-1} &= \lim_{z\to 0}\left[\frac{1}{(2-1)!}\frac{d}{dz}\left\{\frac{\sin z}{z^2+4}\right\}\right] = \lim_{z\to 0}\left[\frac{(z^2+4)\cos z - 2z\sin z}{(z^2+4)^2}\right] = \frac{1}{4} \\ {}_2a_{-1} &= \lim_{z\to i2}\left[\frac{1}{0!}\frac{d^0}{dz^0}\left\{\frac{\sin z}{z^2(z+i2)}\right\}\right] = \lim_{z\to i2}\left[\frac{\sin z}{z^2(z+i2)}\right] = \frac{\sin(i2)}{-i16} = -\frac{\sinh 2}{16} \end{aligned}$$

Hence, $I_1 = i2\pi\,{}_1a_{-1} = i\frac{\pi}{2}$ and $I_2 = i2\pi\,{}_2a_{-1} = -i\frac{\pi}{8}\sinh 2$ (as before).
Accordingly:

$$I = I_1 + I_2 = i\frac{\pi}{2} - i\frac{\pi}{8}\sinh 2 = i\pi\left(\frac{1}{2} - \frac{\sinh 2}{8}\right) \simeq i0.1465$$

Note: as we see, the above techniques (which are supposedly different) are essentially the same although they may look different (superficially).

[254] In the following, f in each line represents the function in the numerator of the integrand of the corresponding integral, i.e. $f = \frac{\sin z}{z^2+4}$ for I_1 and $f = \frac{\sin z}{z^2(z+i2)}$ for I_2.

Chapter 6
Complex Transformations

Earlier in the book, we presented (rather casually) a number of examples of specific transformations of objects (such as straight lines) from the z plane to the w plane (see for instance § 1.11). These transformations (which may also be described as mappings) are achieved by using certain complex functions and relations. In the present chapter, we extend our previous investigation by examining complex transformations more thoroughly and generally (taking account of the available space and scope). However, before we go through the details of this investigation let have some useful background remarks about the subject of investigation and how it is approached and handled:

• In general, any transformation has a geometric aspect which is represented by the action used in its realization such as rotation or reflection, and an algebraic (or mathematical to be more general) aspect which is represented by the mathematical relation used in its representation and implementation such as $w(z) = z^2 - i$. Accordingly, complex transformations can be seen as representations of complex functions[255] and hence they follow in their categorization and classification the types of functions to which they belong. For example, a linear/quadratic/cubic/trigonometric transformation is a mapping conducted (or represented) by using a linear/quadratic/cubic/trigonometric mathematical relation (or function).

• A matter related to the previous remark is that "functions" and "transformations" may be used interchangeably and hence the attributes of transformations may be ascribed to their functions and vice versa.

• In this chapter we will tackle the issue of transformations in the complex plane by complex functions from two sides. One side is the geometric nature of transformations and their relation to their mathematical representations (considering the effect of transformations on a general point in the complex plane), and the other side is the geometric effect of transformations on specific geometric objects (like straight lines and circles) in the complex plane.[256] In other words, one side is about the geometric nature of the transformation as an action inflicted on a point (any point) in the complex plane, while the other side is about the specific geometric effect inflicted by the transformation on selected subsets of the complex plane such as the effect inflicted on a straight line to make it (or map it on) a parabola. In fact, the former is essentially about the geometric nature of the transformation itself (since it essentially reveals its effect on a single point), while the latter is essentially about the nature of the transformed geometric objects (since it essentially reveals its effect on a selected set of points and their geometric relation to each other, e.g. being part of a straight line or being part of a circle).

• Some transformations have simple geometric nature and effect and hence they can be described by simple words. For example, the transformations $w(z) = -z$ and $w(z) = z + i$ can be simply described as rotation by π and translation by 1 unit up. These transformations are generally based on simple algebraic operations (or combinations of such operations). Other transformations have more complex geometric nature and effect and hence they cannot be described by simple words. For example, it is not easy to describe the geometric nature and effect of $\sin z$ or $\tan z$ or $\ln z$. These transformations are mostly based on transcendental mathematical operations (or combinations of such operations) and hence they do not represent actions that we are familiar with in our daily life. Accordingly, most of the examples related to the geometric nature of transformations belong to the previous category where we can provide simple and comprehensible verbal descriptions.

[255] To be more general, we should use "relations" instead of "functions" since complex transformations may be conducted by using mathematical relations that are not functions (in the technical sense of function). However, we tolerate this laxity for simplicity and because the overwhelming majority of complex transformations are conducted by using functions.

[256] For an example of the former side we refer to Problems 2 and 3 of § 6.2, while for an example of the latter side we refer to Problems 4 and 5 of § 6.2.

• Curves and shapes (including regions) in the complex plane have two types of mathematical representations: real form and complex form. In the real form the curves (like straight lines) and shapes (like circles or disks) are represented by mathematical relations that involve real variables only such as $y = f(x)$ or $v = g(u)$, while in the complex form the curves and shapes are represented by mathematical relations that involve complex variables such as $w = f(z)$. For example, in the real form an origin-centered circle is given as $x^2 + y^2 = R^2$ while in the complex form it is given as $|z| = R$. The complex form may not involve complex variables (such as z and w) explicitly but it is facilitated by the use of the imaginary unit i (beside real variables like x and y). For example, a straight line (which is usually given in the real form as $y = ax + b$) may be given in the complex form as $x + i(ax + b)$. In general, we use (here and elsewhere) whatever convenient for the problem at hand and we may even shift from one form to another using the relations between the two forms.
• For the sake of simplicity we dumped the types of general transformations into a single list (represented by the sections of the present chapter) although they actually belong to different classifications and categorizations. For instance, linear and non-linear transformations (which are investigated in § 6.1 and § 6.2) belong to a comprehensive and unique classification but conformal transformation (which is investigated in § 6.4) is a non-comprehensive type of transformation that can be linear or non-linear and hence it does not stand as distinct or opposite to linear or non-linear transformations. This similarly applies to Schwarz-Christoffel transformation (see § 6.5) which is generally a type of conformal non-linear transformation and hence it does not stand (as such) as distinct or opposite to linear or non-linear or conformal transformations.
• As a matter of terminology, we use “source” (or “inverse image”) to refer to an object (in the z plane) that is subject to a transformation, while we use “image” to refer to its map (in the w plane) under that transformation. For example, when we transform a straight line in the z plane to a parabola in the w plane then the straight line is the source (or inverse image) and the parabola is the image.
• Transformations are generally not commutative (i.e. the overall effect depends on the order) and hence in principle the effect of transformation A followed by transformation B is not the same as the effect of transformation B followed by transformation A. Accordingly, when we deal with composite transformations (i.e. those made of basic or sub-transformations) it is important to identify the order of the basic transformations.
• A fixed point z_0 of a given complex transformation $w(z)$ is a point that is mapped by that transformation on itself, i.e. $w(z_0) = z_0$. Some transformations have no fixed points while other transformations have fixed points which could be finite or infinite in number. For example, the identity transformation $w(z) = z$ has infinite number of fixed points since it fixes the entire transformed object (which could be the entire complex plane) due to its geometric nature as “do nothing” operation. Similarly, the reflection transformations across straight lines (some examples of which are given in Problem 1) fix the points on the line of reflection and hence they also have infinite number of fixed points. On the other hand, some transformations do not have fixed points at all. This is seen in particular in the transformations that involve translation due to the nature of translation which moves the points of the transformed object. For example, the transformation $w(z) = z + 1$ has no fixed points since it translates every point in the z plane one unit to the right (i.e. in the positive direction of the real axis) and hence all the points in the z plane that are subjected to this transformation are moved. Many other transformations (such as most types of polynomial transformations) have a finite number of fixed points (see for instance Problems 5 and 6). As we will see (refer to Problem 4), the fixed points of a given transformation $w(z)$ can be easily found (if they exist) by solving the equation $w(z) = z$ (and hence the transformation has no fixed points if this equation has no solution).
• Complex transformations is a big subject and hence what we provide in this chapter is just a glimpse. Therefore, the readers who are interested in acquiring detailed knowledge on this subject should look for more extensive and specialized textbooks and papers.

Problems

1. Make a short list of some basic geometric operations with their corresponding mathematical operations.
   Answer: For example:

- Scaling up or down by a factor $|a|$: multiply by $|a|$, i.e. $w = |a|\, z$.
- Rotation (around the origin) by π: multiply by -1, i.e. $w = -z$.[257]
- Rotation (around the origin) by $\pm\pi/2$: multiply by $\pm i$, i.e. $w = \pm iz$.
- Rotation (around the origin) by $\pm(2n+1)\pi$: multiply by -1, i.e. $w = -z$.
- Rotation (around the origin) by $\pm 2n\pi$: do nothing (or multiply by 1), i.e. $w = z$.
- Rotation (around the origin) by θ: multiply by $e^{i\theta}$, i.e. $w = e^{i\theta} z$.
- Translation by $\pm |a|$ along the real axis: add $\pm |a|$, i.e. $w = z \pm |a|$.
- Translation by $\pm |a|$ along the imaginary axis: add $\pm i\, |a|$, i.e. $w = z \pm i\, |a|$.
- Reflection in the real axis: take conjugate, i.e. $w = z^*$.
- Reflection in the imaginary axis: take negative conjugate, i.e. $w = -z^*$.
- Reflection in the origin: multiply by -1, i.e. $w = -z$.

Note: more complicated operations are made by combining the above basic operations (and their alike). We should also note that for the aforementioned specific rotations around the origin $\big[$i.e. by π, $\pm\pi/2$, $\pm(2n+1)\pi$ and $\pm 2n\pi\big]$ it may be more explicit (and even more appropriate in some cases) to use the polar form representation of these rotations, i.e. multiplying by $e^{i\pi}$, $e^{\pm i\pi/2}$, $e^{\pm i(2n+1)\pi}$ and $e^{\pm i2n\pi}$ (as done for the general rotation around the origin by θ through multiplying by $e^{i\theta}$).

2. In the transformation of curves, how can we determine the direction (or sense) of tracking (i.e. when we track a source curve in a given direction, in what direction its image is tracked)?
Answer:[258] For open curves (i.e. both the source and image curves are open), we take two distinct points (say A and B) on the source curve in the z plane and determine their images (say A′ and B′) on the image curve in the w plane.[259] We can then assume (correctly) that as we go in one direction on the source curve (say from A to B) we should go in the corresponding direction on the image curve (i.e. from A′ to B′). We may confirm this by transforming a point between A and B and determining its image which should be between A′ and B′.
For closed curves (i.e. both the source and image curves are closed), we take three distinct points (say A, B and C) on the source curve in the z plane and determine their images (say A′, B′ and C′) on the image curve in the w plane. The sense of tracking the source curve along the sequence A→B→C (i.e. clockwise or anticlockwise) should then determine the sense of tracking the image curve along the sequence A′→B′→C′. For example, if the circle $|z| = 1$ is mapped onto the circle $|z| = 2$ by a given transformation where the points A$(1, 0)$, B$(0, 1)$ and C$(-1, 0)$ are mapped onto the points A′$(2, 0)$, B′$(0, 2)$ and C′$(-2, 0)$ then it is obvious that as the source curve is tracked in a given sense the image curve will be tracked in the same sense. On the other hand, if the points A$(1, 0)$, B$(0, 1)$ and C$(-1, 0)$ are mapped onto the points A′$(2, 0)$, B′$(0, -2)$ and C′$(-2, 0)$ then it is obvious that as the source curve is tracked in a given sense the image curve will be tracked in the opposite sense.
Note: the three-point method is also used if the source and image curves are different (i.e. one open and the other is closed). In fact, the three-point method can be used in all cases (including when both are open). We should also note that in the above we are generally considering simple curves, and hence more elaborate tests may be needed to determine the sense of tracking in the transformation of more complicated curves (e.g. curves with loops).
3. In the transformation of curves and shapes, how can we determine the mapping of regions bordered by these curves and shapes (e.g. a circle is mapped onto another circle by a given transformation and we want to know if the region inside the source circle is mapped by this transformation onto the inside region or the outside region of the image circle)?
Answer: We may take a given point (or general point) inside the source region and determine its image under this transformation assuming (correctly) that the region to which the source point belongs will be mapped onto the region to which the image point belongs. For example, if we mapped a straight line onto a circle by a given transformation and we want to know if the half-plane to the left of the line will be mapped onto the inside region or the outside region of the circle, then we can take a point inside

[257] As we will note, multiplying by unity in polar form to represent rotation transformations around the origin may be more appropriate and intuitive.
[258] In this Problem (and its alike) we consider continuous curves transformed by continuous functions.
[259] Considering the two end points may be the most appropriate for conducting this test.

the half-plane and determine its image under this transformation and hence if the point is mapped inside/outside the circle then the half-plane is mapped by this transformation onto the inside/outside region of the circle.
In some circumstances, the direction (or sense) of tracking the border curves may be used to determine the mapping of regions where it is assumed that the mapping should follow a certain style. For example, if we mapped a circle onto a circle then we may assume that the region to the left/right of the source circle when it is tracked in a certain sense (say clockwise) will be mapped onto the region to the left/right of the image circle when it is tracked in the corresponding sense (whether clockwise or anticlockwise).

4. How to find out if a given transformation $w(z)$ has fixed points or not, and how to find these points (if it has)?
Answer: By definition, a fixed point z_0 of $w(z)$ is a point that satisfies the relation $w(z_0) = z_0$. Therefore, to find out if $w(z)$ has fixed points or not we need to see if the equation $w(z) = z$ has solution (and hence w has fixed points) or not (and hence w has no fixed point). Accordingly, the fixed points are found (assuming this equation has solution) by solving this equation.
5. Show that the non-identity transformation $w(z) = P_n(z)$ where P_n is an n^{th} order polynomial ($n \geq 1$) has at most n fixed points.
Answer: Referring to § 2.1, an n^{th} order polynomial has the form $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$. Now, having a fixed point means that we have $w(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 = z$, i.e.
$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + (a_1 - 1)z + a_0 = 0 \qquad (211)$$
As we see, this (in principle) is an n^{th} order polynomial equation and hence it has n roots (see Problem 1 of § 7.1). Accordingly, (in principle) we should have n fixed points. However, some of these roots can be repetitive and this should reduce the number of fixed points, i.e. we have at most (but possibly less than) n fixed points. Moreover, when $n = 1$ with $a_1 = 1$ (i.e. when the polynomial is linear of the form $z + a_0$) we get (from Eq. 211) $a_0 = 0$. Now, if a_0 is zero then $w(z)$ is the identity transformation (which we excluded from the question), while if a_0 is not zero then $a_0 = 0$ has no solution and hence there is no fixed point. Accordingly, we conclude that a non-identity polynomial transformation of order $n \geq 1$ has at most n fixed points.
Note: it should be obvious from the above answer that the identity polynomial transformation, i.e. $w(z) = z$, has infinite number of fixed points, while some (non-identity) polynomial transformations have less than n fixed points (including 0 fixed points). So to sum up, we can say: the number of fixed points of a polynomial transformation can be 0 or finite or infinite. We may also say: the number N of fixed points of an n^{th} order polynomial transformation (where $n \geq 1$) is either infinite (i.e. $N = \infty$) or finite but less than or equal n including 0 (i.e. $0 \leq N \leq n$) and hence it cannot be finite and greater than n.[260]
6. Find the fixed points of the following polynomial transformations:
(**a**) $w(z) = z + i$. (**b**) $w(z) = 2z + i$. (**c**) $w(z) = (1 - i)z$. (**d**) $w(z) = 2z^2 + 7z + 6$.
(**e**) $w(z) = z^2 + 3z + 1$. (**f**) $w(z) = z^3$. (**g**) $w(z) = z^4 + z^2 + z$.
Answer:
(**a**) As we see, $z + i = z$ has no solution (noting that $i \neq 0$). This result should be intuitive because this linear transformation includes translation (by i) with no rotation or magnification and hence it has no fixed points.
(**b**) If we have a fixed point then $2z + i = z$, i.e. $z + i = 0$. Hence, we have exactly one fixed point, i.e. $z = -i$.
(**c**) It is obvious that $z = 0$ is a fixed point since $w(0) = 0$. Because it is (non-identity) linear transformation it has at most 1 fixed point, i.e. this transformation has exactly 1 fixed point. This should

[260] It should be noted that the above is based on considering n in general. However, if we consider individual n's then we should say: if $n > 1$ then $1 \leq N \leq n$ while if $n = 1$ then we have three cases: $N = 0$ (when $a_1 = 1$ and $a_0 \neq 0$), $N = \infty$ (when $a_1 = 1$ and $a_0 = 0$), and $N = 1$ (when $a_1 \neq 1$).

be obvious because this linear transformation (which can be represented by $w = \sqrt{2}e^{-i\pi/4}z$) is a combination of rotation by $-\pi/4$ around the origin and magnification by $\sqrt{2}$ (with no translation) and hence it should fix only the origin.
(**d**) If we have a fixed point then $2z^2 + 7z + 6 = z$, i.e. $2z^2 + 6z + 6 = 0$. From the quadratic formula we have:

$$z = \frac{-6 \pm \sqrt{36 - 48}}{4} = \frac{-6 \pm \sqrt{-12}}{4} = \frac{-3 \pm i\sqrt{3}}{2}$$

So, these are the fixed points of this transformation (which are the only fixed points because this transformation is quadratic and hence it has no more than two solutions).
(**e**) If we have a fixed point then $z^2 + 3z + 1 = z$, i.e. $z^2 + 2z + 1 = 0$. This is equivalent to $(z + 1)^2 = 0$ which has only one solution. So, we have only one fixed point, i.e. $z = -1$.
(**f**) If we have a fixed point then $z^3 = z$, i.e. $z^3 - z = 0$. This obviously has three solutions which are 0, 1 and -1. So, we have exactly three fixed points (noting that this is cubic and hence it cannot have more than three solutions).
(**g**) If we have a fixed point then $z^4 + z^2 + z = z$, i.e. $z^4 + z^2 = 0$. On factorizing this we get $z^2 \left(z^2 + 1\right) = 0$ which has three (distinct) solutions: 0, i and $-i$. So, we have three fixed points.

7. What is the transformation that maps the interior of the origin-centered unit disk onto its exterior and vice versa?
Answer: It is the reciprocal transformation $w(z) = 1/z$ (which also maps the perimeter of the disk onto itself although not as an identity map). Also, see Problem 2 of § 1.8.9.[261]

6.1 Linear Transformations

Linear transformations are given by the relation $w = az + b$ where a and b are complex constants (which could be real or imaginary as special cases) and $a \neq 0$. This type of transformation represents a magnification of magnitude $|a|$ associated with a rotation by an angle $\arg(a)$ around the origin and followed by a translation by b.[262] For example, if we have $w = (1 + i)z + (2 - i5)$ then this transformation maps a point z_0 in the z plane onto a point w_0 in the w plane where w_0 represents a magnification of point z_0 (as a position vector) by a factor of $|a| = \sqrt{2}$ associated with a rotation of point z_0 (or rather the magnified z_0) around the origin by an angle $\pi/4$ plus a translation (of the magnified and rotated image) of point z_0 by $(2 - i5)$, i.e. 2 units in the positive x direction (i.e. along the real axis) and 5 units in the negative y direction (i.e. along the imaginary axis). The different kinds of linear transformation are thoroughly investigated in the Problems.

Problems

1. As explained in the text, a linear transformation can be split into three main sub-transformations: magnification, rotation and translation. How are the identity elements of these sub-transformations represented? What is the identity linear transformation?
Answer: The identity of magnification is 1, the identity of rotation is 0 (or $2n\pi$ if we consider non-principal values), and the identity of translation is 0. Accordingly, the identity linear transformation is $w = z$ (which represents a magnification by 1, a rotation by 0, and a translation by 0).
2. Justify the above claim that in the linear transformation $w = az + b$ the multiplication of z by a represents a magnification of z by $|a|$ associated with a rotation of z by $\arg(a)$.
Answer: This should be obvious if we use the polar form of a (i.e. $a = |a|\, e^{i\arg(a)}$) where it is evident that the effect of multiplying z by a is to magnify z by $|a|$ and rotate z by $\arg(a)$ around the origin.
3. The linear transformation $w = az + b$ is made of the three sub-transformations: scaling by $|a|$, rotation by $\arg(a)$ and translation by b. Identify (and justify) the order of these sub-transformations.
Answer: The linear transformation $w = az + b$ means scaling by $|a|$ and rotation by $\arg(a)$ in any order, followed by translation by b. This is because z (which represents the source) is first multiplied

[261] The reciprocal transformation is commonly called inversion.
[262] Magnification in this context should mean scaling up or down or by identity depending on the magnitude of $|a|$, i.e. $|a| > 1$ or $|a| < 1$ or $|a| = 1$.

by a (which means the combination of scaling and rotation) and then translated by b (or rather the scaled and rotated image of z is translated by b).[263] So, when we talk about a linear transformation as scaling and rotation *and* translation we mean scaling and rotation (in any order) *followed by* translation. Accordingly, if we want translation by b followed by scaling and rotation then we should use the linear transformation $w = a(z + b) = az + B$ where $B = ab$ is the actual translation in this case.

4. Give examples of types of transformations that are not linear (and hence they cannot be represented by the relation $w = az + b$).
Answer: Common examples are: the reflection in the real or imaginary axes (i.e. z^* and $-z^*$) or squaring (i.e. z^2) or taking the square root (i.e. $\sqrt{z}$) or reciprocation (i.e. $1/z$) or subjecting to exponential/trigonometric/hyperbolic operations (e.g. $e^z, \sin z, \cosh z$). These transformations will be investigated in § 6.2.

5. Determine the geometric nature of the following transformations whose general form is $w = az + b$:
(**a**) $a = 0$ and $b = 0$. (**b**) $a = 0$ and $b \neq 0$. (**c**) $a \neq 0$ and $b = 0$.
(**d**) $w = 2z$. (**e**) $w = -5z$. (**f**) $w = i6z$.
(**g**) $w = -i\sqrt{\pi}\, z$. (**h**) $w = (e + i\pi)z$. (**i**) $w = -(1 + i\sqrt{7})z$.
(**j**) $w = (-1 + i)z + 3$. (**k**) $w = iez + (6 - i9)$. (**l**) $w = (2 - i6)z - ie^{\pi}$.
(**m**) $w = (11 + i\sqrt{3})z + (1 - i)$. (**n**) $w = (4 + i)z + (5 + i\pi)$. (**o**) $w = z + (8 + i)$.
(**p**) $w = z/(1 + i)$. (**q**) $w = 3i^3(z - 3 + i4)$. (**r**) $w = e^{i\pi}(z + i)$.
Answer: All these transformations are linear except the transformations of parts (a) and (b) which are not linear because in these transformations $a = 0$. In the following, "rotation" means around the origin and we consider the principal argument (for specified rotations).
(**a**) A mapping of the entire z plane onto the origin of the w plane (i.e. $w = 0$).[264]
(**b**) A mapping of the entire z plane onto the point b.
(**c**) A scaling by $|a|$ with a rotation by an angle $\arg(a)$.
(**d**) A scaling by 2.
(**e**) A scaling by 5 with a rotation by π.
(**f**) A scaling by 6 with a rotation by $\pi/2$.
(**g**) A scaling by $\sqrt{\pi}$ with a rotation by $-\pi/2$.
(**h**) A scaling by $\sqrt{e^2 + \pi^2}$ with a rotation by $\arctan(\pi/e) \simeq 0.8575$.
(**i**) A scaling by $\sqrt{8}$ with a rotation by $\arctan\left(-\sqrt{7}/[-1]\right) \simeq -1.9322$.
(**j**) A scaling by $\sqrt{2}$ with a rotation by $3\pi/4$ followed by a translation by 3 in the positive x direction.
(**k**) A scaling by e with a rotation by $\pi/2$ followed by a translation by $(6 - i9)$.[265]
(**l**) A scaling by $\sqrt{40}$ with a rotation by $\simeq -1.2490$ followed by a translation by $-e^{\pi}$ in the y direction.
(**m**) A scaling by $\sqrt{124}$ with a rotation by $\simeq 0.1562$ followed by a translation by $(1 - i)$.
(**n**) A scaling by $\sqrt{17}$ with a rotation by $\simeq 0.2450$ followed by a translation by $(5 + i\pi)$.
(**o**) A translation by $(8 + i)$.
(**p**) We have $w = \frac{z}{1+i} = \left(\frac{1-i}{2}\right) z$ and hence this is a scaling by $1/\sqrt{2}$ with a rotation by $-\pi/4$.
(**q**) A translation by $-3 + i4$ followed by a scaling by 3 and a rotation by $-\pi/2$ (or $3\pi/2$).
(**r**) A translation by i followed by a rotation by π.

6. Write mathematical relations (i.e. $w = az + b$) representing the following linear transformations:
(**a**) Rotation by -8π.
(**b**) Scaling by 3 units followed by translation by 5 units downward.
(**c**) Translation by 5 units downward followed by scaling by 3 units.
(**d**) Reflection in the origin of coordinates.

[263] In fact, the saying "z is first multiplied by a and then translated by b" reflects the precedence of the algebraic operations (i.e. multiplication and addition) in the expression $az + b$.
[264] In such context, "mapping of the entire z plane" should mean mapping of the entire transformed object which could be the entire complex plane.
[265] When we say "a translation by $(6 - i9)$" we mean a translation by 6 units in the positive x (or real) direction and by 9 units in the negative y (or imaginary) direction. This also applies to other similar translations.

(**e**) Scaling by 2 and rotation by $3\pi/2$ followed by translation by π units up and 6 units left.
(**f**) Reflection in the origin of coordinates with rotation by 3π followed by scaling by 10.
(**g**) Rotation by $-3\pi/4$ with scaling by 16 followed by translation by 2 units along the line $y = -x$ (from upper left to lower right).
Answer:
(**a**) $w = e^{-i8\pi}z = z$.
(**b**) $w = 3z - i5$.
(**c**) $w = 3(z - i5) = 3z - i15$.
(**d**) $w = -z$.
(**e**) $w = 2e^{i3\pi/2}z + (i\pi - 6) = -i2z + (i\pi - 6)$.
(**f**) $w = 10(-e^{i3\pi}z) = 10z$.
(**g**) $w = 16e^{-i3\pi/4}z + (\sqrt{2} - i\sqrt{2}) = -8\sqrt{2}(1 + i)z + (\sqrt{2} - i\sqrt{2})$.
Note: as indicated earlier, the order of sub-transformations is important in general (e.g. translation followed by rotation is not the same as rotation followed by translation). This can be easily seen by comparing (b) and (c).

7. Determine how the following curves, shapes and regions are transformed from the z plane to the w plane by the given linear transformations:
(**a**) The straight line $y = 3x - 7$ by the transformation $w = 2z + i5$.
(**b**) The circle $|z| = 2$ by the transformation $w = iz - 6 + i$.
(**c**) The half-plane $2y + 8x - 10 > 0$ by the transformation $w = -8z + 11 - i\pi$.
(**d**) The disk $|z - 1 - i2| \leq 3$ by the transformation $w = (2 - i)\, z + 7$.
Answer:
(**a**) This straight line in the z plane can be represented by the complex equation $z = x+iy = x+i(3x-7)$. On applying the transformation $w = 2z + i5$ on a general point on this line we get:

$$w = 2\,[x + i(3x - 7)] + i5 = 2x + i(6x - 14) + i5 = 2x + i(6x - 9) = u + iv$$

So, we have $u = 2x$ and $v = 6x - 9$ which can be combined (by eliminating x) to obtain $v = 3u - 9$ which is an equation of a straight line in the w plane. This straight line can be represented by the complex equation $w = u + iv = u + i(3u - 9)$.
(**b**) The equation of this circle is $|z| = \sqrt{x^2 + y^2} = 2$ and hence $y = \pm\sqrt{4 - x^2}$. Therefore, this circle can be represented by the complex equation $z = x+iy = x\pm i\sqrt{4 - x^2}$. On applying the transformation $w = iz - 6 + i$ on a general point on this circle we get:

$$w = i\left(x \pm i\sqrt{4 - x^2}\right) - 6 + i = ix \mp \sqrt{4 - x^2} - 6 + i = \left(\mp\sqrt{4 - x^2} - 6\right) + i\,(x + 1) = u + iv$$

So, we have $u = \mp\sqrt{4 - x^2} - 6$ and $v = x + 1$ which can be combined to obtain $(u + 6)^2 = 4 - (v - 1)^2$, that is $(u + 6)^2 + (v - 1)^2 = 4$ which is an equation of a circle (in the w plane) with center $w = -6 + i$ and radius $R = 2$.
(**c**) The equation of the border line of this half-plane is $2y + 8x - 10 = 0$ and hence $y = 5 - 4x$. Therefore, the border can be represented by the complex equation $z = x + iy = x + i\,(5 - 4x)$. On applying the transformation $w = -8z + 11 - i\pi$ on a general point on this border line we get:

$$w = -8\,[x + i\,(5 - 4x)] + 11 - i\pi = -8x + i\,(32x - 40) + 11 - i\pi = (11 - 8x) + i\,(32x - 40 - \pi) = u + iv$$

So, $u = 11 - 8x$ and $v = 32x - 40 - \pi$ which can be combined to obtain $v = 32\left(\frac{11-u}{8}\right) - 40 - \pi = -4u + 4 - \pi$ which is an equation of a straight line in the w plane. So, the half-plane in the z plane is mapped by this transformation onto a half-plane in the w plane. However, we still need to determine which of the half-planes in the w plane is the image, i.e. $v + 4u - 4 + \pi > 0$ or $v + 4u - 4 + \pi < 0$. So, let take a specific point in the z half-plane and determine onto which w half-plane this point is mapped. For example, if we take the point $z = 1 + i2$ (which satisfies the inequality $2y + 8x - 10 > 0$ that represents the z half-plane) then this point is mapped by this transformation onto the point $w = -8(1 + i2) + 11 - i\pi = 3 - i(16 + \pi) = u + iv$ which is in the w half-plane $v + 4u - 4 + \pi < 0$.

So, the z half-plane $2y + 8x - 10 > 0$ is mapped by the transformation $w = -8z + 11 - i\pi$ onto the w half-plane $v + 4u - 4 + \pi < 0$.
(**d**) The equation of the boundary circle of this disk is:

$$|z - 1 - i2| = |x + iy - 1 - i2| = |(x - 1) + i(y - 2)| = \sqrt{(x-1)^2 + (y-2)^2} = 3$$

and hence $y = 2 \pm \sqrt{9 - (x-1)^2}$. Therefore, the boundary can be represented by the complex equation $z = x + iy = x + i\left(2 \pm \sqrt{9 - (x-1)^2}\right)$. On applying the transformation $w = (2 - i)z + 7$ on a general point on this boundary curve we get:

$$\begin{aligned} w &= (2-i)\left[x + i\left(2 \pm \sqrt{9-(x-1)^2}\right)\right] + 7 = x(2-i) + i(2-i)\left(2 \pm \sqrt{9-(x-1)^2}\right) + 7 \\ &= \left(2x + 9 \pm \sqrt{9-(x-1)^2}\right) + i\left(-x + 4 \pm 2\sqrt{9-(x-1)^2}\right) \end{aligned}$$

So, $u = 2x + 9 \pm \sqrt{9-(x-1)^2}$ and $v = -x + 4 \pm 2\sqrt{9-(x-1)^2}$ which can be combined to obtain $(u-11)^2 + (v-3)^2 = 45$ which is an equation of a circle (in the w plane) with center $w = 11 + i3$ and radius $R = \sqrt{45} = 3\sqrt{5}$. This circle is represented by the complex equation $|w - 11 - i3| = \sqrt{45}$. It is obvious that the interior of the z disk should be mapped onto the interior of the w disk. However, we can check this by transforming a point inside the z disk to see if it is mapped inside or outside the w disk. For example, if we transform the point $z = 0$ (which is inside the z disk since $|0 - 1 - i2| = \sqrt{5} \le 3$) by the transformation $w = (2 - i)z + 7$ we get $w = 7$ (which is inside the w disk since $|7 - 11 - i3| = |-4 - i3| = 5 \le \sqrt{45}$). So in brief, the z disk $|z - 1 - i2| \le 3$ is mapped by the transformation $w = (2 - i)z + 7$ onto the w disk $|w - 11 - i3| \le \sqrt{45}$.
Note: for obvious pedagogical reasons, we followed in the above solutions the standard method in solving this type of problems. However, linear transformations are obviously characterized by preserving the basic shapes of the transformed objects (see Problem 1 of § 6.2) and hence we can solve linear transformation problems by more simple methods using this fact. For example, we can solve the problems of straight lines and their alike (such as half-planes) by mapping two points and connecting them by a straight line. Similarly, we can solve the problems of circles and disks by mapping the center and guessing the mapped radius. For instance, we can solve part (d) of the present Problem by arguing that a disk transformed by a linear transformation (from z plane to w plane) should be mapped onto another disk where the center of the z disk is mapped onto the center of the w disk while the radius of the w disk is a scaled version of the radius of the z disk where the scaling factor comes from the applied transformation. Accordingly, the center of the w disk is the image of the center of the z disk (i.e. $z = 1 + i2$) under the above transformation, i.e. $w = (2 - i)(1 + i2) + 7 = 11 + i3$. Moreover, the factor $(2 - i)$ in the transformation $w = (2 - i)z + 7$ represents a scaling by a factor $\sqrt{5}$ and hence the radius of the z disk (which is 3) should be scaled by $\sqrt{5}$ to obtain the radius of the w disk. This means that the image of the z disk is a w disk with center $w = 11 + i3$ and radius $R = 3\sqrt{5}$ (which can be represented by the complex equation $|w - 11 - i3| \le \sqrt{45}$ as obtained above).

8. The following are images obtained by the given linear transformations. Determine the sources (or inverse images) of these images.
(**a**) The straight line $v = 5 - u$ obtained by the transformation $w = z - 3 - i2$.
(**b**) The circle $|w - i4| = 2$ obtained by the transformation $w = 5 + i2z$.
Answer:
(**a**) We have $w = u + iv = z - 3 - i2 = (x - 3) + i(y - 2)$, i.e. $u = x - 3$ and $v = y - 2$. So, the source of the straight line $v = 5 - u$ should be $y - 2 = 5 - (x - 3)$, i.e. the straight line $y = 10 - x$ which in complex form is $z = x + i(10 - x)$. This can be easily verified by transforming the line $y = 10 - x$ by the given transformation to obtain the line $v = 5 - u$.
(**b**) We have $w = u + iv = 5 + i2z = 5 + i2x - 2y = (5 - 2y) + i2x$. So, the source of the circle

$|w - i4| = 2$ should be $|(5 - 2y) + i2x - i4| = 2$, that is:

$$\begin{aligned}
|(5-2y) + i(2x-4)| &= 2 \\
|(5-2y) + i(2x-4)|^2 &= 4 \\
(5-2y)^2 + (2x-4)^2 &= 4 \\
\left(\frac{5}{2} - y\right)^2 + (x-2)^2 &= 1 \\
(x-2)^2 + \left(y - \frac{5}{2}\right)^2 &= 1
\end{aligned}$$

This is a circle of radius 1 and center $\left(2, \frac{5}{2}\right)$. So, the source of the circle $|w - i4| = 2$ under the given transformation is the circle $\left|z - 2 - i\frac{5}{2}\right| = 1$. This can be easily verified by transforming the circle $\left|z - 2 - i\frac{5}{2}\right| = 1$ by this transformation to obtain the circle $|w - i4| = 2$.

6.2 Non-Linear Transformations

There are many types of non-linear transformations and hence this is a generic and general category. For example, there are non-linear polynomial transformations which are transformations conducted by using non-linear polynomial functions like quadratic or cubic (see § 2.1). Non-linear transformations may also be of trigonometric or hyperbolic or exponential or logarithmic types (among other types). In brief, complex transformations (or mappings) follow in their categories the types of functions used in their realization. Hence, from this aspect the category of linear transformations is not different from other types of transformations apart from its geometric and mathematical simplicity as well as wide applicability and usability (also see Problem 1).

Problems

1. Considering their geometric aspect, what distinguishes linear transformations from non-linear transformations when used to transform geometric objects (such as straight lines or circles) from the z plane to the w plane?

   Answer: Linear transformations are distinguished by preserving the basic shape of these geometric objects since scaling, rotation and translation do not affect the main characteristic features of these objects. For example, by linear transformations we map straight lines onto straight lines, circles onto circles and so on. This is not the case in the non-linear transformations which usually (but not necessarily) distort the basic shapes, e.g. a straight line may be mapped onto a parabola or a circle may be mapped onto an ellipse and so on.

   Note: as indicated above, some non-linear transformations (such as reflections across lines) may also preserve the basic shapes. Hence, linear transformations preserve the shape while non-linear transformations may or may not preserve the shape.

2. Find the mathematical relations $\left[\text{i.e. } w = f(z) \text{ or } w = f(x, y)\right]$ that represent the following non-linear transformations:[266]

   (**a**) Reflection in the real axis.
   (**b**) Reflection in the imaginary axis.
   (**c**) Mapping points in the z plane onto their magnitude in the w plane (i.e. on the positive u axis).
   (**d**) Mapping points in the z plane onto their negative magnitude in the w plane.
   (**e**) Mapping points in the z plane onto their (positive) magnitude on the v axis.
   (**f**) Mapping points in the z plane onto their negative magnitude on the v axis.
   (**g**) Reflection in the line $y = x$.
   (**h**) Reflection in the line $y = -x$.
   (**i**) Reflection in the line $x = a$ (with a being a real constant).

[266] It should be noted that the transformation in part (k) is essentially linear but it is included here for structural and practical considerations.

(**j**) Reflection in the line $y = b$ (with b being a real constant).
(**k**) Rotation by a given angle θ around a given point in the z plane.
(**l**) Projection onto the real axis.
(**m**) Projection onto the imaginary axis.
(**n**) Projection onto the line $y = ax + b$ (with a and b being real constants).
(**o**) Reflection in the line $y = ax + b$ (with a and b being real constants).
Answer:[267]
(**a**) $w = z^*$. This can also be seen as a rotation by an angle $-2 \arg z$.[268]
(**b**) $w = -z^*$. This can also be seen as a rotation by an angle $\pi - 2 \arg z$ (i.e. a combination of part a and a reflection in the origin).
(**c**) $w = |z| = \sqrt{zz^*}$. This can also be seen as a rotation by an angle $-\arg z$.
(**d**) $w = -|z| = -\sqrt{zz^*}$. This can also be seen as a rotation by an angle $\pi - \arg z$.
(**e**) $w = i|z| = i\sqrt{zz^*}$. This can also be seen as a rotation by an angle $-\arg z + \pi/2$ (i.e. a combination of part c and an anticlockwise rotation by $\pi/2$).
(**f**) $w = -i|z| = -i\sqrt{zz^*}$. This can also be seen as a rotation by an angle $-\arg z - \pi/2$ (i.e. a combination of part c and a clockwise rotation by $\pi/2$) or $\pi - \arg z + \pi/2 = 3\pi/2 - \arg z$ (i.e. a combination of part d and an anticlockwise rotation by $\pi/2$).
(**g**) $w = \text{Im}(z) + i\text{Re}(z) = \frac{z-z^*}{i2} + i\frac{z+z^*}{2}$. This is an exchange of the real and imaginary parts.
(**h**) $w = -\text{Im}(z) - i\text{Re}(z) = \frac{z^*-z}{i2} - i\frac{z+z^*}{2}$. This is an exchange of the negatives of the real and imaginary parts as if we reflect in the line $y = x$ with a rotation by π.
(**i**) $w = -[\text{Re}(z) - a] + a + i\text{Im}(z) = [2a - \text{Re}(z)] + i\text{Im}(z)$. This is a translation of the real part by $-a$ $\big[$i.e. $\text{Re}(z) - a\big]$ followed by a reflection in the imaginary axis (i.e. $-[\text{Re}(z) - a]$) followed by a translation of the real part by a (i.e. $+a$) to annul the effect of the previous translation.
(**j**) $w = \text{Re}(z) - i[\text{Im}(z) - b] + ib = \text{Re}(z) + i[2b - \text{Im}(z)]$. This is a translation of the imaginary part by $-b$ $\big[$i.e. $\text{Im}(z) - b\big]$ followed by a reflection in the real axis (i.e. $-[\text{Im}(z) - b]$) followed by a translation of the imaginary part by b (i.e. $+b$) to annul the effect of the previous translation.
(**k**) As seen earlier (refer to Problem 1 of § 6), a rotation of a point z_1 by an angle θ around the origin is achieved by multiplying z_1 by $e^{i\theta}$. So, to achieve a rotation around a given point z_2 we simply translate the origin to z_2 (to make z_2 a new origin), rotate z_1 around the new origin (i.e. z_2) then translate the origin back to its original position in the complex plane, that is:

$$\begin{aligned}
w &= e^{i\theta}(z_1 - z_2) + z_2 = (\cos\theta + i\sin\theta)(x_1 + iy_1 - x_2 - iy_2) + x_2 + iy_2 \\
&= (x_2 + x_1\cos\theta - x_2\cos\theta - y_1\sin\theta + y_2\sin\theta) + i(y_2 + y_1\cos\theta - y_2\cos\theta + x_1\sin\theta - x_2\sin\theta)
\end{aligned}$$

(**l**) $w = \text{Re}(z) = \frac{z+z^*}{2}$, i.e. taking the real part (or component).
(**m**) $w = i\,\text{Im}(z) = \frac{z-z^*}{2}$, i.e. taking the imaginary component.
(**n**) If z_2 is the projection of a point z_1 onto the line $y = ax + b$ then the line C that connects z_1 and z_2 should be perpendicular to the line $y = ax + b$. Noting that the product of the slopes of perpendicular lines is -1, we conclude that the equation of C is:

$$y = -\frac{x}{a} + B \qquad (B \text{ is real constant})$$

Now, since C passes through z_1 then we have $y_1 = -\frac{x_1}{a} + B$ and hence $B = y_1 + \frac{x_1}{a}$ and the equation of C becomes $y = -\frac{x}{a} + y_1 + \frac{x_1}{a}$. Moreover, z_2 is on both lines and hence we have:

$$\begin{aligned}
y_2 &= y_2 \\
ax_2 + b &= -\frac{x_2}{a} + y_1 + \frac{x_1}{a}
\end{aligned}$$

[267] To enhance clarity and improve presentation, we used in some parts indexed variables to represent the general variable z. More specifically, in parts (k), (n) and (o) $z_1 = x_1 + iy_1$ stands for $z = x + iy$ in the relation $w = f(z)$ or $w = f(x, y)$.
[268] Although the transformations in parts (a)-(f) can be seen as rotations, this does not qualify them to be linear transformations because $\arg z$ is variable in general and hence these transformations cannot be represented by $w = az + b$ (with a and b being constants).

$$\begin{aligned} ax_2 + \frac{x_2}{a} &= y_1 + \frac{x_1}{a} - b \\ x_2 &= \frac{a}{a^2+1}\left(y_1 + \frac{x_1}{a} - b\right) = \frac{ay_1 + x_1 - ab}{a^2+1} \end{aligned}$$

Hence, using the equations of the two lines respectively we have:

$$\begin{aligned} y_2 &= ax_2 + b = \frac{a^2y_1 + ax_1 - a^2b}{a^2+1} + b = \frac{a^2y_1 + ax_1 + b}{a^2+1} \\ \text{or} \quad y_2 &= -\frac{\frac{ay_1+x_1-ab}{a^2+1}}{a} + y_1 + \frac{x_1}{a} = -\frac{ay_1 + x_1 - ab - y_1a(a^2+1) - x_1(a^2+1)}{a(a^2+1)} = \frac{a^2y_1 + ax_1 + b}{a^2+1} \end{aligned}$$

Therefore, $w = \left(\frac{ay_1+x_1-ab}{a^2+1}\right) + i\left(\frac{a^2y_1+ax_1+b}{a^2+1}\right)$.

(**o**) If a point z_3 is the reflection of a point z_1 in the line $y = ax+b$ then z_3 can be obtained by rotating z_1 around its projection z_2 on the line $y = ax + b$ by an angle π. So, all we need to do is to combine the results of parts (k) and (n), i.e. by rotating z_1 around z_2 by π to obtain z_3 (according to part k) where z_2 is obtained by projecting z_1 onto the line $y = ax + b$ (according to part n), that is:

$$\begin{aligned} w &= (x_2 + x_1\cos\pi - x_2\cos\pi - y_1\sin\pi + y_2\sin\pi) + i\,(y_2 + y_1\cos\pi - y_2\cos\pi + x_1\sin\pi - x_2\sin\pi) \\ &= (x_2 - x_1 + x_2) + i\,(y_2 - y_1 + y_2) = (2x_2 - x_1) + i\,(2y_2 - y_1) \\ &= \left(\frac{2ay_1 + 2x_1 - 2ab}{a^2+1} - x_1\right) + i\left(\frac{2a^2y_1 + 2ax_1 + 2b}{a^2+1} - y_1\right) \\ &= \left(\frac{2ay_1 + x_1 - 2ab - a^2x_1}{a^2+1}\right) + i\left(\frac{a^2y_1 + 2ax_1 + 2b - y_1}{a^2+1}\right) \end{aligned}$$

3. Determine the geometric nature of the following non-linear transformations (with z_0 being a given number):

(**a**) $w = -6z^* - i2$. (**b**) $w = 1/z$. (**c**) $w = 1/(z - z_0)$. (**d**) $w = |z|^2$.
(**e**) $w = \sqrt{z}$. (**f**) $w = z^2$. (**g**) $w = z^3$. (**h**) $w = \sqrt{z + z_0}$.
(**i**) $w = (z + z_0)^2$. (**j**) $w = \sqrt{z} - z_0$. (**k**) $w = 6z^2 - ez$. (**l**) $w = \sqrt[3]{z}$.
(**m**) $w = \sqrt{5z}$. (**n**) $w = 5\sqrt{z}$. (**o**) $w = \pi\sqrt[3]{z - i}$. (**p**) $w = \sqrt[3]{z_0 z}$.
(**q**) $w = \sqrt{z} + z^2$. (**r**) $w = 6z^2 - 3\sqrt[3]{z}$.

Answer: We note first that in the parts that contain roots we consider in our answer the principal root only.

(**a**) This is a combination of a reflection in the imaginary axis (i.e. $-z^*$; see part b of Problem 2) associated with a magnification by 6 (i.e. $\times 6$) followed by a translation by 2 along the negative imaginary axis (i.e. $-i2$).

(**b**) We have:

$$w = \frac{1}{z} = \frac{z^*}{zz^*}$$

As we know, the transformation $w = z^*$ is a reflection in the real axis (see part a of Problem 2). Moreover, zz^* is the square of the modulus of z, i.e. $zz^* = |z|^2$. So the transformation $1/z$ maps a point in the z plane onto its reflection across the real axis in the w plane but scaled by the reciprocal of its squared modulus. For example, the point $z = 2+i3$ will be mapped onto the point $w = (2-i3)/13$, i.e. the reflection represented by taking the conjugate $(2 - i3)$ combined with the scaling by $1/13$.

(**c**) This is essentially the same as the transformation of part (b) but before conducting the transformation of part (b) the transformed point is translated by $-z_0$. For example, if $z_0 = (2 - i5)$ then the point $z = 1 + i$ will be mapped by this transformation onto the point:

$$w = \frac{1}{(1+i) - (2-i5)} = \frac{1}{-1+i6} = \frac{-1-i6}{37}$$

i.e. the point $z = 1 + i$ is first translated 2 units left and 5 units up to be mapped on the point $z_1 = (-1+i6)$ which will then be mapped by the transformation of part (b) onto the point $w = (-1-i6)/37$.
(**d**) In this transformation we take first the modulus $|z|$ (which is equivalent to the rotation of z by an angle $-\arg z$ so that it maps on the positive real axis; see part c of Problem 2). We then square the modulus to obtain $|z|^2$ (which is equivalent to scaling $|z|$ by $|z|$). For example, the point $\pi - i6$ in the z plane is first mapped onto its modulus $\sqrt{\pi^2+36}$ on the real axis. This modulus is then scaled by $\sqrt{\pi^2+36}$ to map finally onto the point π^2+36 on the real line in the w plane.
(**e**) Using the polar form of z we have:

$$w = \sqrt{re^{i\theta}} = \left(re^{i\theta}\right)^{1/2} = r^{1/2}e^{i\theta/2} = \frac{r}{r^{1/2}}e^{i\theta/2} = r^{-1/2}re^{i(\theta-\theta/2)}$$

So, this transformation simply scales the modulus of z by $r^{-1/2}$ and reduces its argument by half.
(**f**) Using the polar form of z we have:

$$w = \left(re^{i\theta}\right)^2 = r^2e^{i2\theta} = rre^{i(\theta+\theta)}$$

So, this transformation simply scales the modulus of z by r and doubles its argument.
(**g**) Using the polar form of z we have:

$$w = \left(re^{i\theta}\right)^3 = r^3e^{i3\theta} = r^2re^{i(\theta+2\theta)}$$

So, this transformation simply scales the modulus of z by r^2 and triples its argument.
(**h**) This is the same as the transformation of part (e) but before applying that transformation we translate z by z_0.
(**i**) This is a translation of z by z_0 followed by applying the transformation of part (f).
(**j**) This is the transformation of part (e) followed by a translation by $-z_0$.
(**k**) This is a combination of a non-linear transformation $6z^2$ (which is the transformation of part f followed by scaling by 6) and a linear transformation $-ez$ (which is a scaling of z by e with a rotation by π). In brief, the geometric effects of these two transformations are added up to make this transformation.[269]
(**l**) Using the polar form of z we have:

$$w = \sqrt[3]{re^{i\theta}} = \left(re^{i\theta}\right)^{1/3} = r^{1/3}e^{i\theta/3} = \frac{r}{r^{2/3}}e^{i\theta/3} = r^{-2/3}re^{i(\theta-2\theta/3)}$$

So, this transformation simply scales the modulus of z by $r^{-2/3}$ and reduces its argument by two thirds.
(**m**) We have $w = \sqrt{5z} = \sqrt{5}\sqrt{z}$ and hence it is a scaled version of the transformation of part (e), i.e. we apply the transformation of part (e) then scale the result by $\sqrt{5}$.
(**n**) This is the same as the transformation of part (m) but the scaling here is by 5 (not by $\sqrt{5}$).
(**o**) In this transformation we simply move z one unit downward (i.e. $-i$), followed by applying the transformation of part (l), followed by scaling the result by π.
(**p**) We have $w = \sqrt[3]{z_0 z} = \sqrt[3]{z_0}\,\sqrt[3]{z}$. So, we first apply the transformation of part (l) followed by multiplying by $\sqrt[3]{z_0}$ (which means scaling by $\left|\sqrt[3]{z_0}\right|$ and rotation by $\arg\sqrt[3]{z_0}$).
(**q**) This is a combination of the transformations of part (e) and part (f), i.e. the resultant geometric effect is the superposition (or addition) of the effects of these parts.
(**r**) This is a combination of the transformation of part (f) scaled by 6 and the transformation of part (l) scaled by -3.

4. Determine how the following shapes and regions are transformed from the z plane to the w plane by the given non-linear transformations:

[269] We consider a combination of linear and non-linear transformations as non-linear (i.e. non-linearity is the stronger element and hence it determines the nature of the transformation as linear or not). In fact, this transformation can be simply described as a single quadratic transformation.

(**a**) A straight line transformed by $w = z^2$ where it is required to investigate the following cases: ($\boldsymbol{\alpha}$) the line is vertical, ($\boldsymbol{\beta}$) the line is horizontal, ($\boldsymbol{\gamma}$) the line passes through the origin, and ($\boldsymbol{\delta}$) the line is not restricted to the previous cases.
(**b**) A triangle with vertices O$(0, 0)$, C$(c, 0)$ and D$(0, d)$ transformed by $w = z^2$.
(**c**) A hyperbola $y = a/x$ transformed by $w = z^2$ (with a being a non-zero real constant and $x \neq 0$).
(**d**) A circle transformed by $w = 1/z$.
(**e**) A straight line transformed by $w = 1/z$.
Answer:
(**a**) We note first that a point z in the z plane will be mapped by this transformation onto a point w in the w plane where:

$$w = z^2 = (x + iy)^2 = \left(x^2 - y^2\right) + i2xy = u + iv$$

So, $u = x^2 - y^2$ and $v = 2xy$. Accordingly:
For case $\boldsymbol{\alpha}$ we have $x = C$ (with C being a real constant) and hence $u = C^2 - y^2$ and $v = 2Cy$. On combining these equations we get $u = C^2 - \frac{v^2}{4C^2}$ ($C \neq 0$) which is a parabola symmetrical about the real axis and opens towards the negative real axis with vertex at $(C^2, 0)$. A special instance of case α is $x = 0$ (i.e. $C = 0$) which yields the degenerative parabola $v = 0$, i.e. the real axis (or rather the non-positive part of it).
For case $\boldsymbol{\beta}$ we have $y = D$ (with D being a real constant) and hence $u = x^2 - D^2$ and $v = 2xD$. On combining these equations we get $u = \frac{v^2}{4D^2} - D^2$ ($D \neq 0$) which is a parabola symmetrical about the real axis and opens towards the positive real axis with vertex at $(-D^2, 0)$. A special instance of case β is $y = 0$ (i.e. $D = 0$) which yields the degenerative parabola $v = 0$, i.e. the real axis (or rather the non-negative part of it).
For case $\boldsymbol{\gamma}$ we have $y = ax$ (with a being a real constant) and hence $u = x^2 - a^2x^2$ and $v = 2ax^2$. On combining these equations we get $v = \frac{2a}{1-a^2}u$ which is an equation of a straight line through the origin (i.e. a z straight line passing through the origin is mapped by the transformation z^2 onto a w straight line passing through the origin).
A special instance of case γ is $a = 0$ which leads to $y = 0$ and $v = 0$ and hence agrees with the result of case β, i.e. the non-negative real axis.
Another special instance of case γ is $a = \infty$ which leads to $x = 0$ (since $x = y/a$) and $v = 0$ (since $v = \frac{2}{(1/a)-a}u$) and hence agrees with the result of case α, i.e. the non-positive real axis.
For case $\boldsymbol{\delta}$ (which is the more general case) the straight line is given by $y = ax + b$ (with a and b being real constants and $a \neq 0$). Now, if a point is on the straight line then we have:

$$\begin{aligned} u &= x^2 - (ax + b)^2 = x^2 - a^2x^2 - 2abx - b^2 = \left(1 - a^2\right)x^2 - 2abx - b^2 & (212) \\ \text{and} \qquad v &= 2x\,(ax + b) = 2ax^2 + 2bx \\ \frac{v}{2a} &= x^2 + \frac{b}{a}x \\ \frac{v}{2a} + \frac{b^2}{4a^2} &= x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2 \\ \text{Hence:} \qquad x &= \frac{-b \pm \sqrt{b^2 + 2av}}{2a} & (213) \end{aligned}$$

where we assume $b^2 + 2av \geq 0$ (since a, b, x, y, u, v are real). On substituting from Eq. 213 into Eq. 212 we get:

$$\begin{aligned} u &= \left(1 - a^2\right)\left(\frac{-b \pm \sqrt{b^2 + 2av}}{2a}\right)^2 - 2ab\left(\frac{-b \pm \sqrt{b^2 + 2av}}{2a}\right) - b^2 \\ &= \left(1 - a^2\right)\left(\frac{-b \pm \sqrt{b^2 + 2av}}{2a}\right)^2 \mp b\sqrt{b^2 + 2av} & (214) \end{aligned}$$

In fact, this equation represents many cases but in general it represents a parabola (potentially degenerative) that could be oriented in any direction.[270] For example, if $a = \pm 1$ (and $b \neq 0$) then $u = \mp b\sqrt{b^2 \pm 2v}$ i.e. $v = \pm 0.5\left(\frac{u^2}{b^2} - b^2\right)$ which is a parabola opening up if $a = 1$ and down if $a = -1$. Also, if $b = 0$ (i.e. passing through the origin) then $u = \frac{1-a^2}{2a}v$, i.e. $v = \frac{2a}{1-a^2}u$ which is what we obtained already in case γ above.

(**b**) From the results of part (a) we know that the side OC is transformed to a straight line segment connecting O to C$'(c^2, 0)$ (see case β) while the side OD is transformed to a straight line segment connecting O to D$'(-d^2, 0)$ (see case α). Regarding the side CD, its equation is $(x/c) + (y/d) = 1$, i.e. $y = d - (d/c)x$. So, from Eq. 214 (with $a = -d/c$ and $b = d$) we get:

$$u = \left(1 - \frac{d^2}{c^2}\right)\left(\frac{cd \mp c\sqrt{d^2 - (2dv/c)}}{2d}\right)^2 \mp d\sqrt{d^2 - (2dv/c)}$$

For example, if $d = \pm c$ (with $c > 0$) then we have $u = \mp d\sqrt{d^2 \mp 2v}$, i.e. $v = \mp 0.5\left(\frac{u^2}{d^2} - d^2\right)$ which is a parabola (opening up if $+$ and opening down if $-$). So, in this case our triangle is mapped on the u axis ($-d^2 \le u \le d^2$) on one side and on the parabola $v = \mp 0.5\left(\frac{u^2}{d^2} - d^2\right)$ on the other side (with $-d^2 \le u \le d^2$). It is obvious that the two closed curves (i.e. the triangle and the "line segment plus parabola") are tracked in the same sense (i.e. clockwise or anticlockwise). It is also obvious that the region of the z plane inside the triangle is mapped by this transformation onto the region of the w plane inside the "line segment plus parabola".[271] This can be verified by mapping a point inside the triangle.[272] For example, if $d = c$ (i.e. $a = -1$) then the point $z = 0.1c + i0.1c$ (which is inside the triangle) will be mapped onto the point $w = i0.02c^2$ which is above the u axis and below the parabola (which cuts the v axis at $w = i0.5d^2 = i0.5c^2$). Similarly, if $d = -c$ (i.e. $a = 1$) then the point $z = 0.1c - i0.1c$ (which is inside the triangle) will be mapped onto the point $w = -i0.02c^2$ which is below the u axis and above the parabola (which cuts the v axis at $w = -i0.5d^2 = -i0.5c^2$).

(**c**) As before, if $z = x + iy$ and $w = z^2$ then $u = x^2 - y^2$ and $v = 2xy$. Now, from the equation of the hyperbola we get $xy = a$ and hence $v = 2a$ which is a constant. This means that the z hyperbola is mapped onto a w horizontal line.

(**d**) If $w = 1/z$ then $z = 1/w$ (noting that the original transformation is one-to-one).[273] Hence:

$$z = \frac{1}{w} = \frac{w^*}{ww^*} = \frac{u - iv}{u^2 + v^2} = x + iy$$

So, $x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$. Now, a circle with center z_0 and radius R is given by $|z - z_0| = R$, that is:

$$\begin{aligned}
|(x + iy) - (x_0 + iy_0)| &= R \\
(x - x_0)^2 + (y - y_0)^2 &= R^2 \\
\left(\frac{u}{u^2 + v^2} - x_0\right)^2 + \left(\frac{-v}{u^2 + v^2} - y_0\right)^2 &= R^2 \\
\frac{u^2}{(u^2 + v^2)^2} - \frac{2x_0 u}{u^2 + v^2} + x_0^2 + \frac{v^2}{(u^2 + v^2)^2} + \frac{2y_0 v}{u^2 + v^2} + y_0^2 &= R^2
\end{aligned} \tag{215}$$

[270] Except horizontally (in some cases) since $a = 0$ is not allowed in this equation although this case was treated already in case β (and even in case γ) where the result was also a parabola (possibly degenerative).

[271] We may call the region bordered by the "line segment plus parabola" parabolic section.

[272] In fact, the sense of tracking should confirm this because as we track the triangle clockwise/anticlockwise the perimeter of the parabolic section is tracked in the same sense and hence the interior of the triangle (which is on the right/left) should map onto the interior of the parabolic section (which is also on the right/left).

[273] For simplicity (and to avoid distraction), we do not consider here the case $z = 0$ (i.e. when the source circle passes through the origin). However, this case will be investigated in part (a) of Problem 6.

$$\begin{aligned}
\frac{u^2+v^2}{(u^2+v^2)^2} - \frac{2x_0u}{u^2+v^2} + \frac{2y_0v}{u^2+v^2} &= R^2 - x_0^2 - y_0^2 \\
\frac{1-2x_0u+2y_0v}{u^2+v^2} &= R^2 - x_0^2 - y_0^2 \qquad (216) \\
u^2 + \frac{2x_0u}{R^2-x_0^2-y_0^2} + v^2 - \frac{2y_0v}{R^2-x_0^2-y_0^2} &= \frac{1}{R^2-x_0^2-y_0^2} \\
\left(u+\frac{x_0}{R^2-x_0^2-y_0^2}\right)^2 + \left(v-\frac{y_0}{R^2-x_0^2-y_0^2}\right)^2 &= \frac{1}{R^2-x_0^2-y_0^2} + \frac{x_0^2+y_0^2}{(R^2-x_0^2-y_0^2)^2} \\
\left(u+\frac{x_0}{R^2-x_0^2-y_0^2}\right)^2 + \left(v-\frac{y_0}{R^2-x_0^2-y_0^2}\right)^2 &= \frac{R^2}{(R^2-x_0^2-y_0^2)^2} \qquad (217)
\end{aligned}$$

This is an equation of a circle with center $w_0 = \left(\frac{-x_0}{R^2-x_0^2-y_0^2}, \frac{y_0}{R^2-x_0^2-y_0^2}\right)$ and radius $= R/\left|R^2-x_0^2-y_0^2\right|$. So, a z circle given by Eq. 215 will be mapped by the transformation $w = 1/z$ onto a w circle given by Eq. 217.

Note: straight lines can be seen as a special case of circles (i.e. circles of infinite radius). Hence, straight lines may be mapped by $w = 1/z$ onto circles and circles may be mapped by $w = 1/z$ onto straight lines (as well as mapping circles onto circles and straight lines onto straight lines). See Problem 6.

(**e**) This is a special case of the previous case (see the note of part d) and hence it should also be mapped by $w = 1/z$ onto a circle (which could be a straight line). However, let do it from the beginning to verify our claim. A straight line is given (in real form) by $y = ax+b$ (with a and b being real constants). On applying the transformation $w = 1/z$ on this line (by using the relations $x = \frac{u}{u^2+v^2}$ and $y = \frac{-v}{u^2+v^2}$ which we obtained in part d) we get:

$$\begin{aligned}
y &= ax+b \\
\frac{-v}{u^2+v^2} &= \frac{au}{u^2+v^2} + b \qquad (218) \\
u^2 + \frac{au}{b} + v^2 + \frac{v}{b} &= 0 \\
\left(u+\frac{a}{2b}\right)^2 + \left(v+\frac{1}{2b}\right)^2 &= \frac{a^2+1}{4b^2} \qquad (219)
\end{aligned}$$

This is an equation of a circle with center $w_0 = \left(\frac{-a}{2b}, \frac{-1}{2b}\right)$ and radius $= \sqrt{\frac{a^2+1}{4b^2}}$.

5. The following are images obtained by the given non-linear transformations. Determine the sources (or inverse images) of these images.

(**a**) The parabola $u = 8 - \frac{(v-1)^2}{32}$ obtained by the transformation $w = 2z^2 + i$.

(**b**) The disk $\left|w - \frac{7}{6} - i\frac{1}{6}\right| \le \sqrt{\frac{25}{18}}$ obtained by the transformation $w = 1/z$.

Answer:

(**a**) We have:

$$w = 2z^2 + i = 2(x+iy)^2 + i = 2\left(x^2-y^2\right) + i(4xy+1) = u + iv$$

i.e. $u = 2\left(x^2-y^2\right)$ and $v = 4xy+1$. Hence, the source of the parabola $u = 8 - \frac{(v-1)^2}{32}$ should be:

$$\begin{aligned}
2\left(x^2-y^2\right) &= 8 - \frac{(4xy+1-1)^2}{32} \\
x^2-y^2 &= 4 - \frac{x^2y^2}{4} \\
x^2\left(1+\frac{y^2}{4}\right) &= 4+y^2
\end{aligned}$$

$$\begin{aligned}
x^2\left(1+\frac{y^2}{4}\right) &= 4\left(1+\frac{y^2}{4}\right)\\
x^2 &= 4\\
x &= \pm 2
\end{aligned}$$

So, the source of the parabola $u = 8 - \frac{(v-1)^2}{32}$ under the given transformation is the vertical lines $x = \pm 2$. This can be easily verified by transforming the vertical lines $x = \pm 2$ by this transformation to obtain the parabola $u = 8 - \frac{(v-1)^2}{32}$.
Note: there is no similar solution for y because:

$$\begin{aligned}
2\left(x^2-y^2\right) &= 8-\frac{(4xy+1-1)^2}{32}\\
x^2-y^2 &= 4-\frac{x^2y^2}{4}\\
\frac{x^2y^2}{4}-y^2 &= 4-x^2\\
\frac{y^2}{4}\left(x^2-4\right) &= -\left(x^2-4\right)\\
\frac{y^2}{4} &= -1
\end{aligned}$$

which has no solution (since y is real).
(**b**) We have:

$$w = \frac{1}{z} = \frac{z^*}{zz^*} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$$

Hence, the source of the disk $\left|w - \frac{7}{6} - i\frac{1}{6}\right| \le \sqrt{\frac{25}{18}}$ should be:

$$\begin{aligned}
\left|\frac{x-iy}{x^2+y^2}-\frac{7}{6}-i\frac{1}{6}\right| &\le \sqrt{\frac{25}{18}}\\
\left|\left(\frac{x}{x^2+y^2}-\frac{7}{6}\right)-i\left(\frac{y}{x^2+y^2}+\frac{1}{6}\right)\right| &\le \sqrt{\frac{25}{18}}\\
\left(\frac{x}{x^2+y^2}-\frac{7}{6}\right)^2+\left(\frac{y}{x^2+y^2}+\frac{1}{6}\right)^2 &\le \frac{25}{18}\\
\frac{x^2}{\left(x^2+y^2\right)^2}-\frac{7x}{3\left(x^2+y^2\right)}+\frac{49}{36}+\frac{y^2}{\left(x^2+y^2\right)^2}+\frac{y}{3\left(x^2+y^2\right)}+\frac{1}{36} &\le \frac{25}{18}\\
\frac{3-7x+y}{x^2+y^2} &\le 0\\
3-7x+y &\le 0\\
y &\le 7x-3
\end{aligned}$$

So, the source of the disk $\left|w - \frac{7}{6} - i\frac{1}{6}\right| \le \sqrt{\frac{25}{18}}$ under the given transformation is the half-plane $y \le 7x - 3$. This can be easily verified by transforming this half-plane by this transformation to obtain the disk $\left|w - \frac{7}{6} - i\frac{1}{6}\right| \le \sqrt{\frac{25}{18}}$ (which can be easily done by reversing the above steps). This can also be verified by using the result of part (e) of Problem 4.

6. Show that the effect of the reciprocal transformation $w(z) = 1/z$ on circles and straight lines is as follows:
(**a**) A circle passing through the origin is mapped onto a straight line not passing through the origin.
(**b**) A circle not passing through the origin is mapped onto a circle not passing through the origin.

(**c**) A straight line passing through the origin is mapped onto a straight line passing through the origin.
(**d**) A straight line not passing through the origin is mapped onto a circle passing through the origin.
Answer: We use in this answer the results of parts (d) and (e) of Problem 4.
(**a**) A circle with center z_0 and radius R is given by $|z - z_0| = R$, and hence if it passes through the origin then we should have:

$$\begin{aligned} |z - z_0| &= |0 - z_0| = |z_0| = \sqrt{x_0^2 + y_0^2} = R \\ \text{that is:} \quad x_0^2 + y_0^2 &= R^2 \end{aligned}$$

Therefore, from Eq. 216 we get $1 - 2x_0 u + 2y_0 v = 0$ which is an equation of a straight line not passing through the origin.[274]
(**b**) A circle with center z_0 and radius R is given by $|z - z_0| = R$, and hence if it does not pass through the origin then $x_0^2 + y_0^2 \neq R^2$ and hence we have (see Eq. 217):

$$\left(u + \frac{x_0}{R^2 - x_0^2 - y_0^2}\right)^2 + \left(v - \frac{y_0}{R^2 - x_0^2 - y_0^2}\right)^2 = \frac{R^2}{\left(R^2 - x_0^2 - y_0^2\right)^2}$$

which is an equation of a circle not passing through the origin (noting that if the circle passes through the origin then we should have $x_0^2 + y_0^2 = R^2$ which is untrue).
(**c**) A straight line passing through the origin is given by $y = ax$, and hence from Eq. 218 (with $b = 0$) we get:

$$\begin{aligned} \frac{-v}{u^2 + v^2} &= \frac{au}{u^2 + v^2} \\ v &= -au \end{aligned}$$

which is an equation of a straight line passing through the origin.[275]
(**d**) A straight line not passing through the origin is given by $y = ax + b$ (with $b \neq 0$), and hence we have (see Eq. 219):

$$\left(u + \frac{a}{2b}\right)^2 + \left(v + \frac{1}{2b}\right)^2 = \frac{a^2 + 1}{4b^2}$$

which is an equation of a circle passing through the origin (noting that if $u = v = 0$ then we get an identity). In fact, the last equation includes even the case when $a = 0$, i.e. when the transformed line is horizontal (not passing through the origin).

6.3 Bilinear Transformation

The bilinear transformation (which may also be called Mobius transformation or linear fractional transformation as well as many other names) is a common and versatile complex transformation that is given by the following relation:[276]

$$w(z) = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0) \qquad (220)$$

where a, b, c, d are complex numbers (which can be real or imaginary as special cases). This transformation is non-linear in general although it can also represent linear transformation as a special case.

Problems

[274] We note that if the transformed circle is not trivial (i.e. single point with zero radius) then at least one of x_0, y_0 is not zero, and hence the equation $1 - 2x_0 u + 2y_0 v = 0$ should represent all the possibilities of a straight line not passing through the origin, i.e. horizontal line (if $x_0 = 0$ and $y_0 \neq 0$), vertical line (if $x_0 \neq 0$ and $y_0 = 0$), and slant line (if $x_0 \neq 0$ and $y_0 \neq 0$).

[275] We note that the equation $v = -au$ (like the equation $y = ax$) represents all the possibilities of a straight line passing through the origin, i.e. horizontal line or real axis (if $a = 0$), vertical line or imaginary axis (if $a = \infty$), and slant line (if $a \neq 0$, i.e. a is finite).

[276] It is "bilinear" because it is linear in numerator and linear in denominator.

1. What is the significance of the condition $ad - bc \neq 0$ which is imposed as part of the definition of the bilinear transformation?
   Answer: To assess the significance of the condition $ad - bc \neq 0$ we need to investigate all the cases that lead to $ad - bc = 0$, that is:
   • If $a = b = c = d = 0$ then $w = 0/0$.
   • If $a = d = b = 0$ ($c \neq 0$) then $w = 0$.
   • If $a = d = c = 0$ ($b \neq 0$) then $w = \infty$.
   • If $a = b = c = 0$ ($d \neq 0$) then $w = 0$.
   • If $d = b = c = 0$ ($a \neq 0$) then $w = \infty$.
   • If $a = b = 0$ ($d \neq 0$, $c \neq 0$) then $w = 0$.
   • If $a = c = 0$ ($d \neq 0$, $b \neq 0$) then $w = b/d$.
   • If $d = b = 0$ ($a \neq 0$, $c \neq 0$) then $w = a/c$.
   • If $d = c = 0$ ($a \neq 0$, $b \neq 0$) then $w = \infty$.
   So, the significance of the condition $ad - bc \neq 0$ is to exclude the above troubling or unwanted cases, i.e. $w = 0/0$, $w = \infty$, and $w =$ constant (whether the constant is zero or b/d or a/c).
2. List some of the properties of the bilinear transformation.
   Answer: We note the following:
   • The linear transformation (i.e. $w = az + b$ which corresponds to $cz + d = 1$ with $a \neq 0$) and the reciprocal transformation (i.e. in the general form $w = \frac{1}{cz+d}$ which corresponds to $az + b = 1$ with $c \neq 0$) are special cases of the bilinear transformation.
   • The bilinear transformation (excluding the identity transformation) has at most two fixed points, i.e. it could have no fixed point or one fixed point or two fixed points (but no more). For example, the transformation $w(z) = z - 3$ has no fixed point (since it represents a translation with no rotation or scaling), the transformation $w(z) = 3z$ and $w(z) = iz$ have exactly one fixed point which is 0 (since they represent scaling and rotation with no translation and hence they fix only the origin), and the transformation $w(z) = 1/z$ has exactly two fixed points which are 1 and -1 since $w(1) = 1$ and $w(-1) = -1$. It should be obvious that the identity transformation $w(z) = z$ (which is an instance of the bilinear transformation) has infinite number of fixed points since it fixes all the points of the complex plane.[277] So, if we include the identity transformation then we should say: the number of fixed points of the bilinear transformation is 0 or 1 or 2 or infinite (see Problem 3).
   • Each bilinear transformation has an inverse which is also bilinear. This is because if $w = \frac{az+b}{cz+d}$ then we have:

$$\begin{aligned} czw + dw &= az + b \\ (cw - a)z &= -dw + b \\ z &= \frac{-dw + b}{cw - a} \end{aligned}$$

   As we see, the transformation in the last equation $\big[$which is the inverse of the transformation $w(z)$ since it transforms w to $z\big]$ is also bilinear since it has the form of a bilinear transformation (as given by Eq. 220) and it satisfies the condition $(-d)(-a) - bc = ad - bc \neq 0$ (which is inherited from the original transformation).
   • The composition of two bilinear transformations is a bilinear transformation (see Problem 4).
   • The set of all bilinear transformations is a (non-commutative infinite) group under the operation of composition since it satisfies closure, associativity, inverse and identity $\big[$where closure and inverse are shown in the previous points, the identity is $w(z) = z$, and the associativity can be easily (although algebraically messily) demonstrated$\big]$.
   • As we will see later (refer to Problem 4 of § 6.4), the bilinear transformation is conformal.
3. Show that the number of fixed points of the bilinear transformation (including the identity transformation) is 0 or 1 or 2 or infinite.

[277] When we say "all the points of the complex plane" in such context it should mean any point in the complex plane that is subjected to this transformation (and this can represent the entire complex plane).

Answer: Having a fixed point means that we have $w(z) = \frac{az+b}{cz+d} = z$, i.e. $cz^2 + (d-a)z - b = 0$. Now, if $c = 0$ then we have $(d-a)z - b = 0$ which either has no solution (if $d = a$ and $b \neq 0$), or has one solution (if $d \neq a$), or has infinitely-many solutions (if $d = a$ and $b = 0$). On the other hand, if $c \neq 0$ then we have a quadratic equation which has either one (repetitive) solution or two solutions (within the complex plane; see Problem 5 of § 6). So in brief, we have 0 or 1 or 2 or infinite number of solutions (i.e. fixed points).

Note: the details in the above answer are given for clarity; otherwise we can say: $cz^2+(d-a)z-b = 0$ is (at most) a quadratic polynomial and hence by the result of Problem 5 of § 6 it should have 0 or 1 or 2 or infinite number of fixed points (where the latter case corresponds to the identity).

4. Show that the composition of two bilinear transformations is a bilinear transformation.

Answer: If we have the following bilinear transformations:

$$w_1(z) = \frac{a_1z + b_1}{c_1z + d_1} \quad (a_1d_1 - b_1c_1 \neq 0) \qquad \text{and} \qquad w_2(z) = \frac{a_2z + b_2}{c_2z + d_2} \quad (a_2d_2 - b_2c_2 \neq 0)$$

and $w(z)$ is the composition $w_1(w_2)$ of these transformations then we have:

$$\begin{aligned} w(z) &= \frac{az+b}{cz+d} = w_1(w_2(z)) = \frac{a_1\left(\frac{a_2z+b_2}{c_2z+d_2}\right) + b_1}{c_1\left(\frac{a_2z+b_2}{c_2z+d_2}\right) + d_1} = \frac{a_1(a_2z+b_2) + b_1(c_2z+d_2)}{c_1(a_2z+b_2) + d_1(c_2z+d_2)} \\ &= \frac{(a_1a_2 + b_1c_2)z + (a_1b_2 + b_1d_2)}{(a_2c_1 + c_2d_1)z + (b_2c_1 + d_1d_2)} \end{aligned}$$

As we see, w has the form of a bilinear transformation (with a, b, c, d corresponding to the coefficients in the above equation). So, all we need to do is to verify the condition $ad - bc \neq 0$, that is:

$$\begin{aligned} ad - bc &= (a_1a_2 + b_1c_2)(b_2c_1 + d_1d_2) - (a_1b_2 + b_1d_2)(a_2c_1 + c_2d_1) \\ &= \cancel{a_1a_2b_2c_1} + a_1a_2d_1d_2 + b_1b_2c_1c_2 + \cancel{b_1c_2d_1d_2} - \cancel{a_1a_2b_2c_1} - a_1b_2c_2d_1 - a_2b_1c_1d_2 - \cancel{b_1c_2d_1d_2} \\ &= a_1a_2d_1d_2 - a_1b_2c_2d_1 + b_1b_2c_1c_2 - a_2b_1c_1d_2 \\ &= (a_1d_1 - b_1c_1)(a_2d_2 - b_2c_2) \neq 0 \end{aligned}$$

where the last step is justified by the conditions $a_1d_1 - b_1c_1 \neq 0$ and $a_2d_2 - b_2c_2 \neq 0$.

5. Find bilinear transformations $w(z) = \frac{az+b}{cz+d}$ that do the following mappings:

(**a**) $w(0) = -i\frac{5}{2}$, $w(1) = \frac{1+i5}{2}$ and $w(i) = \frac{6-i3}{5}$.

(**b**) $w(i) = i$, $w(-i) = -i$ and $w(1) = -1$.

(**c**) $w(A) = A$, $w(-A) = -A$ and $w(B) = -B$ (where A and B are non-zero complex numbers and $A \neq B$).

Answer:

(**a**) We have $w(z) = \frac{az+b}{cz+d}$ and hence:

$$w(0) = \frac{b}{d} = -i\frac{5}{2} \tag{221}$$

$$w(1) = \frac{a+b}{c+d} = \frac{1+i5}{2} \tag{222}$$

$$w(i) = \frac{ia+b}{ic+d} = \frac{6-i3}{5} \tag{223}$$

So, let have $d = 2$ and hence from Eq. 221 we get $b = -i5$. On substituting these values into Eq. 222 and simplifying we get:

$$\frac{a - i5}{c+2} = \frac{1+i5}{2} \qquad \text{and hence} \qquad a = \frac{c + i5c + 2 + i20}{2}$$

On substituting this expression of a (as well as the values of b and d) into Eq. 223 we get:

$$\begin{aligned}
\frac{i\frac{c+i5c+2+i20}{2}-i5}{ic+2} &= \frac{6-i3}{5} \\
\frac{\frac{c+i5c+2+i20}{2}-5}{c-i2} &= \frac{6-i3}{5} \\
7c-i31c &= -28+i124 \\
c\,(7-i31) &= -4\,(7-i31) \\
c &= -4
\end{aligned}$$

and hence $a=\frac{-4-i20+2+i20}{2}=-1$. Accordingly, the required bilinear transformation is:

$$w(z)=\frac{az+b}{cz+d}=\frac{-z-i5}{-4z+2}=\frac{z+i5}{4z-2}$$

(**b**) Referring to Problem 3, we should have exactly two fixed points (i.e. i and $-i$) and hence $c\neq 0$. Moreover, we should have $az+b\neq 0$. These are not very restrictive conditions and hence we can take (while respecting these conditions) $a=d=0$ and $b\neq 0$. So, a possible transformation is $w(z)=\frac{b}{cz}=\frac{b/c}{z}$. Now, from the given mapping $w(1)=-1$ we get $b/c=-1$ and hence our transformation becomes $w(z)=\frac{-1}{z}$ which obviously do the given mappings.

(**c**) If we follow the argument of part (b) then a possible transformation is of the form $w(z)=\frac{b/c}{z}$. Now, from the given mapping $w(B)=-B$ we get $b/c=-B^2$ and hence our transformation becomes $w(z)=\frac{-B^2}{z}$. Moreover, from the given mappings $w(A)=A$ and $w(-A)=-A$ we get $B^2=-A^2$ (i.e. $B=\pm iA$). So, a possible transformation is:

$$w(z)=\frac{A^2}{z} \qquad \text{where } A^2=-B^2$$

For example, if $A=5$ then $w(z)=25/z$ and hence $w(5)=5$, $w(-5)=-5$ and $w(\pm i5)=\mp i5$. Similarly, if $A=i7$ then $w(z)=-49/z$ and hence $w(i7)=i7$, $w(-i7)=-i7$ and $w(\mp 7)=\pm 7$. Also, if $A=1+i2$ then $w(z)=(1+i2)^2/z$ and hence $w(1+i2)=1+i2$, $w(-[1+i2])=-[1+i2]$ and $w(\pm i\,[1+i2])=\mp i\,[1+i2]$.

6. Show that bilinear transformations map circles onto circles (where both "circles" include lines as a special case).

Answer: We note first that the reciprocal transformation, i.e. $w(z)=1/z$, maps circles onto circles (see parts d and e of Problem 4 of § 6.2; also see Problem 6 of § 6.2). This is also true for the linear transformation (see Problem 1 of § 6.2). Now, the bilinear transformation can be written as:

$$w(z)=\frac{az+b}{cz+d}=\frac{az+\frac{ad}{c}+b-\frac{ad}{c}}{cz+d}=\frac{\frac{a}{c}(cz+d)+b-\frac{ad}{c}}{cz+d}=\frac{\frac{a}{c}(cz+d)}{cz+d}+\frac{b-\frac{ad}{c}}{cz+d}=\frac{a}{c}+\frac{b-\frac{ad}{c}}{cz+d}$$

As we see, the transformation in the last equation is a composition of three transformations:

- $w_1(z)=cz+d$ which is a linear transformation and hence it maps circles onto circles.
- $w_2(w_1)=\frac{1}{w_1}$ which is a reciprocal transformation and hence it maps circles onto circles.
- $w_3(w_2)=\frac{a}{c}+\left(b-\frac{ad}{c}\right)w_2$ which is a linear transformation and hence it maps circles onto circles.

Accordingly, the overall transformation $\big[$i.e. the composition $w_3(w_2(w_1(z)))$ of the three transformations $w_1,w_2,w_3\big]$ should also map circles onto circles because no one of these three transformations changes the geometric nature (i.e. being circle) of the transformed object.

Note: as indicated in the question, "circles" include straight lines as a special case (i.e. straight lines are circles of infinite radius). This means that bilinear transformations map straight lines onto straight lines and circles and map circles onto straight lines and circles (see Problem 6 of § 6.2 noting that the reciprocal transformation is an instance of the bilinear transformation). We should also note that in the above equation we assume $c\neq 0$ (noting that if $c=0$ then w is a linear transformation and hence it also maps circles onto circles).

7. Find the images of the following objects under the given bilinear transformations:
(**a**) The circle $x^2+y^2-2x=0$ under the transformation $w=\frac{z-2}{z+5}$.
(**b**) The straight line $y-3x-2=0$ under the transformation $w=\frac{z}{2z-7}$.
(**c**) The upper half of the origin-centered unit circle under the transformation $w=\frac{3}{z+i}$.
Answer:
(**a**) On solving $w=\frac{z-2}{z+5}$ for z we get $z=\frac{5w+2}{1-w}$ and hence:

$$\begin{aligned}
x^2+y^2-2x &= 0\\
zz^*-2\,\mathrm{Re}\,z &= 0\\
\left(\frac{5w+2}{1-w}\right)\left(\frac{5w^*+2}{1-w^*}\right)-\left(\frac{5w+2}{1-w}+\frac{5w^*+2}{1-w^*}\right) &= 0\\
\frac{(5w+2)(5w^*+2)}{(1-w)(1-w^*)}-\frac{(5w+2)(1-w^*)+(5w^*+2)(1-w)}{(1-w)(1-w^*)} &= 0\\
(5w+2)(5w^*+2)-(5w+2)(1-w^*)-(5w^*+2)(1-w) &= 0\\
35ww^*+7w+7w^* &= 0\\
5ww^*+w+w^* &= 0\\
5u^2+5v^2+2u &= 0\\
u^2+\frac{2}{5}u+\frac{1}{25}+v^2 &= \frac{1}{25}\\
\left(u+\frac{1}{5}\right)^2+v^2 &= \frac{1}{25}
\end{aligned}$$

Hence, the image is a w circle with center $(-1/5,0)$ and radius $1/5$.
(**b**) On solving $w=\frac{z}{2z-7}$ for z we get $z=\frac{7w}{2w-1}$ and hence:

$$\begin{aligned}
y-3x-2 &= 0\\
\mathrm{Im}\,z-3\,\mathrm{Re}\,z-2 &= 0\\
\frac{1}{i2}\left(\frac{7w}{2w-1}-\frac{7w^*}{2w^*-1}\right)-\frac{3}{2}\left(\frac{7w}{2w-1}+\frac{7w^*}{2w^*-1}\right)-2 &= 0\\
\left(\frac{7w}{2w-1}-\frac{7w^*}{2w^*-1}\right)-i3\left(\frac{7w}{2w-1}+\frac{7w^*}{2w^*-1}\right)-i4 &= 0\\
\frac{7w(2w^*-1)-7w^*(2w-1)}{(2w-1)(2w^*-1)}-i\frac{21w(2w^*-1)+21w^*(2w-1)}{(2w-1)(2w^*-1)}-\frac{i4(2w-1)(2w^*-1)}{(2w-1)(2w^*-1)} &= 0\\
7w(2w^*-1)-7w^*(2w-1)-i21w(2w^*-1)-i21w^*(2w-1)-i4(2w-1)(2w^*-1) &= 0\\
-i100ww^*+i29w+i29w^*-7w+7w^*-i4 &= 0\\
100ww^*-29w-29w^*-i7w+i7w^*+4 &= 0\\
100u^2+100v^2-58u+14v &= -4\\
u^2+v^2-\frac{29}{50}u+\frac{7}{50}v &= -\frac{1}{25}\\
\left(u-\frac{29}{100}\right)^2+\left(v+\frac{7}{100}\right)^2 &= \frac{49}{1000}
\end{aligned}$$

Hence, the image is a w circle with center $(29/100,-7/100)$ and radius $7/\sqrt{1000}$.
(**c**) The upper half of the origin-centered unit circle is given by $x^2+y^2=1$ (where $-1\le x\le 1$ and $0\le y\le 1$). On solving $w=\frac{3}{z+i}$ for z we get $z=\frac{3-iw}{w}$ and hence:

$$\frac{3-iw}{w} = z$$

$$\begin{aligned}
\left(\frac{3-iw}{w}\right)\left(\frac{3+iw^*}{w^*}\right) &= zz^* \\
\left(\frac{3-iw}{w}\right)\left(\frac{3+iw^*}{w^*}\right) &= x^2+y^2 \\
\left(\frac{3-iw}{w}\right)\left(\frac{3+iw^*}{w^*}\right) &= 1 \\
(3-iw)(3+iw^*) &= ww^* \\
9+ww^*-i3w+i3w^* &= ww^* \\
i3w-i3w^* &= 9 \\
-6\left(\frac{w-w^*}{i2}\right) &= 9 \\
-6v &= 9 \\
v &= -1.5
\end{aligned}$$

Now, if we note that $w(-1) = -1.5 - i1.5$ and $w(1) = 1.5 - i1.5$ we can conclude that the image is the straight line segment $v = -1.5$ between $u = -1.5$ and $u = 1.5$.

8. Show that if z_1, z_2, z_3 are three arbitrary points in the z plane and w_1, w_2, w_3 are three arbitrary points in the w plane then there is a bilinear transformation that maps z_j on w_j $(j = 1, 2, 3)$.
Answer: We note that the following bilinear transformation:
$$B_1(z) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$
maps z_1 on 0, z_2 on 1, and z_3 on ∞. Similarly, the following bilinear transformation:
$$B_2(w) = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}$$
maps w_1 on 0, w_2 on 1, and w_3 on ∞. So, if we use B_1 to get from z_1, z_2, z_3 to $0, 1, \infty$ and then we use the inverse of B_2 to get from $0, 1, \infty$ to w_1, w_2, w_3 then we mapped z_1, z_2, z_3 on w_1, w_2, w_3. In other words, the required bilinear transformation B is the composition of B_1 and the inverse of B_2, that is:
$$B(z) = B_2^{-1} \circ B_1(z)$$
Note: we remind the reader that each bilinear transformation has an inverse which is also bilinear (see Problem 2) and hence the inverse of B_2 does exist and it is bilinear. Moreover, the composition of two bilinear transformations is a bilinear transformation (see Problem 4) and hence B is bilinear (as required).
9. Re-solve Problem 5 using the result of Problem 8.
Answer:
(**a**) Let $0, 1, i$ be z_1, z_2, z_3 and $-i\frac{5}{2}, \frac{1+i5}{2}, \frac{6-i3}{5}$ be w_1, w_2, w_3. Now, since in Problem 8 both B_1 and B_2 map their three points (i.e. z_1, z_2, z_3 for B_1 and w_1, w_2, w_3 for B_2) onto $0, 1, \infty$ then as far as the three points are concerned we can write $B_2(w_1, w_2, w_3) = B_1(z_1, z_2, z_3)$, that is:[278]
$$\begin{aligned}
\frac{\left(w+i\frac{5}{2}\right)\left(\frac{1+i5}{2}-\frac{6-i3}{5}\right)}{\left(w-\frac{6-i3}{5}\right)\left(\frac{1+i5}{2}+i\frac{5}{2}\right)} &= \frac{(z-0)(1-i)}{(z-i)(1-0)} \\
\frac{w+i\frac{5}{2}}{w-\frac{6-i3}{5}} &= \frac{z(1-i)}{(z-i)}\,\frac{\left(\frac{1+i5}{2}+i\frac{5}{2}\right)}{\left(\frac{1+i5}{2}-\frac{6-i3}{5}\right)} \\
\frac{w+i\frac{5}{2}}{w-\frac{6-i3}{5}} &= \frac{z(1-i)}{(z-i)}\left(\frac{3}{2}-i\frac{1}{2}\right)
\end{aligned}$$

[278] In fact, this argument may not be sufficiently rigorous but it is rather intuitive. Moreover, it avoids the use of the “cross ratio” which we do not want to go through its details.

$$\begin{aligned}
\frac{w+i\frac{5}{2}}{w-\frac{6-i3}{5}} &= \frac{z(1-i2)}{z-i} \\
\left(w+i\frac{5}{2}\right)(z-i) &= z(1-i2)\left(w-\frac{6-i3}{5}\right) \\
wz-iw+i\frac{5}{2}z+\frac{5}{2} &= zw-i2zw-\frac{6-i3}{5}z+\frac{6+i12}{5}z \\
w &= \frac{z+i5}{4z-2}
\end{aligned}$$

(**b**) If we repeat our argument in part (a) then we have:

$$\begin{aligned}
\frac{(w-i)(-i+1)}{(w+1)(-i-i)} &= \frac{(z-i)(-i-1)}{(z-1)(-i-i)} \\
\frac{w-i}{w+1} &= \frac{-i(z-i)}{z-1} \\
zw-iz-w+i &= -izw-w-iz-1 \\
zw+izw &= -1-i \\
w &= \frac{-1-i}{z\,(1+i)} = -\frac{1}{z}
\end{aligned}$$

(**c**) If we repeat our argument in part (a) then we have:

$$\begin{aligned}
\frac{(w-A)(-A+B)}{(w+B)(-A-A)} &= \frac{(z-A)(-A-B)}{(z-B)(-A-A)} \\
(w-A)(-A+B)(z-B) &= (z-A)(-A-B)(w+B) \\
-A^2w-B^2w+2Bwz &= -B^2z-A^2z+2A^2B \\
-A^2w+A^2w+2Bwz &= A^2z-A^2z+2A^2B \qquad (A^2=-B^2) \\
w &= \frac{A^2}{z}
\end{aligned}$$

6.4 Conformal Transformation

Conformal transformation is a mapping (i.e. from the z plane to the w plane) that locally preserves angles (i.e. between transformed curves) in size and sense.[279] As we will see, for a transformation to be conformal at a given point in the complex plane the transformation function $w = f(z)$ must be single-valued and analytic at that point, and the derivative of the transformation function dw/dz should not vanish at that point.[280] In this context, a point at which the derivative vanishes is commonly known as a critical point (i.e. of the transformation function). For example, $w = z^2$ has a critical point at $z = 0$ because $dw/dz = 2z = 0$ at that point, while $w = e^z$ has no critical point because $dw/dz = e^z \neq 0$ at any point (i.e. in the finite complex plane).

An obvious example of conformal transformation is linear transformations (see § 6.1) because what they do is to scale, rotate and translate geometric objects (e.g. triangles or squares or circles) in the complex plane and hence they obviously preserve their angles in size and sense because they do not affect the general shape of the transformed objects or subject them to reflections that affect the sense of their angles. Also, from a formal viewpoint their transformation functions (which are of the form $w = az + b$

[279] Angles between straight lines are obvious while angles between curves mean angles between their tangents at the point of intersection. We should also note that "preserve angles" is also described as "isogonal".

[280] In fact, some authors define conformal as: f is conformal at z_0 if f is analytic and has non-vanishing derivative at z_0 (noting that being single-valued my be considered as an implicit condition for being analytic; see footnote [32] on page 17). It is noteworthy that being analytic should also exclude infinite derivative and hence the derivative at the point of conformality should be finite (i.e. neither zero nor infinite).

with $a \neq 0$) are single-valued and analytic over the entire complex plane and their derivative (which is equal to a) does not vanish at any point and hence they are conformal over the entire complex plane. A less obvious example is the quadratic transformation $w = z^2$ which is single-valued and analytic over the entire complex plane and its derivative (which is equal to $2z$) does not vanish except at $z = 0$ and hence it is conformal over the entire complex plane excluding the origin.

Problems

1. List some of the characteristics of the conformal transformation.
   Answer: For example:
   • Conformal transformation can be linear or non-linear.
   • Being conformal is a local property and hence a transformation can be conformal at a given point but non-conformal at another point. In other words, we cannot describe a transformation as conformal or not without reference to a point or region in the complex plane.[281]
   • Because conformal transformation preserves angles, it rotates the tangent vectors of the transformed object at the point of conformality by the same amount (including sense).
   • As we will see, the derivative of a conformal function (representing a conformal transformation) at a given point z_0 is a specific (finite) complex number α and hence it represents scaling and rotation of all the tangent vectors of the transformed object at that point by the same amount independent of the direction of approach to z_0. However, α is local (i.e. it generally depends on the location in the complex plane) and hence we can write $\alpha = \alpha(z)$. In fact, this is another aspect of locality, i.e. the conformality itself is local (and hence a given transformation may be conformal at one location and non-conformal at another location) and the "conformality factor" α is also local (and hence α at one location is generally different from α at another location).
   • Conformal transformation preserves the continuity of lines, i.e. it transforms continuous line to continuous line.
   • Conformal transformation preserves the analyticity of functions, i.e. it transforms analytic function to analytic function.
   • Each conformal transformation has an inverse which is also conformal.
2. Show that analytic functions (as representing transformations) are conformal at their non-critical points.
   Answer: Let have a t-parameterized curve C in the z plane represented by $z(t)$, and let f be an analytic function defined over a region (in the z plane) that contains C. Now, if $z_0 = z_0(t_0)$ is a non-critical point of f on C then from the chain rule (noting that f is analytic over C) we have:

$$\left.\frac{df(z(t))}{dt}\right|_{z=z_0} = \left.\frac{df}{dz}\frac{dz}{dt}\right|_{z=z_0} = f'(z_0) \times \dot{z}(t_0) \tag{224}$$

   where the prime means d/dz and the overdot means d/dt. It should be obvious that $\left.\frac{df(z(t))}{dt}\right|_{z=z_0}$ represents the tangent vector to the image of C at the image of z_0 (i.e. in the w plane), and $\dot{z}(t_0)$ represents the tangent vector to C at z_0 (i.e. in the z plane), while $f'(z_0)$ is the value of the derivative of f at z_0.[282] Now, since z_0 is a non-critical point of f on C and because an analytic function (like f) possesses a definite non-zero derivative at its non-critical points then $f'(z_0)$ is a given (non-zero) complex number and hence it represents a rotation by a specific angle (as well as a scaling by a specific scalar factor).[283] In brief, Eq. 224 means that the tangent vector of the image of C at the image of z_0 is a scaled and rotated version of the tangent vector of C at z_0 (where the scaling and rotation are specified by the modulus and argument of f' which depends on z_0 but not on the direction of approach to z_0). This means that the function f (as a transformation) rotates the tangent vector $\dot{z}(t_0)$ of C at z_0 by a specific angle (which is the argument of f'). Now, if we note that C is arbitrary (i.e. it can

[281] In fact, we do this but relying on certain understandings and contexts.
[282] For clarity and convenience we are applying the concepts and terminology of vectors to complex numbers (see § 1.1 and § 1.3; also see Problem 1 of § 1.8.5).
[283] Since any given (non-zero) complex number has definite modulus and definite argument (considering principal value or polar form), it represents specific scaling and specific rotation.

represent any curve passing through z_0 in any direction) then we can conclude that the function f (as representing a transformation) will rotate the tangents of all curves that pass through z_0 by the same angle and hence the transformation f locally preserves angles at z_0 (i.e. it is conformal at z_0).
Note 1: from the above answer we can see that $f(z)$ is (locally) conformal *iff* f scales and rotates all the tangent vectors of its source (i.e. locally) by the same amount, i.e. $f'(z_0) = \rho_0 e^{i\theta_0}$ with ρ_0 and θ_0 being specific real numbers that solely depend on z_0. We can also see that the property of analytic functions that makes them conformal at their non-critical points is that they preserve angles (as well as scale factors) independent of the direction of approach to these points. We should finally note that the restriction of the derivative to be finite (i.e. neither zero nor infinite) should be justified by the fact that if the derivative is non-finite then scaling and rotation are not defined.
Note 2: noting that the modulus of the derivative f' of an analytic function $f(z)$ at a non-critical point z_0 determines the scaling (up or down) of the tangent at z_0 under the mapping by f, the value of $|f'|$ may be called the dilatation ratio if $|f'| > 1$ and the contraction ratio if $|f'| < 1$. Also, noting that the argument of f' determines the rotation of the tangent (of any curve passing through z_0) under the mapping by f, the magnitude of the argument indicates the amount of the rotation while its sign indicates the sense of rotation.

3. Determine if and where the transformations represented by the following functions are conformal or not:
(**a**) $w(z) = z^3 - 1$. (**b**) $w(z) = \frac{1}{z}$ $(z \neq 0)$. (**c**) $w(z) = \frac{2}{z} + z^2$ $(z \neq 0)$.
(**d**) $w(z) = e^z$. (**e**) $w(z) = z^*$. (**f**) $w(z) = |z|$.
Answer:
(**a**) w is a polynomial function and hence it is single-valued and analytic over the entire (finite) complex plane, moreover it has only one critical point at $z = 0$ and hence it is conformal over the entire complex plane excluding the origin.
(**b**) w is a reciprocal function and hence it is single-valued and analytic over the entire complex plane excluding the origin (since it is not defined there), moreover it has no critical point over its domain and hence it is conformal over the entire complex plane excluding the origin.
(**c**) w is single-valued and analytic over the entire complex plane excluding the origin (since it is not defined there), moreover $\frac{dw}{dz} = -\frac{2}{z^2} + 2z$ which leads to $\frac{dw}{dz} = 0$ at the three critical points: $z = 1$, $z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ (see part b of Problem 1 of § 1.8.11). Hence, w is conformal over the entire complex plane excluding the origin and the three critical points.
(**d**) w is an exponential function and hence it is single-valued and analytic over the entire complex plane, moreover it has no critical point since $dw = e^z \neq 0$ at any point and hence it is conformal over the entire complex plane.
(**e**) w is not analytic at any point in the complex plane (see part d of Problem 7 of § 3.1) and hence it is not conformal at all. In fact, this should be obvious from the geometric nature of conjugation since it reflects vectors across the real axis (see § 1.8.8 and part a of Problem 2 of § 6.2) and hence although it preserves the size of the angles between tangent vectors it reverses the sense of rotation and hence it is not conformal (according to the above definition of conformal).
(**f**) w is not analytic at any point in the complex plane (see part g of Problem 7 of § 3.1) and hence it is not conformal at all. In fact, this should be obvious from the geometric nature of taking the modulus since it effectively rotates a vector by the negative of its argument (see part c of Problem 2 of § 6.2) and hence it does not preserve the angles.
4. Show that the bilinear transformation is conformal.[284]
Answer: The bilinear transformation is given by $w(z) = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$ (see Eq. 220). Hence:

$$\frac{dw}{dz} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{acz + ad - acz - bc}{(cz+d)^2} = \frac{ad - bc}{(cz+d)^2} \neq 0 \qquad (cz + d \neq 0)$$

Accordingly (see Problem 2), the bilinear transformation is conformal over the entire complex plane

[284] As we noted earlier, conformality is a local property and hence when we describe a certain transformation as conformal or not without specifying a point or a curve or a region we mean in general although the transformation may not be so at a few points.

because w has no critical point due to the condition $ad-bc \neq 0$. However, we should exclude the point $z=-d/c$ because w is not defined there due to the condition $cz+d \neq 0$.
Note: If we notice that c and d cannot be both zero (due to the condition $ad-bc \neq 0$) then we have three cases:
• $c=0$ and $d \neq 0$ and hence the transformation is conformal over the entire (finite) complex plane (since w is linear and $-d/c=\infty$).
• $c \neq 0$ and $d=0$ and hence the transformation is conformal over the entire complex plane excluding the point $z=0$ (since $-d/c=0$).
• $c \neq 0$ and $d \neq 0$ and hence the transformation is conformal over the entire complex plane excluding the point $z=-d/c$.

5. Show that the transformation $w(z)=z^2$ maps the grid of vertical and horizontal lines in the z plane onto a grid of mutually orthogonal parabolas in the w plane.
Answer:[285] The transformation $w(z)=z^2$ is single-valued and analytic over the entire (finite) complex plane, moreover it has only one critical point at $z=0$ and hence it is conformal over the entire complex plane excluding the origin. Moreover, we have:

$$w(z)=u+iv=z^2=(x^2-y^2)+i2xy$$

i.e. $u=x^2-y^2$ and $v=2xy$. Now, the vertical lines in the z plane are given by $x=a$ (with a representing real constants) and hence $u=a^2-y^2$ and $v=2ay$ which can be combined to obtain $u=a^2-\frac{v^2}{4a^2}$ which represents parabolas (symmetrical about the real axis) that open to the left. Similarly, the horizontal lines in the z plane are given by $y=b$ (with b representing real constants) and hence $u=x^2-b^2$ and $v=2bx$ which can be combined to obtain $u=\frac{v^2}{4b^2}-b^2$ which represents parabolas (symmetrical about the real axis) that open to the right. Now, since the grid of vertical and horizontal lines are mutually orthogonal and because w is conformal (i.e. it preserves angles during transformation) then the images (i.e. the parabolas that open to the left and those that open to the right) are mutually orthogonal (i.e. at their intersection points which correspond to the intersection points of the grid).

6. Show that if $u(x,y)$ and $v(x,y)$ are harmonic conjugates and z_0 is a non-critical point of $f=u+iv$ then the level curves $u(x,y)=C_1$ and $v(x,y)=C_2$ (where C_1 and C_2 are real constants) at z_0 are orthogonal.[286]
Answer: From the definition of harmonic conjugates as the real and imaginary parts of an analytic function f (see § 3.4), f is analytic and hence at z_0 (which is a non-critical point) f is conformal (see Problem 2). Now, the level curve $u(x,y)=C_1$ will be mapped by f to C_1+iv which is a vertical line in the w plane. Similarly, the level curve $v(x,y)=C_2$ will be mapped by f to $u+iC_2$ which is a horizontal line in the w plane. Accordingly, the images of the level curves $u(x,y)=C_1$ and $v(x,y)=C_2$ are orthogonal in the w plane. Now, since f is conformal at z_0 it should preserve the angles during the transformation and hence the angle between the source curves $\big[$which are the level curves $u(x,y)=C_1$ and $v(x,y)=C_2\big]$ should be preserved during this transformation, i.e. the level curves $u(x,y)=C_1$ and $v(x,y)=C_2$ at z_0 are orthogonal (since their images are orthogonal).
Note: a rather simpler proof may be constructed by using the result of Problem 18 of § 3.1 (i.e. ∇u and ∇v are orthogonal at any point in the domain of f) in conjunction with the fact that ∇u and ∇v

[285] The reader is referred to part (a) of Problem 4 of 6.2.
[286] A level curve of a function $g(x,y)$ can be defined as a cross section of the graph of $g(x,y)$ at a constant value C, i.e. $g(x,y)=C$. Accordingly, the level curves $u(x,y)=C_1$ and $v(x,y)=C_2$ can be seen as curves representing the (constant) values of u and v in a third dimension to the z plane (as in the 3D xyz space with this z representing the third dimension of the space and not the complex number) where the values of u and v are represented on the third axis (i.e. the "z" axis). So, we can imagine $u(x,y)$ as a surface plot S_1 cut by a plane P_1 parallel to the xy plane at level C_1 (which is the value of u at z_0) and hence the level curve $u(x,y)=C_1$ is the intersection of S_1 and P_1. Similarly, we can imagine $v(x,y)$ as a surface plot S_2 cut by a plane P_2 parallel to the xy plane at level C_2 (which is the value of v at z_0) and hence the level curve $v(x,y)=C_2$ is the intersection of S_2 and P_2. We should finally note that although the level curves (according to this visualization) generally do not intersect (since $C_1 \neq C_2$ in general) we can still describe them as orthogonal (considering for example their projection on the xy plane which in fact is what is meant here).

are orthogonal to their corresponding level curves and hence if ∇u and ∇v are orthogonal (i.e. at a given point) then the level curves should also be orthogonal (i.e. at that point).

6.5 Schwarz-Christoffel Transformation

This is a type of conformal transformation in which the interior of a simple polygon in the complex plane is mapped onto the entire upper half of the complex plane while the border of the polygon is mapped onto the entire real axis.[287] It should be remarked that the meaning of "polygon" in this context is extended to include the so-called "open polygon" where some of the polygon vertices is at infinity and hence such a "polygon" is not like the conventional (or closed) polygon. For example, the semi-infinite strip which is bordered on the left by the line $x = -1$ and on the right by the line $x = +1$ while its base is the real axis (i.e. $y = 0$) is an "open triangle" since it has two actual vertices on the real line (i.e. at $x = -1$ and at $x = +1$) and one virtual vertex at infinity. Accordingly, such open polygons are generally treated like closed polygons with regard to this transformation.

To find a suitable Schwarz-Christoffel transformation for a given polygon we need to solve the following differential equation:

$$\frac{dz}{dw} = C\,(w - u_1)^{(\theta_1/\pi)-1} \cdots (w - u_n)^{(\theta_n/\pi)-1} \tag{225}$$

which leads to the following integral:

$$z = C \int (w - u_1)^{(\theta_1/\pi)-1} \cdots (w - u_n)^{(\theta_n/\pi)-1}\, dw \tag{226}$$

where C is a complex constant, $u_1, \cdots, u_n$ are the images on the w real axis of the actual vertices of the polygon and $\theta_1, \cdots, \theta_n$ are the internal angles of the polygon at the actual vertices. As indicated by "actual vertices", virtual vertices are not counted or considered in this formulation. We should also note that $u_1, \cdots, u_n$ are determined by choice and hence there is no unique Schwarz-Christoffel transformation for a given polygon. Yes, if $u_1, \cdots, u_n$ are given in the statement of the required transformation then the transformation is fixed. Some examples of the application of this integral to find suitable Schwarz-Christoffel transformations will be given in Problem 3.

However, before we go through the Problems of this section, we draw the attention of the reader to the following remarks:

• Schwarz-Christoffel transformation may be defined by the reverse mapping, i.e. mapping of the upper half-plane onto the interior of a polygon and the real axis onto the border (noting that since a Schwarz-Christoffel transformation is conformal it should have an inverse which is also conformal; see Problem 1 of § 6.4).

• We are interested here only in open polygons due to the relative complexity of the mathematics of closed polygons and hence the formulation and explanation in this section are based on this case.

• When we describe a given type of transformation (such as bilinear or Schwarz-Christoffel) as conformal we mean "in general" although it may not be (and usually is not) conformal at some points (notably the vertices in the case of Schwarz-Christoffel).

Problems

1. Extract from the definition of Schwarz-Christoffel transformation some practical criteria that should be satisfied by any transformation of this type.

   Answer: We can say:

   • The image of the boundary of the polygon must be real while the image of its interior must be non-real with a positive imaginary part.

   • Tracking the boundary anticlockwise in a single complete circuit from the start point to the end point should generate (continuously) the entire w real axis u from $-\infty$ (corresponding to the start point) to $+\infty$ (corresponding to the end point). This should ensure that the interior of the polygon is mapped onto the entire upper half of the complex plane.

[287] "Simple polygon" means a polygon that does not contain holes or intersect itself.

As we will see, all the Schwarz-Christoffel transformations that we investigate in the following Problems satisfy these criteria (as can be revealed by a simple inspection to the functions that represent these transformations).

2. Determine the images (in the w plane) of the following polygons (in the z plane) under the given transformations $w = f(z)$ and hence determine the nature of these transformations (i.e. if they are Schwarz-Christoffel transformations or not):
(**a**) The semi-infinite strip (which is an "open triangle") bordered on the left by the line $x = -1$ $(\infty > y \ge 0)$, bordered on the bottom by the line $y = 0$ $(-1 \le x \le 1)$ and bordered on the right by the line $x = 1$ $(0 \le y < \infty)$ under the transformation $w = \sin\left(\frac{\pi z}{2}\right)$.
(**b**) The infinite strip (which is an "open rhombus") bordered on the top by the line $y = 1$ $(\infty > x > -\infty)$ and on the bottom by the line $y = -1$ $(-\infty < x < \infty)$ under the transformation $w = ie^{\pi z/2}$.
Answer:
(**a**) From Eq 138 we have:

$$w = \sin\left(\frac{\pi z}{2}\right) = \sin\left(\frac{\pi x}{2} + i\frac{\pi y}{2}\right) = \sin\frac{\pi x}{2}\cosh\frac{\pi y}{2} + i\cos\frac{\pi x}{2}\sinh\frac{\pi y}{2}$$

Accordingly:
• The line $x = -1$ $(\infty > y \ge 0)$ is mapped onto:

$$w = u + iv = \sin\frac{-\pi}{2}\cosh\frac{\pi y}{2} + i\cos\frac{-\pi}{2}\sinh\frac{\pi y}{2} = -\cosh\frac{\pi y}{2} \qquad (\infty > y \ge 0)$$

where the point at $+\infty$ (in the z imaginary direction) is mapped onto the point at $-\infty$ on the w real axis while the point at $x = -1$ on the z real axis is mapped onto the point at $u = -1$ on the w real axis. In other words, as we track the line $x = -1$ $(\infty > y \ge 0)$ up $\to$ down in the z plane we generate its image in the w plane on the real axis $v = 0$ $(-\infty < u \le -1)$ left $\to$ right.
• The line $y = 0$ $(-1 \le x \le 1)$ is mapped onto:

$$w = u + iv = \sin\frac{\pi x}{2}\cosh 0 + i\cos\frac{\pi x}{2}\sinh 0 = \sin\frac{\pi x}{2} \qquad (-1 \le x \le 1)$$

where the point at $x = -1$ on the z real axis is mapped onto the point at $u = -1$ on the w real axis while the point at $x = 1$ on the z real axis is mapped onto the point at $u = 1$ on the w real axis. In other words, as we track the line $y = 0$ $(-1 \le x \le 1)$ left $\to$ right in the z plane we generate its image in the w plane on the real axis $v = 0$ $(-1 \le u \le 1)$ left $\to$ right.
• The line $x = 1$ $(0 \le y < \infty)$ is mapped onto:

$$w = u + iv = \sin\frac{\pi}{2}\cosh\frac{\pi y}{2} + i\cos\frac{\pi}{2}\sinh\frac{\pi y}{2} = \cosh\frac{\pi y}{2} \qquad (0 \le y < \infty)$$

where the point at $x = 1$ on the z real axis is mapped onto the point at $u = 1$ on the w real axis while the point at $+\infty$ (in the z imaginary direction) is mapped onto the point at $+\infty$ on the w real axis. In other words, as we track the line $x = 1$ $(0 \le y < \infty)$ down $\to$ up in the z plane we generate its image in the w plane on the real axis $v = 0$ $(1 \le u < \infty)$ left $\to$ right.
So in brief, as we rotate around this "open triangle" anticlockwise (in a single complete circuit starting from the point at $+\infty$ in the z imaginary direction on the line $x = -1$ and ending on the point at $+\infty$ in the z imaginary direction on the line $x = 1$) we generate the entire w real axis u (moving from the left of u at $-\infty$ to the right of u at $+\infty$).
Finally, we need to determine where the interior of the semi-strip (or open triangle) is mapped, i.e. whether it is mapped onto the upper half of the w plane or onto the lower half. It is obvious that as we rotate around this open triangle anticlockwise the interior of this triangle is on our left hand side while we correspondingly move on its image from left to right (i.e. on the w real axis u) where the upper half of the w plane is on our left hand side. Therefore, the interior of this open triangle is mapped onto the upper half of the w plane. This is supported by the fact that the point $z = i$ (which is inside this open triangle) is mapped by this transformation onto the point $w = \sin\frac{i\pi}{2} = i\sinh\frac{\pi}{2}$ (which is

inside the upper half of the w plane).
As we see, in this transformation the interior of a polygon (i.e. the open triangle) in the z complex plane is mapped onto the entire upper half of the w complex plane while its border is mapped onto the entire w real axis u (with the actual vertices at $z_1 = -1$ and $z_2 = 1$ being mapped respectively onto $w_1 = -1$ and $w_2 = 1$) and hence by definition it is a Schwarz-Christoffel transformation. We also see that this transformation is conformal (noting that w is entire) except at the vertices $z = -1$ and $z = 1$ where the derivative of w $\left[\text{i.e. } \frac{dw}{dz} = \frac{\pi}{2}\cos\left(\frac{\pi z}{2}\right)\right]$ vanishes.
(**b**) We note first that:

$$w = ie^{\pi z/2} = e^{i\pi/2}e^{\pi z/2} = e^{\frac{\pi}{2}(z+i)} = e^{\frac{\pi x}{2}+i\frac{\pi(y+1)}{2}} = e^{\frac{\pi x}{2}}\cos\frac{\pi(y+1)}{2} + ie^{\frac{\pi x}{2}}\sin\frac{\pi(y+1)}{2}$$

Hence, we have:
• The semi-line $y = 1$ $(\infty > x \ge 0)$ is mapped onto:

$$w = u + iv = e^{\frac{\pi x}{2}}\cos\pi + ie^{\frac{\pi x}{2}}\sin\pi = -e^{\frac{\pi x}{2}} \qquad (\infty > x \ge 0)$$

where the point at $+\infty$ (in the z real direction) is mapped onto the point at $-\infty$ on the w real axis while the point at $z = i$ (corresponding to $x = 0$) on the z imaginary axis is mapped onto the point at $u = -1$ on the w real axis. In other words, as we track the semi-line $y = 1$ $(\infty > x \ge 0)$ right $\to$ left in the z plane we generate its image in the w plane on the real axis $v = 0$ $(-\infty < u \le -1)$ left $\to$ right.
• The semi-line $y = 1$ $(0 \ge x > -\infty)$ is mapped onto:

$$w = u + iv = e^{\frac{\pi x}{2}}\cos\pi + ie^{\frac{\pi x}{2}}\sin\pi = -e^{\frac{\pi x}{2}} \qquad (0 \ge x > -\infty)$$

where the point at $z = i$ (corresponding to $x = 0$) on the z imaginary axis is mapped onto the point at $u = -1$ on the w real axis while the point at $-\infty$ (in the z real direction) is mapped onto the point at $u = 0$ on the w real axis. In other words, as we track the semi-line $y = 1$ $(0 \ge x > -\infty)$ right $\to$ left in the z plane we generate its image in the w plane on the real axis $v = 0$ $(-1 \le u \le 0)$ left $\to$ right.
• The semi-line $y = -1$ $(-\infty < x \le 0)$ is mapped onto:

$$w = u + iv = e^{\frac{\pi x}{2}}\cos 0 + ie^{\frac{\pi x}{2}}\sin 0 = e^{\frac{\pi x}{2}} \qquad (-\infty < x \le 0)$$

where the point at $-\infty$ (in the z real direction) is mapped onto the point at $u = 0$ on the w real axis while the point at $z = -i$ (corresponding to $x = 0$) on the z imaginary axis is mapped onto the point at $u = 1$ on the w real axis. In other words, as we track the semi-line $y = -1$ $(-\infty < x \le 0)$ left $\to$ right in the z plane we generate its image in the w plane on the real axis $v = 0$ $(0 \le u \le 1)$ left $\to$ right.
• The semi-line $y = -1$ $(0 \le x < \infty)$ is mapped onto:

$$w = u + iv = e^{\frac{\pi x}{2}}\cos 0 + ie^{\frac{\pi x}{2}}\sin 0 = e^{\frac{\pi x}{2}} \qquad (0 \le x < \infty)$$

where the point at $z = -i$ (corresponding to $x = 0$) on the z imaginary axis is mapped onto the point at $u = 1$ on the w real axis while the point at ∞ (in the z real direction) is mapped onto the point at ∞ on the w real axis. In other words, as we track the semi-line $y = -1$ $(0 \le x < \infty)$ left $\to$ right in the z plane we generate its image in the w plane on the real axis $v = 0$ $(1 \le u < \infty)$ left $\to$ right.
So in brief, as we rotate around this "open rhombus" anticlockwise (in a single complete circuit starting from the point at $+\infty$ in the z real direction on the line $y = 1$ and ending on the point at $+\infty$ in the z real direction on the line $y = -1$) we generate the entire w real axis u (moving from the left of u at $-\infty$ to the right of u at $+\infty$).
Finally, we need to determine where the interior of the infinite strip (or open rhombus) is mapped, i.e. whether it is mapped onto the upper half of the w plane or onto the lower half. It is obvious that as we rotate around this open rhombus anticlockwise the interior of this rhombus is on our left hand

side while we correspondingly move on its image from left to right (i.e. on the w real axis u) where the upper half of the w plane is on our left hand side. Therefore, the interior of this open rhombus is mapped onto the upper half of the w plane. This is supported by the fact that the point $z = 0$ (which is inside this open rhombus) is mapped by this transformation onto the point $w = i$ (which is inside the upper half of the w plane).
As we see, in this transformation the interior of a polygon (i.e. the open rhombus) in the z complex plane is mapped onto the entire upper half of the w complex plane while its border is mapped onto the entire w real axis u (with the actual vertices at $z_1 = i$ and $z_2 = -i$ being mapped respectively onto $w_1 = -1$ and $w_2 = 1$) and hence by definition it is a Schwarz-Christoffel transformation. We also see that this transformation is conformal (noting that w is entire) with no critical points because the derivative of w $\left[\text{i.e. } \frac{dw}{dz} = \frac{i\pi}{2}e^{\pi z/2}\right]$ does not vanish at all (i.e. within the finite complex plane).
Note: detailed inspection of the functions representing the transformations of both parts of this Problem should reveal that these transformations satisfy the practical (and rather formal) criteria of Schwarz-Christoffel transformation that were given in Problem 1.

3. Find (with verification) the Schwarz-Christoffel transformations for the following polygons (with the required mapping of their actual vertices):
(**a**) The semi-infinite strip (i.e. open triangle) bordered on the bottom by the line $y = 0$ $(-\infty < x \le 0)$, bordered on the right by the line $x = 0$ $(0 \le y \le 1)$ and bordered on the top by the line $y = 1$ $(0 \ge x > -\infty)$ where the actual vertices at $z_1 = 0$ and $z_2 = i$ are mapped (respectively) onto $w_1 = -1$ and $w_2 = 1$.
(**b**) The infinite sector (i.e. open quadrilateral) bordered on the left by the line $x = 0$ $(\infty > y \ge 0)$ and on the bottom by the line $y = 0$ $(0 \le x < \infty)$ where the actual vertices at $z_1 = i$, $z_2 = 0$ and $z_3 = 1$ are mapped (respectively) onto $w_1 = -1$, $w_2 = 0$ and $w_3 = 1$.
Answer:
(**a**) We have $u_1 = -1$ and $u_2 = 1$ and $\theta_1 = \theta_2 = \pi/2$. Hence, from Eq. 226 we get:

$$\begin{aligned} z &= C\int (w-u_1)^{(\theta_1/\pi)-1}(w-u_2)^{(\theta_2/\pi)-1}\,dw = C\int (w+1)^{(1/2)-1}(w-1)^{(1/2)-1}\,dw \\ &= C\int \frac{1}{\sqrt{w^2-1}}dw = C_1\int \frac{1}{\sqrt{1-w^2}}dw = C_1 \arcsin w + C_2 \\ \text{that is} \qquad w &= \sin\left(\frac{z-C_2}{C_1}\right) \end{aligned}$$

Now, to determine the complex constants C_1 and C_2 we use our information about the mapping of the actual vertices, that is:

$$\begin{aligned} z_1 = 0 &= C_1 \arcsin w_1 + C_2 = C_1 \arcsin(-1) + C_2 \\ z_2 = i &= C_1 \arcsin w_2 + C_2 = C_1 \arcsin(+1) + C_2 \end{aligned}$$

On subtracting the first equation from the second equation we get $i = C_1[\arcsin(1) - \arcsin(-1)]$ and hence $C_1 = i/\pi$. On substituting this value of C_1 into one of these equations we get $C_2 = i/2$. Hence, the required Schwarz-Christoffel transformation is (see Problems 2 and 5 of § 2.3):

$$\begin{aligned} w &= \sin\left(\frac{z-(i/2)}{(i/\pi)}\right) = \sin\left(\frac{2\pi z - i\pi}{i2}\right) = \sin\left(-i\pi z - \frac{\pi}{2}\right) = \sin(-i\pi z)\cos\frac{\pi}{2} - \cos(-i\pi z)\sin\frac{\pi}{2} \\ &= -\cos(-i\pi z) = -\cos(i\pi z) = -\cosh(\pi z) \end{aligned}$$

Verification: the criteria of Problem 1 are satisfied by this transformation (as can be easily verified by inspecting w closely). In brief, according to this transformation we have:
• The image of the boundary of the polygon is real because: on the line $y = 0$ $(-\infty < x \le 0)$ we have $w = -\cosh(\pi x)$ which is real, on the line $x = 0$ $(0 \le y \le 1)$ we have $w = -\cosh(i\pi y) = -\cos(\pi y)$ which is real, and on the line $y = 1$ $(0 \ge x > -\infty)$ we have $w = -\cosh(\pi x + i\pi) = -\cosh(\pi x)\cos\pi - i\sinh(\pi x)\sin\pi = \cosh(\pi x)$ which is real.

• The image of the interior region of the polygon is non-real with a positive imaginary part because on this region we have $-\infty < x < 0$ and $0 < y < 1$ and hence $w = -\cosh(\pi x + i\pi y) = -\cosh(\pi x)\cos \pi y - i\sinh(\pi x)\sin \pi y = -\cosh(\pi x)\cos \pi y + i\sinh(\pi |x|)\sin \pi y$ which is obviously non-real with a positive imaginary part (considering its domain).
• Tracking the boundary anticlockwise in a single complete circuit generates the entire w real axis u (as will be briefed in the following points by inspecting the mapping of the start and end points as well as the vertices).
• The point at $x = -\infty$ on the z real axis is mapped onto the point at $u = -\infty$ on the w real axis because $w = -\cosh(-\infty) = -\infty$.[288]
• The point $z_1 = 0$ is mapped onto the point $w_1 = -1$ on the w real axis because $w = -\cosh(0) = -1$.
• The point $z_2 = i$ is mapped onto the point $w_2 = 1$ on the w real axis because $w = -\cosh(i\pi) = -\cos \pi = 1$.
• The point at $-\infty$ on the line $y = 1$ is mapped onto the point at $u = \infty$ on the w real axis because $w = -\cosh(-\infty + i\pi) = -\cosh(-\infty)\cos \pi - i\sinh(-\infty)\sin \pi = \cosh(-\infty) = \infty$.
So, as we rotate anticlockwise around this open triangle (in a single complete circuit starting from $x = -\infty$ on the z real axis and ending on the point at $-\infty$ on the line $y = 1$) we track the entire w real axis u from $-\infty$ to $+\infty$ where the upper half of the w plane is on our left hand side (like the interior of this open triangle during our rotation) and hence the interior of the polygon is mapped onto the entire upper half of the complex plane.[289] Thus, the interior of the polygon is mapped onto the entire upper half of the complex plane while its border is mapped onto the entire real axis (in accord with the above definition and criteria of Schwarz-Christoffel transformation). Moreover, the actual vertices at z_1, z_2 are mapped onto w_1, w_2 (as required). Hence, this is the required Schwarz-Christoffel transformation.
(**b**) We note first that this infinite sector is no more than the first quadrant of the z plane (including its boundary). Now, we have $u_1 = -1$, $u_2 = 0$ and $u_3 = 1$ and $\theta_1 = \theta_3 = \pi$ and $\theta_2 = \pi/2$. Hence, from Eq. 226 we get:

$$\begin{aligned}
z &= C\int (w-u_1)^{(\theta_1/\pi)-1}(w-u_2)^{(\theta_2/\pi)-1}(w-u_3)^{(\theta_3/\pi)-1}\,dw \\
&= C\int (w+1)^{1-1}(w-0)^{(1/2)-1}(w-1)^{1-1}\,dw \\
&= C\int w^{-1/2}dw = C_1 w^{1/2} + C_2 \\
\text{that is} \qquad w &= \left(\frac{z - C_2}{C_1}\right)^2
\end{aligned}$$

Now, to determine the complex constants C_1 and C_2 we use our information about the mapping of the actual vertices, that is:

$$\begin{aligned}
z_1 = i &= C_1 w_1^{1/2} + C_2 = C_1(-1)^{1/2} + C_2 = C_1 i + C_2 \\
z_3 = 1 &= C_1 w_3^{1/2} + C_2 = C_1(+1)^{1/2} + C_2 = C_1 + C_2
\end{aligned}$$

On subtracting the first equation from the second equation we get $1 - i = C_1(1 - i)$ and hence $C_1 = 1$. On substituting this value of C_1 into one of these equations we get $C_2 = 0$. Hence, the required Schwarz-Christoffel transformation is:

$$w = z^2$$

[288] We note that ∞ here and in similar equations and expressions means very big number. The purpose of this lax use of this symbol is to ease the notation and explanation.
[289] The mapping of the interior of the polygon onto the upper half of the complex plane (rather than the lower half) is also supported by the fact that the point $z = -0.5 + i0.5$ (which is inside this polygon) is mapped by this transformation onto the point $w \simeq i2.3013$ (which is inside the upper half of the w plane). This is inline with what we verified formally earlier that the image of the interior region is non-real with a positive imaginary part.

which is obviously consistent with the mapping of the actual vertex at $z_2 = 0$ onto $w_2 = 0$.
Verification: the criteria of Problem 1 are satisfied by this transformation (as can be easily verified by inspecting w closely). In brief, according to this transformation we have:
• The image of the boundary of the polygon is real because: on the line $x = 0$ $(\infty > y \geq 0)$ we have $w = (iy)^2 = -y^2$ which is real, and on the line $y = 0$ $(0 \leq x < \infty)$ we have $w = x^2$ which is real.
• The image of the interior region of the polygon is non-real with a positive imaginary part because on this region we have $0 < x < \infty$ and $0 < y < \infty$ and hence $w = (x^2 - y^2) + i2xy$ which is obviously non-real with a positive imaginary part (considering its domain).
• Tracking the boundary anticlockwise in a single complete circuit generates the entire w real axis u (as will be briefed in the following points by inspecting the mapping of the start and end points as well as the vertices).
• The point at $y = \infty$ on the z imaginary axis is mapped onto the point at $u = -\infty$ on the w real axis because $w = (i\infty)^2 = -\infty$.
• The point $z_1 = i$ is mapped onto the point $w_1 = -1$ on the w real axis because $w = i^2 = -1$.
• The point $z_2 = 0$ is mapped onto the point $w_2 = 0$ on the w real axis because $w = 0^2 = 0$.
• The point $z_3 = 1$ is mapped onto the point $w_3 = 1$ on the w real axis because $w = 1^2 = 1$.
• The point at $x = \infty$ on the z real axis is mapped onto the point at $u = \infty$ on the w real axis because $w = \infty^2 = \infty$.
So, as we rotate anticlockwise around this open quadrilateral (in a single complete circuit starting from $y = \infty$ on the z imaginary axis and ending on $x = \infty$ on the z real axis) we track the entire w real axis u from $-\infty$ to $+\infty$ where the upper half of the w plane is on our left hand side (like the interior of this open quadrilateral during our rotation) and hence the interior of the polygon is mapped onto the entire upper half of the complex plane.[290] Thus, the interior of the quadrilateral is mapped onto the entire upper half of the complex plane while its border is mapped onto the entire real axis (in accord with the above definition and criteria of Schwarz-Christoffel transformation). Moreover, the actual vertices at z_1, z_2, z_3 are mapped onto w_1, w_2, w_3 (as required). Hence, this is the required Schwarz-Christoffel transformation.

[290] The mapping of the interior of the polygon onto the upper half of the complex plane (rather than the lower half) is also supported by the fact that the point $z = 1 + i$ (which is inside this polygon) is mapped by this transformation onto the point $w = i2$ (which is inside the upper half of the w plane). This is inline with what we verified formally earlier that the image of the interior region is non-real with a positive imaginary part.

Chapter 7
Applications Of Complex Analysis

In this chapter we present a few (of the many) applications of complex analysis in mathematics. Our original plan was to investigate similar applications in physics and engineering (as well as investigating more applications in mathematics) but we abandoned this plan due to restrictions on the size of the book.

7.1 Algebra

There are many applications for the theorems and techniques of complex analysis in algebra. In the following Problems we present a few simple examples of these applications.

Problems

1. According to the fundamental theorem of algebra, an n^{th} degree polynomial complex function $P_n(z) = \sum_{k=0}^{n} a_k z^k$ ($n > 0$, $a_n \neq 0$) has exactly n complex roots.[291] Outline a proof for this theorem using Liouville's theorem (see § 4.5).
Answer: Let assume that the polynomial $P_n(z)$ has no root (i.e. it does not vanish at any value of z and hence it has no zero). Accordingly, its reciprocal $1/P_n$ must be an entire function. This is because the polynomial is an entire function (see Problem 4 of § 3.1) and hence its reciprocal must be analytic over the entire complex plane except at the zeros of the polynomial (which are the singularities of the reciprocal) and these zeros supposedly do not exist. The reciprocal must also be bounded (i.e. over the *extended* complex plane). This is because the polynomial blows up at infinity (only) and hence its reciprocal should be zero there[292] which means that the reciprocal is bounded in the entire *extended* complex plane (noting that the polynomial supposedly has no zero and hence its reciprocal does not blow up at any point in the complex plane). So in brief, $1/P_n$ is entire and bounded. Hence, by Liouville's theorem $1/P_n$ should be a constant and this means that P_n itself is a constant which contradicts the fact that P_n is an n^{th} degree polynomial ($n > 0$). Therefore, P_n must have at least one root (say $z = z_1$) and hence it can be factorized (using for example the method of long division) as $P_n = (z - z_1)P_{n-1}$. By repeating the above argument on P_{n-1} we also conclude that it should have at least one root (say $z = z_2$) and hence it can be factorized as $P_{n-1} = (z - z_2)P_{n-2}$. So, by applying the above argument n times we conclude that P_n can be factorized as $P_n = (z - z_1)(z - z_2)\cdots(z - z_n)P_0$ (where $P_0 = a_n$ is a constant) and hence P_n has exactly n roots, as required.
Note: "exactly n roots" means it has no more and no less than n roots although some of these roots can be repetitive. Also, "complex roots" in the above statement means they are complex in general (although not necessarily strictly complex since some or all could be real or/and imaginary) noting that this can be seen as a reference to the domain of the polynomial (i.e. it has n roots within the complex domain).
2. As a consequence of the fundamental theorem of algebra (see Problem 1), an n^{th} degree polynomial can be factorized as:
$$P_n(z) = a_n(z - z_1)(z - z_2)\cdots(z - z_n)$$
where $z_1, z_2, \cdots, z_n$ are its roots. Use this fact to factorize the following polynomials accordingly:

[291] This theorem is also commonly stated as: every non-constant polynomial has at least one complex root. Anyway, the above statement is just a direct consequence of this (as we will see next).

[292] This may be expressed more formally as:
$$\lim_{|z|\to\infty} \left|\frac{1}{P_n}\right| = \frac{1}{\lim_{|z|\to\infty} |P_n|} = \frac{1}{\infty} = 0$$

(**a**) $z^4 + i2z^2 - 5$. (**b**) $4z^3 - 11z^2 + 20z - 13$.

Answer:

(**a**) This is a quartic polynomial and hence from the fundamental theorem of algebra it must have four roots. To factorize this polynomial we use first the quadratic formula (with $a = 1$, $b = i2$ and $c = -5$) to obtain solutions for z^2, that is:

$$z^2 = \frac{-i2 \pm \sqrt{-4 + 20}}{2} = \frac{-i2 \pm 4}{2} = \pm 2 - i$$

Now, if $z^2 = 2 - i \simeq \sqrt{5}e^{-i0.4636}$ then (see Eq. 70):

$$z \simeq 5^{1/4} e^{i(-0.4636 + 2n\pi)/2} \qquad (n = 0, 1)$$

i.e. $z_1 \simeq 1.4553 - i0.3436$ and $z_2 \simeq -1.4553 + i0.3436$.
Similarly, if $z^2 = -2 - i \simeq \sqrt{5}e^{-i2.6779}$ then:

$$z \simeq 5^{1/4} e^{i(-2.6779 + 2n\pi)/2} \qquad (n = 0, 1)$$

i.e. $z_3 \simeq 0.3436 - i1.4553$ and $z_4 \simeq -0.3436 + i1.4553$.
Hence, the polynomial should be factorized as:

$$z^4 + i2z^2 - 5 \simeq (z - 1.4553 + i0.3436)(z + 1.4553 - i0.3436)(z - 0.3436 + i1.4553)(z + 0.3436 - i1.4553)$$

Note: to be more brief and accurate, we have: $z_1 = \sqrt{2 - i}$, $z_2 = -\sqrt{2 - i}$, $z_3 = \sqrt{-2 - i}$ and $z_4 = -\sqrt{-2 - i}$ and hence:

$$z^4 + i2z^2 - 5 = (z - \sqrt{2 - i})(z + \sqrt{2 - i})(z - \sqrt{-2 - i})(z + \sqrt{-2 - i})$$

(**b**) This is a cubic polynomial and hence from the fundamental theorem of algebra it must have three roots. By inspection, we can see that $z = 1$ is a root and hence $(z - 1)$ is a factor. Thus, by long division we get:

$$4z^3 - 11z^2 + 20z - 13 = (z - 1)(4z^2 - 7z + 13) = 4(z - 1)\left(z^2 - \frac{7}{4}z + \frac{13}{4}\right)$$

So, all we need to do now is to factorize $z^2 - \frac{7}{4}z + \frac{13}{4}$ which can be done by using the quadratic formula, that is:

$$z = \frac{(7/4) \pm \sqrt{(49/16) - 13}}{2} = \frac{(7/4) \pm \sqrt{-159/16}}{2} = \frac{7 \pm i\sqrt{159}}{8}$$

Hence, the polynomial should be factorized as:

$$4z^3 - 11z^2 + 20z - 13 = 4(z - 1)\left(z - \frac{7 + i\sqrt{159}}{8}\right)\left(z - \frac{7 - i\sqrt{159}}{8}\right)$$

3. Verify the following statements:
(**a**) The zeros of (non-zero) analytic function are isolated.
(**b**) If $f(z)$ and $g(z)$ are analytic functions on a region R in the z plane, and $f = g$ on a connected set S inside R then $f = g$ on the entire R.[293]
(**c**) An entire function that is periodic on the real line (with a given period) is periodic on the entire complex plane and has the same period.

[293] This statement may be called the identity theorem. We should also note that we used the (rather non-rigorous) term "connected set" as opposite to "isolated point" to avoid going through the technicalities of "accumulation point" which may complicate the understanding. Examples of connected set include 1D curve and 2D sub-region such as disk (and can be extended even to countably infinite number of points). In fact, we ignored in the statements of this Problem and the answer many details to avoid unwanted complications and expansions.

Answer:
(**a**) In more technical terms, this statement means that if $f(z)$ is a (non-zero) analytic function on a given region R in the z plane and z_0 is a point inside R such that $f(z_0) = 0$ then there is a deleted neighborhood of z_0 in which $f(z) \neq 0$. This means that if f vanishes on a connected set inside R then f is identically zero on R. So, if $f(z_0) = 0$ then either f is identically zero or f is zero at separate points.
Now, since f is analytic then it should have a Taylor series (see § 5.1) at z_0 that converges to f within a disk $|z - z_0| < \rho$, that is:

$$f(z) = \sum_{n=m}^{\infty} a_n(z - z_0)^n = \sum_{n=0}^{\infty} a_{n+m}(z - z_0)^{n+m} = (z - z_0)^m \sum_{n=0}^{\infty} a_{n+m}(z - z_0)^n = (z - z_0)^m \phi(z)$$

where $m > 0$ because $f(z_0) = 0 = a_0$. Noting that other a_n's (e.g. a_1 and a_2) can also be zero, let m be the lowest integer for which $a_m \neq 0$. Also, noting that $\phi(z)$ is a sum of power terms,[294] it is obviously analytic on $|z - z_0| < \rho$ and hence it is continuous there (see part a of Problem 7 of § 1.9). Moreover, $\phi(z_0) = a_m \neq 0$. So, from the continuity plus the fact that $\phi(z_0) = a_m$ we can conclude that there is a disk $|z - z_0| < \varepsilon$ $(0 < \varepsilon < \rho)$ such that for all z inside this disk we have $|\phi(z) - a_m| < |a_m|/2$. Also, from the fact that $\phi(z_0) = a_m \neq 0$ we can conclude that $\phi(z)$ cannot be zero inside the ε-disk because otherwise we have $|0 - a_m| = |a_m| < |a_m|/2$ which is nonsensical. Now, from the last equality of the above equation we can conclude that $f(z) = 0$ inside the ε-disk only if $(z - z_0)^m = 0$ and this can occur only at $z = z_0$. This means that inside the ε-disk $f(z) = 0$ only at $z = z_0$ and hence z_0 is necessarily an isolated zero.
(**b**) Since f and g are analytic over R, their difference $f - g$ is analytic over R. Now, $f - g$ is zero on S (which is not an isolated point) and hence from the result of part (a) we conclude that $f - g$ is zero over R, i.e. $f = g$ over R.
(**c**) Let $f(z)$ be such a function. Since f is periodic on the real line then there is a real constant C (i.e. the period) such that $f(z + C) = f(z)$ and hence $f(z + C) - f(z) = 0$ on the real line. Therefore, from the result of part (b) (noting that the real line is a connected set and f is entire) we conclude that $f(z + C) - f(z) = 0$ over the entire complex plane, i.e. $f(z + C) = f(z)$ over the entire complex plane which means that f is periodic on the entire complex plane and has the same period.
Note: the result of part (c) confirms our previous result about the periodicity of the trigonometric cosine and sine functions (see Problem 15 of § 2.3) because these functions (which are entire) are periodic on the real line with a period of 2π and hence they should be periodic on the complex plane with the same period. We should also note that the statement of part (c) also applies to any line or curve similar to the real line such as the imaginary line. For example, in Problem 18 of § 2.2 we found that e^z is periodic with a period of $i2\pi$ which can be easily concluded from our result here (applied to the imaginary line) because e^z (which is entire) is periodic on the imaginary line with a period of $i2\pi$ (since $e^{z+i2\pi} = e^z e^{i2\pi} = e^z \cos 2\pi + ie^z \sin 2\pi = e^z$) and hence it should be periodic on the complex plane with the same period. A similar example is $\cosh z$ and $\sinh z$ which are periodic (as well as entire) with a period of $i2\pi$ (see Problem 16 of § 2.3). In fact, the statement of part (c) is rather weak (based on our needs and objectives); otherwise we can make a similar (but stronger) statement in which we lift or modify some of the restrictions in the above statement.

4. Discuss the implication of the result of part (b) of Problem 3.
Answer: This result (with some modification and generalization) should lead to (or at least suggest) what is called **analytic continuation** where a function f_1 of a given domain of definition D_1 coincides (in value) with a function f_2 of a larger domain of definition D_2 over a connected set (e.g. a curve of finite length) and hence the two functions should be identical over the larger domain which means the extension of the validity of f_1 over the larger domain (which is the domain of f_2). For example, let f_1 be an analytic function defined by a closed algebraic form or a series or an integral over the origin-centered unit disk D_1 and let f_2 be an analytic function defined by a (different) closed algebraic form or a series or an integral over the origin-centered disk of radius 2 D_2 and the two functions (despite their

[294] As remarked in § 5.1, the power series $\sum_{n=0}^{\infty} a_n(z - z_0)$ is an analytic function (in itself) inside its disk of convergence.

different forms) are identical in value over the real interval $0 \le x \le 0.5$. Accordingly, we can conclude that f_1 and f_2 are identical (i.e. they are essentially representing the same function in different forms) over not only D_1 but even over D_2 which means that we extended the validity of f_1 to the entire D_2, i.e. f_1 is *analytically continued* in validity beyond D_1 to include the entire D_2. In fact, this analytic continuation can be of more than one side (where each form extends the validity of the other form to its non-common part and hence both forms will be extended to the union of their domains) and may involve more than two functions (or rather forms), which make analytic continuation more powerful and useful.

5. Show that the singularities of rational functions are isolated.
   Answer: The singularities of a rational function are the zeros of its denominator which is a non-constant polynomial (see Problem 7 of § 2.1). Hence, by the result of part (a) of Problem 3 the singularities must be isolated.
6. Show that the sum of the n n^{th} roots ($n > 1$) of 1 is zero, and hence show that the sum of the n n^{th} roots of any (non-zero) complex number is zero.
   Answer: The n n^{th} roots of 1 are the solutions (or roots) of the equation $z^n = 1$, i.e. $z^n - 1 = 0$. So, if the n n^{th} roots of 1 are $z_1, \cdots, z_n$ and we factorize the latter equation (see Problem 1) then we have:

$$z^n - 1 = (z - z_1) \cdots (z - z_n)$$

   On multiplying the factors on the right hand side of this equation we can see that the sum of the roots (i.e. $z_1 + \cdots + z_n$) is equal to minus the coefficient of z^{n-1}. Now, if we compare the coefficients on the two sides (noting that there is no z^{n-1} term on the left hand side and hence its coefficient is zero) we can conclude that the coefficient of z^{n-1} on the right hand side is equal to zero and hence $z_1 + \cdots + z_n = 0$, i.e. the sum of the n n^{th} roots of 1 is zero.
   Now, the n n^{th} roots of any (non-zero) complex number can be obtained from the n n^{th} roots of 1 by multiplying the roots of 1 by a modulus factor $r^{1/n}$ and an argument factor $e^{i\theta/n}$ (see Problem 8 of § 1.8.11). Since these factors are common to all the n^{th} roots (and hence they can be taken out as a common factor for the sum) then the sum of the n n^{th} roots of any complex number is also zero, i.e.

$$r^{1/n} e^{i\theta/n} z_1 + \cdots + r^{1/n} e^{i\theta/n} z_n = r^{1/n} e^{i\theta/n} (z_1 + \cdots + z_n) = 0$$

7.2 Geometry and Trigonometry

The versatility and flexibility of complex analysis make it ideal for establishing many geometric and trigonometric results and identities. In this section we present a few examples of the use of the techniques of complex numbers and analysis in Euclidean geometry and trigonometry (including some hyperbolic).[295]

Problems

1. Prove the law of cosines using the techniques of complex numbers.
   Answer:[296] Referring to Figure 34, we have:

$$\begin{aligned}
c^2 &= |z_2 - z_1|^2 \\
&= (z_2 - z_1)(z_2 - z_1)^* \\
&= (z_2 - z_1)(z_2^* - z_1^*) \\
&= z_2 z_2^* - z_1 z_2^* - z_1^* z_2 + z_1 z_1^* \\
&= |z_1|^2 + |z_2|^2 - (z_1^* z_2 + z_1 z_2^*) \\
&= |z_1|^2 + |z_2|^2 - \left(|z_1| e^{-i\theta_1} |z_2| e^{i\theta_2} + |z_1| e^{i\theta_1} |z_2| e^{-i\theta_2} \right)
\end{aligned}$$

[295] We should note that in § 1.8.10 we outlined the method of using De Moivre's formula to obtain trigonometric expressions which is one of the applications of complex numbers and analysis in trigonometry (see Problem 4 of § 1.8.10).

[296] As indicated by the question, this Problem (and its alike) is based on the algebra of complex numbers (rather than complex analysis).

$$\begin{aligned} &= |z_1|^2 + |z_2|^2 - |z_1||z_2|\left(e^{-i\theta_1}e^{i\theta_2} + e^{i\theta_1}e^{-i\theta_2}\right) \\ &= |z_1|^2 + |z_2|^2 - |z_1||z_2|\left(e^{i(\theta_2-\theta_1)} + e^{-i(\theta_2-\theta_1)}\right) \\ &= |z_1|^2 + |z_2|^2 - |z_1||z_2|\left(2\cos(\theta_2-\theta_1)\right) \end{aligned}$$

where in the last line we used the definition of cosine (see Eq. 131). Now, if we note that $|z_1| = a$, $|z_2| = b$, $(\theta_2 - \theta_1) = \theta$ then the last equation is no more than the law of cosines, i.e. $c^2 = a^2 + b^2 - 2ab\cos\theta$. **Note**: Pythagoras theorem is a special case for the law of cosines (corresponding to $\theta = \pi/2$) and hence the above proof is also a proof for Pythagoras theorem (by the techniques of complex numbers).[297]

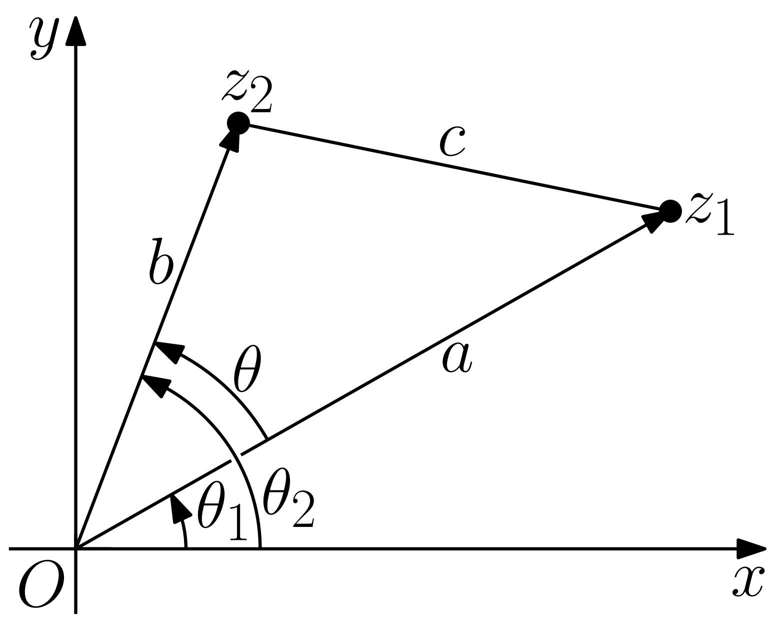

Figure 34: Graphic illustration of the setting of the law of cosines. See Problem 1 of § 7.2.

2. Prove the following trigonometric and hyperbolic identities using the techniques of complex numbers:
(**a**) $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\frac{(2n+1)\theta}{2}}{2\sin\frac{\theta}{2}}$ $\quad (0 < \theta < 2\pi)$.[298]
(**b**) $\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \frac{\cos\frac{\theta}{2} - \cos\frac{(2n+1)\theta}{2}}{2\sin\frac{\theta}{2}}$ $\quad (0 < \theta < 2\pi)$.
(**c**) $\cos^3\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$.[299]
(**d**) $\cosh^3 x = \frac{1}{4}(\cosh 3x + 3\cosh x)$.
(**e**) $\sinh^4 x = \frac{1}{8}(\cosh 4x - 4\cosh 2x + 3)$.
(**f**) $\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$.
Answer:
(**a**) First, if $S = 1 + z + z^2 + \cdots + z^n$ then $S - zS = S(1-z) = 1 - z^{n+1}$ and hence:

$$S = 1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \qquad (z \neq 1) \tag{227}$$

[297] In fact, some of the above formulations may rest on Pythagoras theorem and hence it may not be a valid proof for Pythagoras theorem due to circularity.
[298] This is known as Lagrange's trigonometric identity.
[299] Although θ here and in part f (as well as x in parts d and e) may suggest to be real, it is more general. The use of these symbols is because they are usually used in identities like these and hence their form is more familiar with these symbols.

Now, if $z = e^{i\theta}$ (with $0 < \theta < 2\pi$ since $z \neq 1$) then from Eq. 227 we have:

$$\begin{aligned}
1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} &= \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \\
&= \frac{1 - e^{i(n+1)\theta}}{e^{i\theta/2}\left(e^{-i\theta/2} - e^{i\theta/2}\right)} \\
&= \frac{1 - e^{i(n+1)\theta}}{e^{i\theta/2}\left(-i2\sin\frac{\theta}{2}\right)} \\
&= i\frac{e^{-i\theta/2}\left[1 - e^{i(n+1)\theta}\right]}{2\sin\frac{\theta}{2}} \\
&= i\frac{e^{-i\theta/2} - e^{i(n+\frac{1}{2})\theta}}{2\sin\frac{\theta}{2}} \\
&= i\frac{\left[\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right] - \left[\cos(n+\frac{1}{2})\theta + i\sin(n+\frac{1}{2})\theta\right]}{2\sin\frac{\theta}{2}} \\
&= i\frac{\left[\cos\frac{\theta}{2} - \cos(n+\frac{1}{2})\theta\right] - i\left[\sin\frac{\theta}{2} + \sin(n+\frac{1}{2})\theta\right]}{2\sin\frac{\theta}{2}} \\
&= \frac{i\left[\cos\frac{\theta}{2} - \cos(n+\frac{1}{2})\theta\right] + \left[\sin\frac{\theta}{2} + \sin(n+\frac{1}{2})\theta\right]}{2\sin\frac{\theta}{2}} \\
&= \frac{\sin\frac{\theta}{2} + \sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} + i\frac{\cos\frac{\theta}{2} - \cos(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} \\
&= \left[\frac{1}{2} + \frac{\sin\frac{(2n+1)\theta}{2}}{2\sin\frac{\theta}{2}}\right] + i\left[\frac{\cos\frac{\theta}{2} - \cos\frac{(2n+1)\theta}{2}}{2\sin\frac{\theta}{2}}\right] \qquad (228)
\end{aligned}$$

We also have:

$$1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = \left[1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta\right] + i\left[\sin\theta + \sin 2\theta + \cdots + \sin n\theta\right] \quad (229)$$

On comparing the real parts of Eqs. 228 and 229 we get the required identity.
(**b**) This identity can be obtained from comparing the imaginary parts of Eqs. 228 and 229.
(**c**) From Eq. 131 we have:

$$\begin{aligned}
\cos^3\theta &= \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^3 = \frac{e^{i3\theta} + 3e^{i2\theta}e^{-i\theta} + 3e^{i\theta}e^{-i2\theta} + e^{-i3\theta}}{8} = \frac{e^{i3\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-i3\theta}}{8} \\
&= \frac{1}{4}\left(\frac{e^{i3\theta} + e^{-i3\theta}}{2} + 3\frac{e^{i\theta} + e^{-i\theta}}{2}\right) = \frac{1}{4}\left(\cos 3\theta + 3\cos\theta\right)
\end{aligned}$$

(**d**) From the result of part (c) with ix replacing θ we have:

$$\begin{aligned}
\cos^3 ix &= \frac{1}{4}\left(\cos i3x + 3\cos ix\right) \\
\cosh^3 x &= \frac{1}{4}\left(\cosh 3x + 3\cosh x\right)
\end{aligned}$$

where in line 2 we used $\cos(iz) = \cosh z$ (see Problem 5 of § 2.3).
(**e**) From Eq. 133 we have:

$$\sinh^4 x = \left(\frac{e^x - e^{-x}}{2}\right)^4 = \frac{e^{4x} - 4e^{3x}e^{-x} + 6e^{2x}e^{-2x} - 4e^x e^{-3x} + e^{-4x}}{16}$$

$$\begin{aligned} &= \frac{e^{4x} - 4e^{2x} + 6 - 4e^{-2x} + e^{-4x}}{16} = \frac{1}{8}\left(\frac{e^{4x} + e^{-4x}}{2} - 4\frac{e^{2x} + e^{-2x}}{2} + 3\right) \\ &= \frac{1}{8}(\cosh 4x - 4\cosh 2x + 3) \end{aligned}$$

(**f**) From the result of part (e) with $i\theta$ replacing x we have:

$$\begin{aligned} \sinh^4 i\theta &= \frac{1}{8}(\cosh i4\theta - 4\cosh i2\theta + 3) \\ (i\sin\theta)^4 &= \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3) \\ \sin^4\theta &= \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3) \end{aligned}$$

where in line 2 we used $\sinh(iz) = i\sin z$ and $\cosh(iz) = \cos z$ (see Problem 5 of § 2.3).

3. Verify the following (where z is unity, i.e. $z = e^{i\theta}$):

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad \text{and} \qquad z^n - \frac{1}{z^n} = i2\sin n\theta$$

Answer: From the definition of cosine and sine functions (see Eq. 131) we have:

$$\begin{aligned} z^n + \frac{1}{z^n} &= z^n + z^{-n} = e^{in\theta} + e^{-in\theta} = 2\frac{e^{in\theta} + e^{-in\theta}}{2} = 2\cos n\theta \\ z^n - \frac{1}{z^n} &= z^n - z^{-n} = e^{in\theta} - e^{-in\theta} = i2\frac{e^{in\theta} - e^{-in\theta}}{i2} = i2\sin n\theta \end{aligned}$$

7.3 Differential and Integral Calculus

Complex analysis is "analysis" and hence it is essentially the differential and integral calculus of complex variables. However, complex analysis can be used even in the differential and integral calculus of real variables (i.e. real analysis) as we saw, for instance, in § 5.4 in the evaluation of difficult definite real integrals by using the residue theorem. In the Problems of the present section we present a sample of model problems of *real* calculus that can be solved by the techniques of complex analysis.[300] However, before going through these Problems we draw the attention to the following useful remarks:

• Whenever complex analysis is used to solve a real-valued problem (e.g. evaluating a real improper integral) the result should obviously be real (because the problem is real) and hence if the result obtained by complex analysis techniques is not real (i.e. imaginary or complex) then it should be rejected without further ado, and this should be adopted as a first validity check.

• Complex analysis techniques can offer a substantial help in solving difficult real-valued problems as the complex analysis techniques are generally more flexible and powerful (and may even be easier to apply) than the real analysis techniques. Accordingly, it is an advantage for those who work on real analysis problems to be (at least) aware of the availability of complex analysis techniques as an alternative for solving their problems (at least to double-check the results obtained by real analysis techniques). In fact, this awareness can even be a necessity to avoid some traps (especially for those who are addicted to using computer algebra systems to solve their problems).[301] To clarify my intention, I give a simple

[300] It is noteworthy that these model problems (which are about integral calculus) represent different categories (noting that we generally explain the general method of tackling each category although we still rely on the ability of the reader to extract some details of the method from these examples). In fact, there are many categories and cases to be considered but due to limitations on space and scope we will investigate only some of these. We also note that similar Problems (about using complex analysis techniques in differential and integral calculus) have been given earlier (see for example § 5.4).

[301] As a general advice, the use of computer algebra systems (at least in their current state of development) should always be associated with caution and double-check as the results can be misleading and even wrong. It should be understood that computer algebra systems (especially for solving difficult problems) are not a substitute for proper mathematical knowledge and skills (although they undeniably can offer a substantial help). Hence, only those with sufficient knowledge, experience and skills should rely in their work on these systems.

example (noting that there are many other examples from my own personal experience as well as from the experiences of others). Let have an improper real integral that has only Cauchy principal value (see Problem 3) and we used a computer algebra software to evaluate this integral. If the software uses real analysis techniques (which is very common) the evaluation will fail and we conclude (wrongly) that the integral is divergent (regardless of being in the real sense or in a more general sense). However, if we are aware of the complex analysis techniques for solving real improper integrals (as will be explained in the following Problems) then we can obtain a value for this integral and avoid this error.

Problems

1. Evaluate the following real indefinite integrals (where $x, a, b \in \mathbb{R}$):

   (**a**) $\int e^{ax}\cos(bx)\,dx$. (**b**) $\int e^{ax}\sin(bx)\,dx$.

   Answer: This Problem represents the category of real indefinite integrals involving exponential and trigonometric functions. Instead of using the (rather messy) method of integration by parts we can exploit the (complex) relation between the exponential and trigonometric functions. Accordingly, the above integrals are (correspondingly) the real and imaginary parts of the integral $\int e^{(a+ib)x}\,dx$ and hence we can integrate $\int e^{(a+ib)x}\,dx$ (which is straightforward) and then take the real and imaginary parts of the result (noting that the real/imaginary part of the result corresponds to the integral of the real/imaginary part of the integrand), that is:

$$\begin{aligned}
\int \Big[e^{ax}\cos(bx) + ie^{ax}\sin(bx)\Big]\,dx &= \int e^{(a+ib)x}\,dx \\
\int e^{ax}\cos(bx)\,dx + i\int e^{ax}\sin(bx)\,dx &= \frac{e^{(a+ib)x}}{a+ib} + C \\
\int e^{ax}\cos(bx)\,dx + i\int e^{ax}\sin(bx)\,dx &= \frac{a-ib}{a^2+b^2}\Big[e^{ax}\cos(bx) + ie^{ax}\sin(bx)\Big] + C_1 + iC_2 \\
\int e^{ax}\cos(bx)\,dx + i\int e^{ax}\sin(bx)\,dx &= \left[\frac{ae^{ax}\cos(bx) + be^{ax}\sin(bx)}{a^2+b^2} + C_1\right] + \\
&\quad\; i\left[\frac{ae^{ax}\sin(bx) - be^{ax}\cos(bx)}{a^2+b^2} + C_2\right]
\end{aligned}$$

   where the real part of the last equation is the solution of part (a) of this Problem while the imaginary part of the last equation is the solution of part (b) of this Problem. As we see, the above method did not only make the calculations easier (due to the ease of integrating exponential functions) but it also reduced the amount of work because instead of calculating two (real) integrals we calculated just one (complex) integral.

2. Evaluate the following real definite integrals:

   (**a**) $\int_0^{2\pi}\frac{d\theta}{1+a^2-2a\cos\theta}$ ($\theta, a \in \mathbb{R}$ and $0 < a < 1$). (**b**) $\int_0^{2\pi}\frac{\cos 2\theta}{2ab\cos\theta - a^2 - b^2}\,d\theta$ ($\theta, a, b \in \mathbb{R}$ and $0 < a < b$).

   Answer: This Problem represents the category of real definite integrals involving trigonometric functions in the above rational-like form. This type of integrals can be seen as a contour integral over the origin-centered unit circle C which is represented by $z = e^{i\theta}$ and hence $dz = ie^{i\theta}d\theta = izd\theta$ (i.e. $d\theta = -idz/z$). We can then use the results of Problem 3 of § 7.2 to express the trigonometric functions in terms of z^n and z^{-n}. Alternatively, we can use the definition of trigonometric functions (as given by Eqs. 131 and 132) directly as we did in Problem 9 of § 5.4. We then obtain a contour integral over C in terms of z where the value of this (complex) integral is equal to the value of the original (real) integral. Hence, by evaluating this complex integral by the techniques of complex analysis we obtain the value of the original integral.

   (**a**) From Problem 3 of § 7.2 we have $\cos\theta = \frac{z+z^{-1}}{2}$. On substituting from this equation (as well as the above equations) into the integral we get:

$$\int_0^{2\pi}\frac{d\theta}{1+a^2-2a\cos\theta} = \oint_C \frac{1}{1+a^2-a(z+z^{-1})}\left(\frac{-idz}{z}\right) = i\oint_C \frac{dz}{az^2-(1+a^2)z+a}$$

$$= i\oint_C \frac{dz}{(z-a)(az-1)} = iI$$

As we see, inside C the integrand of the integral I has only one (simple) pole at $z_0 = a$ (noting that $1/a$ is outside C since $a < 1$). Therefore, from Eq. 202 we get:

$$I = i2\pi a_{-1}$$

where a_{-1} is the residue of the integrand corresponding to its Laurent series expansion around z_0. Now, from Eq. 204 we have:

$$a_{-1} = \lim_{z\to a}\left[\frac{1}{(1-1)!}\frac{d^{1-1}}{dz^{1-1}}\left\{(z-a)\frac{1}{(z-a)(az-1)}\right\}\right] = \lim_{z\to a}\left[\frac{1}{az-1}\right] = \frac{1}{a^2-1}$$

On combining the above equations we get:

$$\int_0^{2\pi}\frac{d\theta}{1+a^2-2a\cos\theta} = iI = i\left(i2\pi a_{-1}\right) = i\left(\frac{i2\pi}{a^2-1}\right) = \frac{-2\pi}{a^2-1} = \frac{2\pi}{1-a^2}$$

(**b**) From Problem 3 of § 7.2 we have $\cos\theta = \frac{z+z^{-1}}{2}$ and $\cos 2\theta = \frac{z^2+z^{-2}}{2}$. On substituting from these equations (as well as the above equations) into the integral we get:

$$\begin{aligned}\int_0^{2\pi}\frac{\cos 2\theta}{2ab\cos\theta - a^2 - b^2}\,d\theta &= \oint_C \frac{(z^2+z^{-2})/2}{ab(z+z^{-1})-a^2-b^2}\left(\frac{-idz}{z}\right) = \frac{-i}{2ab}\oint_C \frac{z^4+1}{z^4-\left(\frac{a}{b}+\frac{b}{a}\right)z^3+z^2}\,dz \\ &= \frac{-i}{2ab}\oint_C \frac{z^4+1}{z^2\left(z-\frac{a}{b}\right)\left(z-\frac{b}{a}\right)}\,dz = \frac{-i}{2ab}I\end{aligned}$$

As we see, inside C the integrand of the integral I has a (double) pole at $z_1 = 0$ and a (simple) pole at $z_2 = a/b$ (noting that b/a is outside C since $a < b$). Therefore, from Eq. 203 we get:

$$I = i2\pi\left({}_1a_{-1} + {}_2a_{-1}\right)$$

where ${}_1a_{-1}$ and ${}_2a_{-1}$ are the residues of the integrand corresponding to its Laurent series expansion around z_1 and z_2 respectively. Now, from Eq. 205 we have:

$$\begin{aligned}{}_1a_{-1} &= \lim_{z\to 0}\left[\frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left\{(z-0)^2\frac{z^4+1}{z^2\left(z-\frac{a}{b}\right)\left(z-\frac{b}{a}\right)}\right\}\right] = \lim_{z\to 0}\left[\frac{d}{dz}\left\{\frac{z^4+1}{\left(z-\frac{a}{b}\right)\left(z-\frac{b}{a}\right)}\right\}\right] \\ &= \lim_{z\to 0}\left[\frac{4z^3\left(z-\frac{a}{b}\right)\left(z-\frac{b}{a}\right)-\left(z^4+1\right)\left\{\left(z-\frac{a}{b}\right)+\left(z-\frac{b}{a}\right)\right\}}{\left(z-\frac{a}{b}\right)^2\left(z-\frac{b}{a}\right)^2}\right] = \frac{a}{b}+\frac{b}{a} \\ {}_2a_{-1} &= \lim_{z\to\frac{a}{b}}\left[\frac{1}{(1-1)!}\frac{d^{1-1}}{dz^{1-1}}\left\{\left(z-\frac{a}{b}\right)\frac{z^4+1}{z^2\left(z-\frac{a}{b}\right)\left(z-\frac{b}{a}\right)}\right\}\right] = \lim_{z\to\frac{a}{b}}\left[\frac{z^4+1}{z^2\left(z-\frac{b}{a}\right)}\right] = \frac{a^4+b^4}{a^3b-ab^3}\end{aligned}$$

On combining the above equations we get:

$$\begin{aligned}\int_0^{2\pi}\frac{\cos 2\theta}{2ab\cos\theta-a^2-b^2}\,d\theta &= \frac{-i}{2ab}I = \frac{-i}{2ab}\left[i2\pi\left({}_1a_{-1}+{}_2a_{-1}\right)\right] = \frac{-i}{2ab}\left[i2\pi\left(\frac{a}{b}+\frac{b}{a}+\frac{a^4+b^4}{a^3b-ab^3}\right)\right] \\ &= \frac{2\pi a^2}{a^2b^2-b^4}\end{aligned}$$

3. Discuss briefly the term "Cauchy principal value" which is used extensively in complex analysis in the context of evaluating improper real integrals by complex analysis methods (like the calculus of residues).

Answer: The value of the improper integrals obtained by the complex analysis methods (as will be explained in the following Problems) is called Cauchy principal value (and may be symbolized by acronyms like PV or P.V. or CPV). In fact, this term may be restricted to the value of the improper integrals (as obtained by these techniques) when they are not defined otherwise (noting that the term may also be used in other similar meanings although the general concept is essentially the same). Anyway, when the improper integrals are defined otherwise (i.e. their value can be obtained by techniques other than complex analysis) then their Cauchy principal value should be identical to the value obtained by the other techniques (such as by calculus or by numerical integration). So in brief, Cauchy principal value is no more than the value of the integral (according to the definition of the "value of an integral") although it may be the only way for evaluating the integral (i.e. we have Cauchy principal value but not "ordinary value"). Accordingly, Cauchy principal value can be seen as an extension to the standard definition of the "value of an integral" (as given in calculus) where its legitimacy in the cases where no "ordinary value" exists is based on its agreement with the "ordinary value" in the cases where both values exist. In fact, this can be seen as another example of the superiority of complex analysis over real analysis where some problems can be solved only by the former (e.g. improper integrals that have only Cauchy principal value). It may also lend support to the proposal that complex analysis is an extension to real analysis since concepts like the "value of an integral" are extended and generalized by complex analysis.
Note: Cauchy principal value may be defined within real analysis (i.e. independent of any reference to any complex analysis concept or technique). For example, the real improper integral $\int_a^c \frac{dx}{x-b}$ (with $a < b < c$) is divergent (since the integrand has a singularity at b inside the interval of integration) and hence it cannot be evaluated as it is. However, we may evaluate this integral by the following manipulation (with δ being a positive infinitesimal number):

$$\begin{aligned}\int_a^c \frac{dx}{x-b} &= \lim_{\delta\to 0}\left(\int_a^{b-\delta}\frac{dx}{x-b} + \int_{b+\delta}^{c}\frac{dx}{x-b}\right) = \lim_{\delta\to 0}\left(\Big[\ln|x-b|\Big]_a^{b-\delta} + \Big[\ln|x-b|\Big]_{b+\delta}^{c}\right)\\ &= \lim_{\delta\to 0}\Big(\ln\delta - \ln|a-b| + \ln|c-b| - \ln\delta\Big) = \lim_{\delta\to 0}\Big(-\ln|a-b| + \ln|c-b|\Big) = \ln\left|\frac{c-b}{a-b}\right|\end{aligned}$$

This value (which is well defined within real analysis) may be called in real analysis "Cauchy principal value" without reference to any concept or technique of complex analysis.

4. Evaluate the following real improper integrals (of two infinite limits):
(**a**) $I_o = \int_{-\infty}^{+\infty} \frac{x^2}{(x^2+1)^2}\,dx$. (**b**) $I_o = \int_{-\infty}^{+\infty} \frac{1}{x^6+1}\,dx$.
Answer: This Problem represents the category of real rational improper integrals whose both limits are infinite (with some conditions and restrictions on this category that will be explained in the following).[302] In this type of integrals we can replace the real variable (i.e. x in the above examples) by the complex variable z and hence we get a complex-like integral whose (real) limits should be modified to make it fully-fledged complex integral whose value is equal to the value of the original (real) integral. Now, if the integrand of the complex-like integral is analytic over the upper half of the complex plane[303] except at a few poles none of which is on the real axis and the integral of this integrand over an origin-centered semi-circle in the upper half tends to zero as the radius R of the semi-circle tends to infinity[304] then a suitable contour C that can replace the (real) limits of the original integral will be the combination of this semi-circle C_s plus the line segment C_r on the real axis

[302] It should be noted that we assume these improper integrals to be convergent according to the known criteria (as investigated and determined in the literature of real analysis). Anyway, all the improper integrals that we investigate in this book meet these criteria (and hence there should be no worry about this condition).

[303] The lower half of the complex plane may be used instead in this argument and formulation (as will be indicated later). However, in some cases a specific half must be used (or at least it is advantageous). In general, the selection of the contour and in which part of the plane it should be must be determined according to an informed and wise choice and this depends on the nature of the problem.

[304] The above condition (i.e. the integral tends to zero as R tends to infinity) may be expressed more technically as: the maximum of $|f|$ times R on the semi-circle tends to zero as R tends to infinity (with f being the integrand). The condition may also be given (more practically) by demanding the order of the denominator of f to be at least two

between $-R$ and R. This contour is illustrated in Figure 35. Now, the fully-fledged complex integral over C is the sum of the complex integral over C_s and the complex integral over C_r. So, if we extend R to infinity (noting that the value of the integral over C is independent of the value of R as long as C encloses the poles in the upper half) the complex integral over C_s vanishes and hence the complex integral over C becomes equal to the complex integral over C_r. Also, as we extend R to infinity C_r will become the entire real axis (i.e. from $-\infty$ to $+\infty$) and hence the value of the complex integral over C (which can be obtained by the techniques of complex analysis) becomes equal to the value of the original (real) integral. So in brief, as $R \to \infty$ the value of the complex integral over C will be equal to the value of the complex integral over C_r and this latter value is the same as the value of the original (real) integral. The logic and procedure of this method of evaluating this type of integrals will be clarified further in the following.

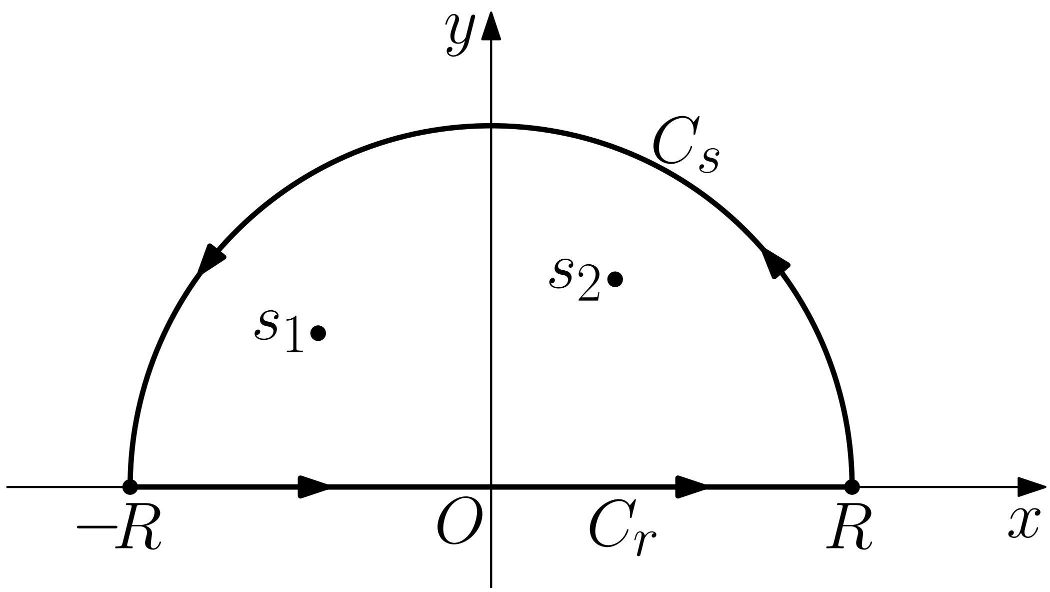

Figure 35: The (anticlockwise) contour C made of the union $C_s \cup C_r$ where C_s is the origin-centered semi-circle (in the upper half of the complex plane) of radius R and C_r is the line segment on the real axis between $-R$ and R. The contour C encloses two poles (s_1 and s_2) none of which is on the real axis. See Problem 4 of § 7.3.

(**a**) The integrand of the complex-like integral $\int_{-\infty}^{+\infty} \frac{z^2}{(z^2+1)^2}\,dz$ has two double poles at $z_{1,2} = \pm i$, i.e. the zeros of $(z^2+1)^2 = [(z-i)(z+i)]^2 = (z-i)^2(z+i)^2$. Only one of these poles (i.e. $z_1 = i$) is in the upper half of the z plane and none of the poles is on the real axis. So, if C is the curve described above with $R > 1$ then C encloses the pole at z_1 and hence we can consider the (fully-fledged) complex integral:[305]

$$I_C = \oint_C \frac{z^2}{(z^2+1)^2}\,dz = \oint_{C_r} \frac{z^2}{(z^2+1)^2}\,dz + \oint_{C_s} \frac{z^2}{(z^2+1)^2}\,dz = I_{C_r} + I_{C_s}$$

where I_C (on the left) can be evaluated by the techniques of complex analysis (i.e. the calculus of residues as we will do). Now, if R goes to infinity then C_r will become the entire real axis (i.e. from

degrees higher than the order of its numerator (which may also be given as: $z \times f$ tends to zero as $|z|$ tends to infinity; as well as other similar forms).

[305] The orientation of C_r is obvious and hence we use the integral symbol $\oint_{C_r}$ although C_r is a straight line. Moreover, C_r (as a part of C which is anticlockwise oriented) can be regarded as anticlockwise oriented. Anyway, we do not need distraction by such trivial issues.

$-\infty$ to $+\infty$) and hence the value of I_{C_r} becomes equal to the value of the original (real) integral I_o, that is:

$$I_{C_r} = I_o$$

Moreover, as R goes to infinity I_{C_s} vanishes (see the upcoming note 1) and hence:

$$I_C = I_{C_r}$$

On combining the last two equations we get $I_o = I_C$. So, all we need to do to find the value of I_o is to evaluate I_C using the techniques of complex analysis.
Now, from Eq. 202 we have:

$$I_C = i2\pi a_{-1}$$

where a_{-1} is the residue of the integrand of I_C corresponding to its Laurent series expansion around i. This residue is obtained by Eq. 204, that is:

$$\begin{aligned} a_{-1} &= \lim_{z\to i}\left[\frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left\{(z-i)^2\frac{z^2}{(z^2+1)^2}\right\}\right] = \lim_{z\to i}\left[\frac{d}{dz}\left\{\frac{z^2}{(z+i)^2}\right\}\right] \\ &= \lim_{z\to i}\left[\frac{2z(z+i)^2 - 2z^2(z+i)}{(z+i)^4}\right] = -\frac{i}{4} \end{aligned}$$

On combining the above results we get:

$$\int_{-\infty}^{+\infty}\frac{x^2}{(x^2+1)^2}\,dx \equiv I_o = I_C = i2\pi a_{-1} = i2\pi\left(-\frac{i}{4}\right) = \frac{\pi}{2}$$

Note 1: the condition that $I_{C_s} \to 0$ as $R \to \infty$ is obviously satisfied according to the above-given criteria (see footnote [304]).
Note 2: the above method of solving the above integral can be generalized to all integrals of the form $\int_{-\infty}^{+\infty}\frac{x^2}{(x^2+a)^2}\,dx$ ($a \in \mathbb{R}$, $a > 0$), that is:

$$\begin{aligned} \int_{-\infty}^{+\infty}\frac{x^2}{(x^2+a)^2}\,dx &= \frac{1}{a^2}\int_{-\infty}^{+\infty}\frac{x^2}{\left(\frac{x^2}{a}+1\right)^2}\,dx = \frac{1}{a^2}\int_{-\infty}^{+\infty}\frac{a\xi^2}{(\xi^2+1)^2}\sqrt{a}\,d\xi \\ &= \frac{a^{3/2}}{a^2}\int_{-\infty}^{+\infty}\frac{\xi^2}{(\xi^2+1)^2}\,d\xi = \frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty}\frac{\xi^2}{(\xi^2+1)^2}\,d\xi = \frac{\pi}{2\sqrt{a}} \end{aligned}$$

where we used $\xi = x/\sqrt{a}$ (and hence $dx = \sqrt{a}\,d\xi$) in the second equality and used the result that we already obtained for I_o in the last equality.
Note 3: if we use the lower half of the complex plane then the pole enclosed by the contour (i.e. $C = C_s \cup C_r$ where C_s now is the origin-centered semi-circle in the lower half of the complex plane) will be at $-i$ and the real axis will be tracked from $+\infty$ to $-\infty$ (assuming an anticlockwise sense) resulting in a minus sign and this should be accounted for by the change of the sign of the residue (which is positive in this case, i.e. $i/4$). Therefore, we should obtain the same result.
(**b**) The integrand of the complex-like integral $\int_{-\infty}^{+\infty}\frac{1}{z^6+1}\,dz$ has six simple poles (i.e. the zeros of z^6+1) which are $e^{i\pi/6}, e^{i3\pi/6}, e^{i5\pi/6}, e^{i7\pi/6}, e^{i9\pi/6}, e^{i11\pi/6}$ (i.e. the six 6^{th} roots of -1).[306] Only three of these poles (i.e. $e^{i\pi/6}, e^{i3\pi/6}, e^{i5\pi/6}$) are in the upper half of the z plane and none of the poles is on the real axis. So, if C is the curve described above with $R > 1$ then C encloses these three poles and hence we can consider the (fully-fledged) complex integral:

$$I_C = \oint_C \frac{1}{z^6+1}\,dz = \oint_{C_r}\frac{1}{z^6+1}\,dz + \oint_{C_s}\frac{1}{z^6+1}\,dz = I_{C_r} + I_{C_s}$$

[306] Within the range $-\pi < \theta \le \pi$, the last three numbers are $e^{-i5\pi/6}, e^{-i3\pi/6}, e^{-i\pi/6}$.

where I_C (on the left) can be evaluated by the techniques of complex analysis (i.e. the calculus of residues as we will do). Now, if R goes to infinity then C_r will become the entire real axis (i.e. from $-\infty$ to $+\infty$) and hence the value of I_{C_r} becomes equal to the value of the original (real) integral I_o, that is:

$$I_{C_r} = I_o$$

Moreover, as R goes to infinity I_{C_s} vanishes (see the upcoming note 1) and hence:

$$I_C = I_{C_r}$$

On combining the last two equations we get $I_o = I_C$. So, all we need to do to find the value of I_o is to evaluate I_C using the techniques of complex analysis.
Now, from Eq. 203 we have:

$$I_C = i2\pi \left({}_1a_{-1} + {}_2a_{-1} + {}_3a_{-1} \right)$$

where ${}_1a_{-1}, {}_2a_{-1}, {}_3a_{-1}$ are the residues of the integrand of I_C corresponding to its Laurent series expansions around $e^{i\pi/6}, e^{i3\pi/6}, e^{i5\pi/6}$ respectively. These residues are obtained by Eq. 205, that is:

$$\begin{aligned}
{}_1a_{-1} &= \lim_{z \to e^{i\pi/6}} \left[\frac{1}{(1-1)!} \frac{d^{1-1}}{dz^{1-1}} \left\{ (z - e^{i\pi/6}) \frac{1}{z^6+1} \right\} \right] = \lim_{z \to e^{i\pi/6}} \left[\frac{z - e^{i\pi/6}}{z^6+1} \right] = \frac{e^{-i5\pi/6}}{6} \\
{}_2a_{-1} &= \lim_{z \to e^{i3\pi/6}} \left[\frac{1}{(1-1)!} \frac{d^{1-1}}{dz^{1-1}} \left\{ (z - e^{i3\pi/6}) \frac{1}{z^6+1} \right\} \right] = \lim_{z \to e^{i3\pi/6}} \left[\frac{z - e^{i3\pi/6}}{z^6+1} \right] = -\frac{i}{6} \\
{}_3a_{-1} &= \lim_{z \to e^{i5\pi/6}} \left[\frac{1}{(1-1)!} \frac{d^{1-1}}{dz^{1-1}} \left\{ (z - e^{i5\pi/6}) \frac{1}{z^6+1} \right\} \right] = \lim_{z \to e^{i5\pi/6}} \left[\frac{z - e^{i5\pi/6}}{z^6+1} \right] = \frac{e^{-i\pi/6}}{6}
\end{aligned}$$

On combining the above results we get:

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{1}{x^6+1}\, dx &\equiv I_o = I_C = i2\pi \left({}_1a_{-1} + {}_2a_{-1} + {}_3a_{-1} \right) = i2\pi \left(\frac{e^{-i5\pi/6}}{6} - \frac{i}{6} + \frac{e^{-i\pi/6}}{6} \right) \\
&= i2\pi \left(-\frac{i}{3} \right) = \frac{2\pi}{3}
\end{aligned}$$

Note 1: the condition that $I_{C_s} \to 0$ as $R \to \infty$ is obviously satisfied according to the above-given criteria (see footnote [304]).
Note 2: the above method of solving the above integral can be generalized to all integrals of the form $\int_{-\infty}^{+\infty} \frac{1}{x^6+a}\, dx$ ($a \in \mathbb{R}$, $a > 0$), that is:

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{1}{x^6+a}\, dx &= \frac{1}{a} \int_{-\infty}^{+\infty} \frac{1}{(x/a^{1/6})^6+1}\, dx = \frac{1}{a} \int_{-\infty}^{+\infty} \frac{1}{\xi^6+1}\, a^{1/6} d\xi \\
&= \frac{a^{1/6}}{a} \int_{-\infty}^{+\infty} \frac{1}{\xi^6+1}\, d\xi = \frac{1}{a^{5/6}} \int_{-\infty}^{+\infty} \frac{1}{\xi^6+1}\, d\xi = \frac{2\pi}{3a^{5/6}}
\end{aligned}$$

where we used $\xi = x/\sqrt[6]{a}$ (and hence $dx = \sqrt[6]{a}\, d\xi$) in the second equality and used the result that we already obtained for I_o in the last equality.
Note 3: if we use the lower half of the complex plane then the poles enclosed by the contour (i.e. $C = C_s \cup C_r$ where C_s now is the origin-centered semi-circle in the lower half of the complex plane) will be at $e^{-i5\pi/6}, e^{-i3\pi/6}, e^{-i\pi/6}$ and the real axis will be tracked from $+\infty$ to $-\infty$ (assuming an anticlockwise sense) resulting in a minus sign and this should be accounted for by the change of the sign of the sum of residues (which is positive in this case, i.e. $i/3$). Therefore, we should obtain the same result.

5. Discuss briefly how to evaluate real improper integrals of only one infinite limit (assuming it is convergent).
Answer: In many cases, this type of integrals is the same as that of Problem 4 but only one of the

limits of the integral is infinite (and hence we may be able to use the method of Problem 4). In fact, there are different cases some of which can be tackled by a similar method to that of Problem 4 while other cases require different treatment (e.g. by combining a number of methods and techniques based for instance on splitting the integral). Anyway, in this brief discussion we consider only one simple case in which the method of Problem 4 can be exploited. In brief, if one limit is infinite and the other limit is 0 while the integrand is symmetric with respect to the imaginary axis (i.e. the integrand is even) then in this case we can assume both limits to be infinite and evaluate that integral (as we did in Problem 4) and then use the symmetry to obtain the value of the original integral (which is half the value of the integral with two infinite limits). For example:

$$\int_{-\infty}^{0} \frac{x^2}{(x^2+1)^2}\,dx = \int_{0}^{+\infty} \frac{x^2}{(x^2+1)^2}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x^2}{(x^2+1)^2}\,dx = \frac{\pi}{4}$$

where we exploited the result of part (a) of Problem 4 (aided by the symmetry). Similarly:

$$\int_{-\infty}^{0} \frac{1}{x^6+1}\,dx = \int_{0}^{+\infty} \frac{1}{x^6+1}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{x^6+1}\,dx = \frac{\pi}{3}$$

where we exploited the result of part (b) of Problem 4 (aided by the symmetry).

6. Evaluate the following real improper integrals:

(**a**) $I_o = \int_{-\infty}^{+\infty} \frac{1}{(x-1)(x^2+4)^2}\,dx$. (**b**) $I_o = \int_{-\infty}^{+\infty} \frac{1}{x^6-1}\,dx$.

Answer: This Problem represents the same type as that of Problem 4 but with some simple poles being on the real axis. Accordingly, the method of tackling this type is similar to the method of Problem 4 but with a slight modification by indenting C with a tiny semi-circle(s) C_ρ of radius ρ in the upper half of the complex plane around the pole(s) on the real axis to exclude that pole(s) only (i.e. without excluding any pole in the upper half). This is illustrated in Figure 36. Therefore, C now is the union $C_s \cup C_r \cup C_\rho$ and hence $I_C = I_{C_s} + I_{C_r} + I_{C_\rho}$. If we now extend R to infinity and repeat the argument of Problem 4 then I_{C_s} vanishes and we get $I_C = I_{C_r} + I_{C_\rho}$ (i.e. $I_{C_r} = I_C - I_{C_\rho}$). Moreover, if we make ρ infinitesimally tiny so that the diameter of the semi-circular arc (on the real axis) is negligibly small that it does not make any contribution to the range of the real integral I_o then we get $I_{C_r} = I_o$. On combining these equations we get:

$$I_o = I_{C_r} = I_C - I_{C_\rho}$$

So, all we need to do to obtain I_o is to evaluate I_C and I_{C_ρ} and take their difference. Now, I_C can be evaluated by the method of residues while I_{C_ρ} can be evaluated by the result of Problem 10 of § 5.4. Noting that the sense of tracking C_ρ is clockwise and its angular measurement is π we have $\theta_\rho = -\pi$ and hence:

$$I_o = I_C - I_{C_\rho} = I_C - ia_{-1}\theta_\rho = I_C - ia_{-1}(-\pi) = I_C + i\pi a_{-1} \tag{230}$$

where we used Eq. 210 in the second equality (with a_{-1} being the residue of the integrand corresponding to its Laurent series expansion around the simple pole on the real axis). The method can be easily extended when we have n simple poles on the real axis, that is:

$$I_o = I_C - \sum_{k=1}^{n} I_{C_{\rho k}} = I_C - i\sum_{k=1}^{n} {}_k a_{-1}\theta_{\rho k} = I_C - i\sum_{k=1}^{n} {}_k a_{-1}(-\pi) = I_C + i\pi\sum_{k=1}^{n} {}_k a_{-1} \tag{231}$$

where ${}_k a_{-1}$ $(k = 1, \cdots, n)$ are the residues of the integrand corresponding (respectively) to its Laurent series expansions around the simple poles s_k $(k = 1, \cdots, n)$ on the real axis. The logic and procedure of this method of evaluating this type of integrals will be clarified further in the following.[307]

(**a**) The integrand of the complex-like integral $\int_{-\infty}^{+\infty} \frac{dz}{(z-1)(z^2+4)^2}$ has a simple pole on the real axis at

[307] Improper integrals of this type but with only one infinite limit may be treated similarly in some cases (see Problem 5).

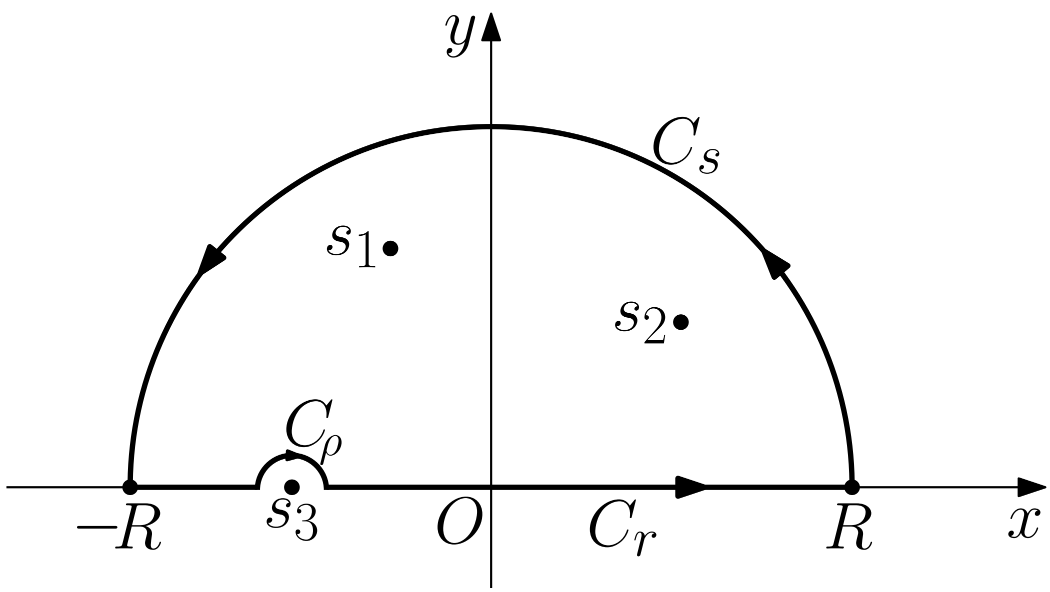

Figure 36: The (anticlockwise) contour C made of the union $C_s \cup C_r \cup C_\rho$ where C_s is the origin-centered semi-circle (in the upper half of the complex plane) of radius R, C_ρ is the upper half of the tiny circle centered on the simple pole s_3 on the real axis, and C_r is the line segment on the real axis between $-R$ and R (excluding the diameter of C_ρ). The contour C encloses two poles (s_1 and s_2) none of which is on the real axis. See Problem 6 of § 7.3.

$z_1 = 1$ as well as two double poles at $z_{2,3} = \pm i2$ only one of which (i.e. $z_2 = i2$) is in the upper half of the complex plane. So, if C is the curve described above with $R > 2$ then C encloses the pole in the upper half at z_2 and hence we can consider the (fully-fledged) complex integral:[308]

$$\begin{aligned} I_C &= \oint_C \frac{dz}{(z-1)(z^2+4)^2} = \oint_{C_r} \frac{dz}{(z-1)(z^2+4)^2} + \oint_{C_\rho} \frac{dz}{(z-1)(z^2+4)^2} + \oint_{C_s} \frac{dz}{(z-1)(z^2+4)^2} \\ &= I_{C_r} + I_{C_\rho} + I_{C_s} \end{aligned}$$

where C_ρ is the semi-circle around z_1. Now, if R goes to infinity and ρ (of C_ρ) approaches zero then C_r will become the entire real axis (i.e. from $-\infty$ to $+\infty$ noting that the diameter of C_ρ becomes negligibly small) and hence the value of I_{C_r} becomes equal to the value of the original (real) integral I_o, that is:

$$I_{C_r} = I_o$$

Moreover, as R goes to infinity I_{C_s} vanishes (according to the given criteria; see footnote [304] on page 289) and hence:

$$I_C = I_{C_r} + I_{C_\rho}$$

On combining the last two equations we get $I_o = I_C - I_{C_\rho}$.
Now, from Eq. 202 we have:

$$I_C = i2\pi \, {}_2a_{-1}$$

[308] We note that C_ρ (as an independent curve) is a clockwise curve and hence we should use the integral symbol $\oint$ in its integral. However, we use the integral symbol $\oint_{C_\rho}$ to avoid distraction. Anyway, we consider the orientation of C_ρ in the evaluation of its integral. We may also justify the symbol $\oint_{C_\rho}$ by the fact that C_ρ is part of C which is oriented anticlockwise. This note also applies to other similar curves in the upcoming questions.

where ${}_2a_{-1}$ is the residue of the integrand of I_C corresponding to its Laurent series expansion around $z_2 = i2$. This residue is obtained by Eq. 204, that is:

$$\begin{aligned} {}_2a_{-1} &= \lim_{z\to i2}\left[\frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left\{(z-i2)^2\frac{1}{(z-1)(z^2+4)^2}\right\}\right] = \lim_{z\to i2}\left[\frac{d}{dz}\left\{\frac{1}{(z-1)(z+i2)^2}\right\}\right] \\ &= \lim_{z\to i2}\left[\frac{-3z+2-i2}{(z-1)^2(z+i2)^3}\right] = \frac{-16+i13}{800} \end{aligned}$$

Hence:

$$I_C = i2\pi\,{}_2a_{-1} = i2\pi\frac{-16+i13}{800} = \frac{-13\pi - i16\pi}{400}$$

We also have (see Eq. 230):

$$I_{C_\rho} = -i\pi\,{}_1a_{-1}$$

where ${}_1a_{-1}$ is the residue of the integrand corresponding to its Laurent series expansion around $z_1 = 1$ (noting that the minus sign is because C_ρ is clockwise). This residue is obtained by Eq. 204, that is:

$${}_1a_{-1} = \lim_{z\to 1}\left[\frac{1}{(1-1)!}\frac{d^{1-1}}{dz^{1-1}}\left\{(z-1)\frac{1}{(z-1)(z^2+4)^2}\right\}\right] = \lim_{z\to 1}\left[\frac{1}{(z^2+4)^2}\right] = \frac{1}{25}$$

Hence:

$$I_{C_\rho} = -i\pi\;{}_1a_{-1} = -\frac{i\pi}{25} = -\frac{i16\pi}{400}$$

Therefore:

$$I_o = I_C - I_{C_\rho} = \frac{-13\pi - i16\pi}{400} - \left(-\frac{i16\pi}{400}\right) = -\frac{13\pi}{400}$$

(**b**) The integrand of the complex-like integral $\int_{-\infty}^{+\infty}\frac{dz}{z^6-1}$ has six simple poles (i.e. the zeros of $z^6 - 1$) which are $z_{1,2,3,4,5,6} = 1, e^{i\pi/3}, e^{i2\pi/3}, -1, e^{i4\pi/3}, e^{i5\pi/3}$ (i.e. the six 6^{th} roots of $+1$).[309] Only two of these poles (i.e. $e^{i\pi/3}, e^{i2\pi/3}$) are in the upper half of the z plane while two of these simple poles are on the real axis (i.e. ± 1). So, if C is the curve described above with $R > 1$ then C encloses the two poles in the upper half and hence we can consider the (fully-fledged) complex integral:

$$I_C = \oint_C \frac{dz}{z^6-1} = \oint_{C_r}\frac{dz}{z^6-1} + \oint_{C_{\rho 1}}\frac{dz}{z^6-1} + \oint_{C_{\rho 4}}\frac{dz}{z^6-1} + \oint_{C_s}\frac{dz}{z^6-1} = I_{C_r} + I_{C_{\rho 1}} + I_{C_{\rho 4}} + I_{C_s}$$

where $C_{\rho 1}, C_{\rho 4}$ are the semi-circles around z_1, z_4. Now, if R goes to infinity and ρ (of $C_{\rho 1}, C_{\rho 4}$) approaches zero then C_r will become the entire real axis (i.e. from $-\infty$ to $+\infty$ noting that the diameters of $C_{\rho 1}, C_{\rho 4}$ become negligibly small) and hence the value of I_{C_r} becomes equal to the value of the original (real) integral I_o, that is:

$$I_{C_r} = I_o$$

Moreover, as R goes to infinity I_{C_s} vanishes (according to the given criteria; see footnote [304] on page 289) and hence:

$$I_C = I_{C_r} + I_{C_{\rho 1}} + I_{C_{\rho 4}}$$

On combining the last two equations we get $I_o = I_C - \left(I_{C_{\rho 1}} + I_{C_{\rho 4}}\right)$.
Now, from Eq. 203 we have:

$$I_C = i2\pi\left({}_2a_{-1} + {}_3a_{-1}\right)$$

where ${}_2a_{-1}, {}_3a_{-1}$ are the residues of the integrand of I_C corresponding to its Laurent series expansions around z_2, z_3 (i.e. $e^{i\pi/3}, e^{i2\pi/3}$) respectively. These residues are obtained by Eq. 205, that is:

$${}_2a_{-1} = \lim_{z\to e^{i\pi/3}}\left[\frac{1}{(1-1)!}\frac{d^{1-1}}{dz^{1-1}}\left\{(z-e^{i\pi/3})\frac{1}{z^6-1}\right\}\right] = \lim_{z\to e^{i\pi/3}}\left[\frac{z-e^{i\pi/3}}{z^6-1}\right] = \frac{e^{i\pi/3}}{6}$$

[309] Within the range $-\pi < \theta \le \pi$, the last two numbers are $e^{-i2\pi/3}, e^{-i\pi/3}$.

$$ {}_3a_{-1} = \lim_{z\to e^{i2\pi/3}} \left[\frac{1}{(1-1)!} \frac{d^{1-1}}{dz^{1-1}} \left\{ (z-e^{i2\pi/3}) \frac{1}{z^6-1} \right\} \right] = \lim_{z\to e^{i2\pi/3}} \left[\frac{z-e^{i2\pi/3}}{z^6-1} \right] = \frac{e^{i2\pi/3}}{6} $$

Hence:

$$ I_C = i2\pi \left({}_2a_{-1} + {}_3a_{-1} \right) = i2\pi \left(\frac{e^{i\pi/3}}{6} + \frac{e^{i2\pi/3}}{6} \right) = -\frac{\pi}{\sqrt{3}} $$

We also have (see Eq. 231):

$$ I_{C_{\rho 1}} + I_{C_{\rho 4}} = -i\pi \left({}_1a_{-1} + {}_4a_{-1} \right) $$

where ${}_1a_{-1}, {}_4a_{-1}$ are the residues of the integrand corresponding to its Laurent series expansions around ± 1 respectively (noting that the minus sign is because $C_{\rho 1}$ and $C_{\rho 4}$ are clockwise). These residues are obtained by Eq. 205, that is:

$$ {}_1a_{-1} = \lim_{z\to +1} \left[\frac{1}{(1-1)!} \frac{d^{1-1}}{dz^{1-1}} \left\{ (z-1) \frac{1}{z^6-1} \right\} \right] = \lim_{z\to +1} \left[\frac{z-1}{z^6-1} \right] = +\frac{1}{6} $$

$$ {}_4a_{-1} = \lim_{z\to -1} \left[\frac{1}{(1-1)!} \frac{d^{1-1}}{dz^{1-1}} \left\{ (z+1) \frac{1}{z^6-1} \right\} \right] = \lim_{z\to -1} \left[\frac{z+1}{z^6-1} \right] = -\frac{1}{6} $$

Hence:

$$ I_{C_{\rho 1}} + I_{C_{\rho 4}} = -i\pi \left({}_1a_{-1} + {}_4a_{-1} \right) = -i\pi \left(\frac{1}{6} - \frac{1}{6} \right) = 0 $$

Therefore:

$$ I_o = I_C - \left(I_{C_{\rho 1}} + I_{C_{\rho 4}} \right) = -\frac{\pi}{\sqrt{3}} - 0 = -\frac{\pi}{\sqrt{3}} $$

Note: similar to what we did in note 2 of part (b) of Problem 4, we can generalize the result obtained in this part of the present Problem to all integrals of the form $\int_{-\infty}^{+\infty} \frac{1}{x^6-a}\, dx$ ($a \in \mathbb{R}$, $a > 0$), that is:

$$ \int_{-\infty}^{+\infty} \frac{1}{x^6-a}\, dx = -\frac{\pi}{\sqrt{3}\, a^{5/6}} $$

7. Let $f(z)$ be an analytic function in the upper half of the complex plane (except *possibly* at a finite number of poles in the upper half), and let the contour C_s be as defined and illustrated earlier (see Figures 35 and 36). Also, assume that the maximum of $|f|$ on C_s tends to zero as R tends to infinity. Show that:

$$ I_{C_s} \to 0 \quad \text{as} \quad R \to \infty \qquad \left(I_{C_s} = \oint_{C_s} e^{iaz} f\, dz \right) $$

where $a \in \mathbb{R}$ and $a > 0$.
Answer: We note first that this theorem is called Jordan's lemma and is commonly used in evaluating certain types of improper integrals (as we will see in Problem 8). We should also note that we have (see the upcoming note 2):

$$ \theta \ge \sin\theta \ge \frac{2\theta}{\pi} \qquad \left(0 \le \theta \le \frac{\pi}{2} \right) \tag{232} $$

Now, on C_s we have:

$$ \left| e^{iaz} \right| = \left| e^{ia(R\cos\theta + iR\sin\theta)} \right| = \left| e^{iaR\cos\theta - aR\sin\theta} \right| = \left| e^{iaR\cos\theta} \right| \left| e^{-aR\sin\theta} \right| = \left| e^{-aR\sin\theta} \right| = e^{-aR\sin\theta} $$

Therefore:

$$ \begin{aligned} |I_{C_s}| &= \left| \oint_{C_s} e^{iaz} f\, dz \right| \\ &\le \oint_{C_s} \left| e^{iaz} \right| |f|\, |dz| \qquad \text{(see Eq. 45)} \end{aligned} $$

$$
\begin{aligned}
&= \int_0^{\pi} e^{-aR\sin\theta}\ |f|\ Rd\theta \\
&\le \int_0^{\pi} e^{-aR\sin\theta}\, M\, Rd\theta && (M \text{ is the maximum of } |f| \text{ on } C_s) \\
&= MR\int_0^{\pi} e^{-aR\sin\theta}\, d\theta \\
&= 2MR\int_0^{\pi/2} e^{-aR\sin\theta}\, d\theta && (e^{-aR\sin\theta} \text{ is symmetric with respect to } \theta = \frac{\pi}{2}) \\
&\le 2MR\int_0^{\pi/2} e^{-aR\left(\frac{2\theta}{\pi}\right)}\, d\theta && (\text{see Eq. } 232) \\
&= 2MR\left[-\pi\frac{e^{-2aR\theta/\pi}}{2aR}\right]_0^{\pi/2} \\
&= 2MR\left[-\pi\frac{e^{-aR}}{2aR} + \frac{\pi}{2aR}\right] \\
&= \frac{\pi M}{a}\left(1 - e^{-aR}\right) \\
&\le \frac{\pi M}{a}
\end{aligned}
$$

that is: $\quad |I_{C_s}| \le \dfrac{\pi M}{a}$

Now, $M \to 0$ as $R \to \infty$ (according to the given assumption) and hence $|I_{C_s}| \to 0$ (according to the last line). This means that $I_{C_s} \to 0$ as $R \to \infty$ (as required).

Note 1: the condition in Jordan's lemma that requires "the maximum of $|f|$ on C_s tends to zero as R tends to infinity" is weaker than the condition required earlier (in evaluating the improper integrals of the types investigated in Problems 4 and 6) that "the maximum of $|f| \times R$ on C_s tends to zero as R tends to infinity" (see footnote [304] on page 289).[310] The relaxation of this condition makes a considerable advantage as it enables us to evaluate certain integrals (as we will see in Problem 8) that cannot be evaluated if we have to observe the stronger condition.

Note 2: for $\theta = 0$ we have $\sin\theta = 0 = \frac{2\theta}{\pi}$ and for $\theta = \pi/2$ we have $\sin\theta = 1 = \frac{2\theta}{\pi}$. Moreover, for $0 < \theta < \pi/2$ the curve of $\sin\theta$ is above the straight line $\frac{2\theta}{\pi}$ which passes through the points $(0,0)$ and $(\pi/2, 1)$ because the second derivative of $\sin\theta$ (i.e. $-\sin\theta$) is negative (i.e. the curve concaves downwards). Accordingly, we can write $\sin\theta \ge \frac{2\theta}{\pi}$ $(0 \le \theta \le \pi/2)$. Also, the curve $\theta - \sin\theta$ is zero at $\theta = 0$ and is increasing over $0 < \theta \le \pi/2$ because its first derivative (i.e. $1 - \cos\theta$) is positive and hence $\theta - \sin\theta$ is positive which means $\theta > \sin\theta$. Accordingly, we can write $\theta \ge \sin\theta$ $(0 \le \theta \le \pi/2)$. On combining these results we get:

$$\theta \ge \sin\theta \ge \frac{2\theta}{\pi} \qquad\qquad \left(0 \le \theta \le \frac{\pi}{2}\right)$$

8. Evaluate the following real improper integrals (where $a, b \in \mathbb{R}$ and $a > 0$):

   (**a**) $I_1 = \int_{-\infty}^{+\infty} \frac{\cos ax}{x-b}\, dx$. (**b**) $I_2 = \int_{-\infty}^{+\infty} \frac{\sin ax}{x-b}\, dx$.

   Answer: Let consider the complex-like integral $\int_{-\infty}^{+\infty} \frac{e^{iaz}}{z-b}\, dz$ where we justify this by the fact that the numerators of the above integrals look like the real and imaginary parts of e^{iaz} (with $z = x$). Now, if we consider a contour C like the one used in Problem 6 (and illustrated in Figure 36) and consider

[310] Being "weaker" is because if $|f| \times R$ tends to zero then $|f|$ should tend to zero (noting that R is a magnifying factor) while if $|f|$ tends to zero then $|f| \times R$ may not tend to zero (noting again that R is a magnifying factor). Hence, some integrals can be evaluated under the weaker condition but not under the stronger condition (and that is why it is "weaker" because it tolerates more integrals to be included). Finally, it should be noted that f in the two conditions are not the same (i.e. being the integrand and part of the integrand).

the (fully-fledged) complex integral $\oint_C \frac{e^{iaz}}{z-b}\,dz$ around C then we have:

$$I_C = \oint_C \frac{e^{iaz}}{z-b}dz = \oint_{C_r} \frac{e^{iaz}}{z-b}dz + \oint_{C_\rho} \frac{e^{iaz}}{z-b}dz + \oint_{C_s} \frac{e^{iaz}}{z-b}dz = I_{C_r} + I_{C_\rho} + I_{C_s}$$

where C_ρ is the semi-circle around $z = b$ on the real axis. Noting that the integrand $\frac{e^{iaz}}{z-b}$ has no singularity inside C, we have $I_C = 0$ (by Cauchy's theorem; see § 4.2). Also, noting that the integral I_{C_s} meets all the conditions of Jordan's lemma (as investigated in Problem 7), we have $I_{C_s} = 0$ as $R \to \infty$. Moreover, if $\rho \to 0$ (with $R \to \infty$), C_r becomes the entire real axis (i.e. from $-\infty$ to $+\infty$) and hence the value of I_{C_r} becomes equal to $\int_{-\infty}^{+\infty} \frac{e^{iax}}{x-b}dx$. On combining these equations we get:

$$\begin{aligned} I_{C_r} + I_{C_\rho} &= 0 \\ I_{C_r} &= -I_{C_\rho} \\ \int_{-\infty}^{+\infty} \frac{e^{iax}}{x-b}dx &= i\pi a_{-1} \end{aligned} \tag{233}$$

where we used $I_{C_\rho} = -i\pi a_{-1}$ (see Problem 10 of § 5.4) noting that a_{-1} is the residue of the integrand corresponding to its Laurent series expansion around $z = b$ and C_ρ is clockwise. Now, from Eq. 204 (noting that $z = b$ is a simple pole of the integral) we have:

$$a_{-1} = \lim_{z\to b}\left[\frac{1}{(1-1)!}\frac{d^{1-1}}{dz^{1-1}}\left\{(z-b)\frac{e^{iaz}}{z-b}\right\}\right] = \lim_{z\to b}\left[e^{iaz}\right] = e^{iab}$$

Hence, from Eq. 233 we get:

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{e^{iax}}{x-b}dx &= i\pi e^{iab} \\ \left(\int_{-\infty}^{+\infty} \frac{\cos ax}{x-b}dx\right) + i\left(\int_{-\infty}^{+\infty} \frac{\sin ax}{x-b}dx\right) &= (-\pi \sin ab) + i\,(\pi \cos ab) \end{aligned}$$

i.e. $I_1 = -\pi \sin ab$ and $I_2 = \pi \cos ab$.

7.4 Summation of Infinite Series

This is one of the common and amusing applications of complex analysis where the calculus of residues is used to find the sum of infinite (real) series and hence the discrete mathematics gets help from the continuous mathematics (or the real mathematics gets help from the complex mathematics). The idea of the method is rather complicated but its essence is simply to convert the problem of "finding the sum of infinite series" to a problem of "finding the sum of a finite number of residues" by exploiting the techniques of the calculus of residues. Accordingly, the problem is reformulated (by finding or designing certain functions and contours) in a way that makes it eligible for the application of the residue theorem which facilitates the aforementioned conversion, as will be explained in the following.

To be specific and elaborate, let S be an infinite series of the form:

$$S(n) = \sum_{n=-\infty}^{n=+\infty} g(n) \qquad\qquad (n \text{ is integer}) \tag{234}$$

where $g(z)$ is a complex function analytic over the complex plane except at some finite number of poles, and for simplicity let assume that none of these poles is at real integer points (i.e. $z = x = n = 0, \pm 1, \pm 2, \cdots$) although this assumption will be lifted (or dealt with) later on. Also, let $f(z)$ be another function analytic over the complex plane except at real integer points where it has a simple pole at each one of these points with a residue equal to 1. Now, from the result of Problem 11 of § 5.4 we can conclude that the residue

${}_nA_{-1}$ of the product fg at each integer point $z = x = n$ is equal to the residue of f (which is 1) times the value of $g(z)$ at that point [which is $g(n)$ since g is analytic at $z = n$], that is:

$$ {}_nA_{-1} = 1 \times g(n) = g(n) \tag{235} $$

Now, let C be a contour over which fg is analytic and C encloses all the singularities of fg (whether these singularities belong to f or g) and let R_g be the sum of the residues of fg at the singularities of g. By the extended Cauchy's theorem we have:

$$ \oint_C fg\,dz = i2\pi R_g + i2\pi \sum_{n=-\infty}^{n=+\infty} {}_nA_{-1} = i2\pi \left(R_g + \sum_{n=-\infty}^{n=+\infty} g(n) \right) \tag{236} $$

$$ \text{that is:} \quad \sum_{n=-\infty}^{n=+\infty} g(n) = \left(\frac{1}{i2\pi} \oint_C fg\,dz \right) - R_g \tag{237} $$

Now, if we can choose C such that the contour integral in the last equation is zero then we get:

$$ \sum_{n=-\infty}^{n=+\infty} g(n) = -R_g \tag{238} $$

In simple terms, the last equation means: the sum of our series is equal to minus the sum of the residues of the function fg at the singularities of g. So, our task is to find a suitable f and a suitable C such that the contour integral vanishes and hence the series is evaluated by finding $-R_g$.

We should now draw the attention to the following useful remarks (before going through the Problems which will apply and clarify the above-described method):

• Apart from its general features, this method is more of an art than a science, and hence the details in each case should be worked out with cunning and flexibility (not by rigidity and formality).

• If $g(z)$ has a pole at an integer point n_0 then n_0 is a pole of f (since n_0 is an integer) as well as being a pole of g, and hence we have a common pole that requires modification to the above strategy and method (although we need to note that even though n_0 is a pole of f and a pole of g it is just one pole for fg and not two poles because it is just a single point). Accordingly, in the above formulation (specifically in Eq. 236) we should either consider n_0 a pole of f or consider n_0 a pole of g. This is to avoid counting this pole twice (in the right hand side of Eq. 236) since for fg (which is integrated on the left hand side of Eq. 236) n_0 is just one pole. The easiest and safest method is to consider n_0 a pole of g (i.e. its contribution is included in R_g) and hence it is excluded from being a pole of f by excluding n_0 from the index of the series, i.e. the series index runs over all the integers excluding n_0. Accordingly, Eq. 236 becomes:

$$ \oint_C fg\,dz = i2\pi R_g + i2\pi \sum_{\substack{n=-\infty \\ n\neq n_0}}^{n=+\infty} {}_nA_{-1} = i2\pi \left(R_g + \sum_{\substack{n=-\infty \\ n\neq n_0}}^{n=+\infty} g(n) \right) \tag{239} $$

$$ \text{that is:} \quad \sum_{\substack{n=-\infty \\ n\neq n_0}}^{n=+\infty} g(n) = \left(\frac{1}{i2\pi} \oint_C fg\,dz \right) - R_g \tag{240} $$

Again, if we can choose C such that the contour integral in the last equation is zero then we get:

$$ \sum_{\substack{n=-\infty \\ n\neq n_0}}^{n=+\infty} g(n) = -R_g \tag{241} $$

Finally, the value of the (complete) series is obtained by adding the excluded term (corresponding to $n = n_0$) to the value obtained from the last equation. An example of this type of series is given in parts

(b) and (d) of Problem 4. It should be obvious that this treatment can be extended easily if more than one pole of g occur at a number of integer points.

• The index of some series may run between a finite limit and an infinite limit (e.g. $\sum_{n=1}^{n=\infty}$) rather than between two infinite limits (i.e. $\sum_{n=-\infty}^{n=+\infty}$) as presumed above. In this case we can split the $\sum_{n=-\infty}^{n=+\infty}$ series to two parts and exclude or include some terms (corresponding to certain indices) separately.[311] For example, if we have to deal with the series $\sum_{n=1}^{n=\infty} \frac{1}{n^2+1}$ then we may write:

$$\begin{aligned}\sum_{n=1}^{n=\infty} \frac{1}{n^2+1} &= \frac{1}{2}\left(\sum_{n=1}^{n=\infty} \frac{1}{n^2+1} + \sum_{n=1}^{n=\infty} \frac{1}{n^2+1}\right) = \frac{1}{2}\left(\sum_{n=-\infty}^{n=-1} \frac{1}{n^2+1} + \sum_{n=1}^{n=\infty} \frac{1}{n^2+1}\right) \\ &= \frac{1}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{n=\infty} \frac{1}{n^2+1} = \frac{1}{2}\left(\left[\sum_{n=-\infty}^{n=+\infty} \frac{1}{n^2+1}\right] - 1\right) = -\frac{1}{2} + \sum_{n=-\infty}^{n=+\infty} \frac{1}{2\left(n^2+1\right)}\end{aligned} \tag{242}$$

and hence we deal again with a series of the type $\sum_{n=-\infty}^{n=+\infty}$. An example of this type of series is given in parts (b) and (d) as well as in the note of part (c) of Problem 4.

Similarly, if we have to deal with the series $\sum_{n=0}^{n=\infty} \frac{1}{n^2+1}$ then we may write:

$$\begin{aligned}\sum_{n=0}^{n=\infty} \frac{1}{n^2+1} &= \frac{1}{2}\left(\sum_{n=1}^{n=\infty} \frac{1}{n^2+1} + \sum_{n=1}^{n=\infty} \frac{1}{n^2+1}\right) + 1 = \frac{1}{2}\left(2 + \sum_{n=-\infty}^{n=-1} \frac{1}{n^2+1} + \sum_{n=1}^{n=\infty} \frac{1}{n^2+1}\right) \\ &= \frac{1}{2}\left(1 + \sum_{n=-\infty}^{n=+\infty} \frac{1}{n^2+1}\right) = \frac{1}{2} + \sum_{n=-\infty}^{n=+\infty} \frac{1}{2\left(n^2+1\right)}\end{aligned} \tag{243}$$

An example of this type of series is given in the note of part (c) of Problem 4.

• If the series is alternating $\left[\text{i.e. it has the form } \sum_{n=-\infty}^{n=+\infty} (-1)^n g(n)\right]$ then the function $f(z)$ should have alternating residues $\left[\text{i.e. } (-1)^n\right]$ and hence Eq. 238 becomes:

$$\sum_{n=-\infty}^{n=+\infty} (-1)^n g(n) = -R_g \tag{244}$$

An example of this type of series is given in part (d) of Problem 4.

• If the series has more than one odd feature then the above treatments (as outlined in the previous points) should be combined. For example, if $g(z)$ has a pole at an integer point n_0 (say $n_0 = 0$) and the series is alternating and its index is running from 1 to ∞ then we can combine the above treatments (as explained in the last three points) and write $\left[\text{assuming } g(n) = g(-n) \text{ and } R_g \text{ includes the contribution of the pole at } n_0\right]$:

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{n=+\infty} (-1)^n g(n) = -R_g \tag{245}$$

$$\sum_{n=1}^{n=\infty} (-1)^n g(n) = -\frac{R_g}{2} \tag{246}$$

An example of this type of series is given in part (d) of Problem 4.

• If the contour integral is not zero (but it is convergent) then its value should be considered (i.e. added) accordingly (noting that being zero is a special case which is the simplest and most common).

Problems

[311] In fact, this method can be applied only to series of certain types (as will be given in the Problems).

1. Give an example of a complex function $f(z)$ that is analytic over the complex plane except at (real) integer points where it has a simple pole at each one of these points with a residue equal to 1.
Answer: In part (d) of Problem 7 of § 5.4 we found that the residue of $\cot z$ at each one of its singularities (i.e. at $z = n\pi = 0, \pm\pi, \pm 2\pi, \cdots$) is 1. So, we can use $\cot z$ as a prototype although we should scale it to put the singularities at $z = n = 0, \pm 1, \pm 2, \cdots$ as we will do in the following. From Eq. 205 $\big[$with ${}_n a_{-1}$ representing the residues of $\cot(\pi z)$ at its singularities of integer points$\big]$ we have:

$$\begin{aligned}
{}_n a_{-1} &= \lim_{z\to n}\left[\frac{1}{(1-1)!}\frac{d^{1-1}}{dz^{1-1}}\left\{(z-n)^1\cot(\pi z)\right\}\right] = \lim_{z\to n}\big[(z-n)\cot(\pi z)\big] \\
&= \frac{1}{\pi}\lim_{z\to n}\big[(\pi z-\pi n)\cot(\pi z)\big] = \frac{1}{\pi}\lim_{z\to n}\left[\frac{\cos(\pi z)}{\frac{\sin(\pi z)}{(\pi z-\pi n)}}\right] = \frac{1}{\pi}\lim_{z\to n}\left[\frac{\cos(\pi z)}{\frac{\sin(\pi z)-0}{(\pi z-\pi n)}}\right] \\
&= \frac{1}{\pi}\lim_{z\to n}\left[\frac{\cos(\pi z)}{\frac{\sin(\pi z)-\sin(\pi n)}{(\pi z-\pi n)}}\right] = \frac{1}{\pi}\left[\frac{\lim_{z\to n}\cos(\pi z)}{\lim_{z\to n}\frac{\sin(\pi z)-\sin(\pi n)}{(\pi z-\pi n)}}\right] \\
&= \frac{1}{\pi}\frac{\cos(\pi n)}{\frac{d\sin(\pi z)}{d(\pi z)}\Big|_{z=n}} = \frac{1}{\pi}\frac{\cos(\pi n)}{\cos(\pi z)\Big|_{z=n}} = \frac{1}{\pi}\frac{\cos(\pi n)}{\cos(\pi n)} = \frac{1}{\pi}
\end{aligned}$$

So, the residues of $\cot(\pi z)$ at its singularities of integer points are $\frac{1}{\pi}$ and hence if we normalize $\cot(\pi z)$ by multiplying it by π we get the required function, i.e. $f(z) = \pi\cot(\pi z)$ is a function that is analytic except at integer points n where it has a simple pole at each n with a residue equal to 1.
2. Give an example of a complex function $f(z)$ that is analytic over the complex plane except at (real) integer points $z = x = n$ where it has a simple pole at each one of these points with a residue equal to $(-1)^n$.
Answer: In part (e) of Problem 7 of § 5.4 we found that the residue of $\csc z$ at each one of its singularities (i.e. at $z = n\pi = 0, \pm\pi, \pm 2\pi, \cdots$) is $(-1)^n$. So, all we need is to put these singularities on integer points by scaling (as we did in Problem 1), that is:

$$\begin{aligned}
{}_n a_{-1} &= \lim_{z\to n}\big[(z-n)\csc(\pi z)\big] = \frac{1}{\pi}\lim_{z\to n}\left[\frac{\pi z-\pi n}{\sin(\pi z)}\right] = \frac{1}{\pi}\left[\frac{1}{\lim_{z\to n}\frac{\sin(\pi z)-\sin(\pi n)}{\pi z-\pi n}}\right] \\
&= \frac{1}{\pi}\frac{1}{\frac{d\sin(\pi z)}{d(\pi z)}\Big|_{z=n}} = \frac{1}{\pi}\frac{1}{\cos(\pi z)\Big|_{z=n}} = \frac{1}{\pi}\frac{1}{\cos(\pi n)} = \frac{(-1)^n}{\pi}
\end{aligned}$$

Now, if we normalize $\csc(\pi z)$ by multiplying it by π we get the required function, i.e. $f(z) = \pi\csc(\pi z)$ is a function that is analytic except at integer points n where it has a simple pole at each n with a residue equal to $(-1)^n$.
3. Let C be an origin-centered circle with radius $R > 0$ that does not go through the integer points of the real axis (i.e. $R \neq 1, 2, \cdots$). Show that $\cot(\pi z)$ is bounded on C.
Answer: We have (see Eqs. 139 and 140):

$$|\cot(\pi z)|^2 = \left|\frac{\cos(\pi z)}{\sin(\pi z)}\right|^2 = \frac{|\cos(\pi z)|^2}{|\sin(\pi z)|^2} = \frac{\cosh^2(\pi y) - \sin^2(\pi x)}{\cosh^2(\pi y) - \cos^2(\pi x)} = \frac{1 - \frac{\sin^2(\pi x)}{\cosh^2(\pi y)}}{1 - \frac{\cos^2(\pi x)}{\cosh^2(\pi y)}}$$

As we see, the numerator in the last equality is bounded (noting that $R \neq 0, 1, 2, \cdots$ and hence $0 \le \frac{\sin^2(\pi x)}{\cosh^2(\pi y)} \le 1$) and the denominator does not vanish (noting that $R \neq 0, 1, 2, \cdots$ and hence $0 \le \frac{\cos^2(\pi x)}{\cosh^2(\pi y)} < 1$). Hence, $|\cot(\pi z)|^2$ is bounded and thus $|\cot(\pi z)|$ is bounded which means that $\cot(\pi z)$ is bounded on C.

Note: we can similarly show that $\csc(\pi z)$ is bounded on C, that is:

$$|\csc(\pi z)|^2 = \left|\frac{1}{\sin(\pi z)}\right|^2 = \frac{1}{|\sin(\pi z)|^2} = \frac{1}{\cosh^2(\pi y) - \cos^2(\pi x)} = \frac{\frac{1}{\cosh^2(\pi y)}}{1 - \frac{\cos^2(\pi x)}{\cosh^2(\pi y)}}$$

As we see, the numerator in the last equality is bounded $\left[\text{since } \cosh^2(\pi y) \geq 1\right]$ and the denominator does not vanish (noting that $R \neq 0, 1, 2, \cdots$ and hence $0 \leq \frac{\cos^2(\pi x)}{\cosh^2(\pi y)} < 1$). Hence, $|\csc(\pi z)|^2$ is bounded and thus $|\csc(\pi z)|$ is bounded which means that $\csc(\pi z)$ is bounded on C.

4. Evaluate the following infinite (real) series using the calculus of residues:

(**a**) $\sum_{n=-\infty}^{n=+\infty} \frac{1}{(n-b)^2}$ (b is real non-integer). (**b**) $\sum_{n=1}^{n=\infty} \frac{1}{n^2}$.

(**c**) $\sum_{n=-\infty}^{n=+\infty} \frac{1}{n^2+b^2}$ (b is real positive). (**d**) $\sum_{n=1}^{n=\infty} \frac{(-1)^n}{n^4}$.

Answer:

(**a**) Let $g(z) = \frac{1}{(z-b)^2}$ and $f(z) = \pi \cot(\pi z)$ and hence $fg = \frac{\pi \cot(\pi z)}{(z-b)^2}$ (noting that f is an analytic function that has a simple pole at each integer point with a residue equal to 1 as established in Problem 1). Moreover, let C be an origin-centered circle with radius $R > 0$ that is big enough to enclose the point $z = b$ and C does not go through the integer points of the real axis (i.e. $R \neq 1, 2, \cdots$), and consider the integral:

$$\oint_C fg\, dz = \oint_C \frac{\pi \cot(\pi z)}{(z-b)^2}\, dz < M \oint_C \frac{dz}{(z-b)^2}$$

where M is an upper bound of $\pi \cot(\pi z)$ noting that $\cot(\pi z)$ is bounded on C (as shown in Problem 3). Now, if $R \to \infty$ then all the singularities of fg will become enclosed inside C, moreover the integral $\oint_C \frac{dz}{(z-b)^2}$ will vanish[312] and hence we get $\oint_C fg\, dz = 0$. Hence, from Eq. 238 (whose all conditions are satisfied) we get:

$$\sum_{n=-\infty}^{n=+\infty} g(n) = \sum_{n=-\infty}^{n=+\infty} \frac{1}{(n-b)^2} = -R_g$$

So, all we need to do to find the sum of our series is to find R_g. Now, g has only one double pole at $z = b$ with a residue of fg there given by:

$$\begin{aligned} a_{-1} &= \lim_{z\to b}\left[\frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\Big\{(z-b)^2 fg\Big\}\right] = \lim_{z\to b}\left[\frac{d}{dz}\left\{(z-b)^2\frac{\pi\cot(\pi z)}{(z-b)^2}\right\}\right] \\ &= \lim_{z\to b}\left[\frac{d}{dz}\{\pi\cot(\pi z)\}\right] = \lim_{z\to b}\left[-\pi^2\csc^2(\pi z)\right] = -\frac{\pi^2}{\sin^2(\pi b)} \end{aligned}$$

Therefore, $-R_g = -a_{-1} = \frac{\pi^2}{\sin^2(\pi b)}$ and hence:

$$\sum_{n=-\infty}^{n=+\infty} \frac{1}{(n-b)^2} = \frac{\pi^2}{\sin^2(\pi b)}$$

(**b**) Let $g(z) = \frac{1}{z^2}$ and $f(z) = \pi \cot(\pi z)$ and hence $fg = \frac{\pi \cot(\pi z)}{z^2}$ (noting that f is as in part a). Also, let C be the curve described in part (a). Hence, g has only one double pole at $z = 0$ and f has a simple pole at each integer point. However, despite the simplicity of its form, this series has two problems. First, its index n runs from 1 to $+\infty$ rather than from $-\infty$ to $+\infty$. Second, the function g (which represents the form of the series terms) has a singularity at an integer point (i.e. $z = n = 0$) which is also a singularity of f. So, we need to deal with these problems to apply the above method

[312] This can be justified by the criteria given earlier (see footnote [304] on page 289). In fact, it may also be justified by the result of Problem 6 of § 4.2.1.

of conversion.
The first problem can be tackled as follows:

$$\sum_{n=1}^{n=\infty}\frac{1}{n^2}=\frac{1}{2}\left(\sum_{n=1}^{n=\infty}\frac{1}{n^2}+\sum_{n=1}^{n=\infty}\frac{1}{n^2}\right)=\frac{1}{2}\left(\sum_{n=-\infty}^{n=-1}\frac{1}{n^2}+\sum_{n=1}^{n=\infty}\frac{1}{n^2}\right)=\frac{1}{2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{n=+\infty}\frac{1}{n^2}$$

Since we excluded $n=0$ in the last equation then the second problem is addressed automatically by the modified formulation of Eq. 238 (which we explained in the remarks) as given by Eq. 241 (noting that as $R\to\infty$ the contour C will enclose all the singularities of fg and $\oint_C fg\,dz=0$ as explained and justified in part a), that is:

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{n=+\infty}\frac{1}{n^2}=-R_g \qquad\to\qquad 2\sum_{n=1}^{n=\infty}\frac{1}{n^2}=-R_g \qquad\to\qquad \sum_{n=1}^{n=\infty}\frac{1}{n^2}=-\frac{R_g}{2}$$

So, all we need to do to find the sum of our series is to find R_g. Now, g has only one double pole at $z=0$ and hence we need to find the residue R_g of fg at $z=0$. As we know from Problem 1, $\cot(\pi z)$ has a simple pole at $z=0$ and hence its Laurent series around $z=0$ is of the form $a_{-1}(\pi z)^{-1}+a_0+a_1(\pi z)+\cdots$. We also know that $\cot(\pi z)=\cos(\pi z)/\sin(\pi z)$ and hence $\sin(\pi z)\cot(\pi z)=\cos(\pi z)$, that is (see Eqs. 192 and 193):

$$\begin{aligned}\left[(\pi z)-\frac{(\pi z)^3}{3!}+\frac{(\pi z)^5}{5!}-\cdots\right]\left[a_{-1}(\pi z)^{-1}+a_0+a_1(\pi z)+\cdots\right] &= 1-\frac{(\pi z)^2}{2!}+\frac{(\pi z)^4}{4!}-\cdots\\ a_{-1}+a_0(\pi z)+\left[-\frac{a_{-1}}{3!}+a_1\right](\pi z)^2+\cdots &= 1-\frac{(\pi z)^2}{2!}+\frac{(\pi z)^4}{4!}-\cdots\end{aligned}$$

On comparing the coefficients on the two sides of the last equation we get:

$$a_{-1}=1 \qquad\qquad a_0=0 \qquad\qquad -\frac{a_{-1}}{3!}+a_1=-\frac{1}{2}$$

and hence $a_1=-\frac{1}{3}$. Therefore, the Laurent series of $fg=\frac{\pi\cot(\pi z)}{z^2}$ around $z=0$ is of the form:

$$\frac{\pi\cot(\pi z)}{z^2}=\frac{\pi}{z^2}\left[(\pi z)^{-1}-\frac{\pi z}{3}+\cdots\right]=\frac{1}{z^3}-\frac{\pi^2}{3z}+\cdots$$

and hence the residue of fg at $z=0$ is $R_g=-\frac{\pi^2}{3}$. Therefore:

$$\sum_{n=1}^{n=\infty}\frac{1}{n^2}=-\frac{R_g}{2}=\frac{\pi^2}{6}$$

(**c**) Let $g(z)=\frac{1}{z^2+b^2}$ and $f(z)=\pi\cot(\pi z)$ and hence $fg=\frac{\pi\cot(\pi z)}{z^2+b^2}$ (noting that f is as in part a). Also, let C be the curve described in part (a). Hence, g has two simple poles at $z=\pm ib$ and f has a simple pole at each integer point (and hence f and g do not share any pole). Accordingly, we use Eq. 238 (noting that as $R\to\infty$ the contour C will enclose all the singularities of fg and $\oint_C fg\,dz=0$ as explained and justified in part a although the second justification for vanishing the integral does not apply here). So, we need to find R_g and this requires calculating the residues of fg at $z=\pm ib$, that is (see Eq. 205):

$$\begin{aligned}{}_1a_{-1} &= \lim_{z\to ib}\left[(z-ib)\frac{\pi\cot(\pi z)}{z^2+b^2}\right]=\lim_{z\to ib}\left[\frac{\pi\cot(\pi z)}{z+ib}\right]=\frac{\pi\cot(ib\pi)}{i2b}=\frac{-i\pi\coth(b\pi)}{i2b}=\frac{-\pi\coth(b\pi)}{2b}\\ {}_2a_{-1} &= \lim_{z\to -ib}\left[(z+ib)\frac{\pi\cot(\pi z)}{z^2+b^2}\right]=\lim_{z\to -ib}\left[\frac{\pi\cot(\pi z)}{z-ib}\right]=\frac{\pi\cot(-ib\pi)}{-i2b}=\frac{-\pi\coth(b\pi)}{2b}\end{aligned}$$

and hence $R_g = {}_1a_{-1} + {}_2a_{-1} = \frac{-\pi \coth(b\pi)}{b}$. Therefore, from Eq. 238 we get:

$$\sum_{n=-\infty}^{n=+\infty} \frac{1}{n^2+b^2} = -R_g = \frac{\pi \coth(b\pi)}{b}$$

Note: if the series is $\sum_{n=1}^{n=+\infty} \frac{1}{n^2+b^2}$ then we can write (using the above result):

$$\sum_{n=1}^{n=\infty} \frac{1}{n^2+b^2} = \frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{n=\infty} \frac{1}{n^2+b^2} = \left(\frac{1}{2} \sum_{n=-\infty}^{n=+\infty} \frac{1}{n^2+b^2} \right) - \frac{1}{2b^2} = \frac{\pi \coth(b\pi)}{2b} - \frac{1}{2b^2}$$

Similarly, if the series is $\sum_{n=0}^{n=+\infty} \frac{1}{n^2+b^2}$ then we can write (using the above result):

$$\sum_{n=0}^{n=\infty} \frac{1}{n^2+b^2} = \left(\frac{1}{2} \sum_{n=-\infty}^{n=+\infty} \frac{1}{n^2+b^2} \right) + \frac{1}{2b^2} = \frac{\pi \coth(b\pi)}{2b} + \frac{1}{2b^2}$$

(**d**) Let $g(z) = \frac{1}{z^4}$ and $f(z) = \pi \csc(\pi z)$ and hence $fg = \frac{\pi \csc(\pi z)}{z^4}$ $\big[$noting that f is an analytic function that has a simple pole at each integer point with a residue equal to $(-1)^n$ as established in Problem 2$\big]$. Also, let C be the curve described in part (a) and hence as $R \to \infty$ the contour C will enclose all the singularities of fg and $\oint_C fg\, dz = 0$ as explained and justified in part (a).[313] Hence, g has only one quadruple pole at $z = 0$ and f has a simple pole at each integer point with an alternating residue $(-1)^n$. Now, if we write:

$$\sum_{n=1}^{n=\infty} \frac{(-1)^n}{n^4} = \frac{1}{2} \left(\sum_{n=1}^{n=\infty} \frac{(-1)^n}{n^4} + \sum_{n=1}^{n=\infty} \frac{(-1)^n}{n^4} \right) = \frac{1}{2} \left(\sum_{n=-\infty}^{n=-1} \frac{(-1)^n}{n^4} + \sum_{n=1}^{n=\infty} \frac{(-1)^n}{n^4} \right) = \frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{n=+\infty} \frac{(-1)^n}{n^4}$$

and use Eq. 245 then we have:

$$\begin{aligned}
\sum_{\substack{n=-\infty \\ n\neq 0}}^{n=+\infty} (-1)^n g(n) &= -R_g \\
\sum_{\substack{n=-\infty \\ n\neq 0}}^{n=+\infty} \frac{(-1)^n}{n^4} &= -R_g \\
\frac{1}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{n=+\infty} \frac{(-1)^n}{n^4} &= -\frac{R_g}{2} \\
\sum_{n=1}^{n=\infty} \frac{(-1)^n}{n^4} &= -\frac{R_g}{2}
\end{aligned}$$

So, all we need is to find R_g and this requires calculating the residue of fg at the pole of g, i.e. at $z = 0$. As we know from Problem 2, $\csc(\pi z)$ has a simple pole at $z = 0$ and hence its Laurent series around $z = 0$ is of the form $a_{-1}(\pi z)^{-1} + a_0 + a_1(\pi z) + a_2(\pi z)^2 + a_3(\pi z)^3 + \cdots$. We also know that $\csc(\pi z) = 1/\sin(\pi z)$ and hence $\sin(\pi z)\csc(\pi z) = 1$, that is (see Eq. 193):

$$\left[(\pi z) - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} - \cdots \right] \left[a_{-1}(\pi z)^{-1} + a_0 + a_1(\pi z) + a_2(\pi z)^2 + a_3(\pi z)^3 + \cdots \right] = 1$$

[313] However, it should be noted that here we have $f = \pi \csc(\pi z)$ although this will not change the argument of part (a) taking into account the note of Problem 3, i.e. $\csc(\pi z)$ is bounded on C like $\cot(\pi z)$.

$$a_{-1} + a_0(\pi z) + \left[a_1 - \frac{a_{-1}}{3!}\right](\pi z)^2 + \left[a_2 - \frac{a_0}{3!}\right](\pi z)^3 + \left[a_3 - \frac{a_1}{3!} + \frac{a_{-1}}{5!}\right](\pi z)^4 + \cdots \quad = \quad 1$$

On comparing the coefficients on the two sides of the last equation we get:

$$a_{-1} = 1 \qquad a_0 = 0 \qquad a_1 - \frac{a_{-1}}{3!} = 0 \qquad a_2 - \frac{a_0}{3!} = 0 \qquad a_3 - \frac{a_1}{3!} + \frac{a_{-1}}{5!} = 0$$

and hence $a_1 = \frac{1}{6}$, $a_2 = 0$ and $a_3 = \frac{7}{360}$. Therefore, the Laurent series of $fg = \frac{\pi \csc(\pi z)}{z^4}$ around $z = 0$ is of the form:

$$\frac{\pi \csc(\pi z)}{z^4} = \frac{\pi}{z^4}\left[(\pi z)^{-1} + \frac{(\pi z)}{6} + \frac{7(\pi z)^3}{360} + \cdots\right] = \frac{1}{z^5} + \frac{\pi^2}{6z^3} + \frac{7\pi^4}{360z} + \cdots$$

and hence the residue of fg at $z = 0$ is $R_g = \frac{7\pi^4}{360}$. Therefore:

$$\sum_{n=1}^{n=\infty} \frac{(-1)^n}{n^4} = -\frac{R_g}{2} = -\frac{7\pi^4}{720}$$

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Note: as well as the above references, we consulted during our work on the preparation of this book many other books, research and review papers and general articles about this subject.

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